Chapter 8: Self Test

Special Functions of Mathematical Physics

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Roadmap

- **Problems 1–3:** analyticity, isolated singularities, residues, poles of $\csc z$, and poles/residues of $\Gamma(z)$.
- **Problems 4–5:** real integrals via Γ and B, with an independent keyhole-contour check for Problem 5.
- **Problem 6:** short definitions at core level—branch cuts, singularity types, analytic continuation, Fuchs' theorem, Sturm–Liouville self-adjointness, and $J_m(x)$ as $x \to 0$.
- **Problem 7:** essential singularity at 0 times simple poles at $\pm i$; residue bookkeeping and a real-line integral.
- **Problems 8–10:** Hermite and Chebyshev by Frobenius (termination and recurrences), then Hermite weighted integrals via orthogonality.

Where we are going

This chapter is a closed-book style self-test for the *core path* of Chapters 1–7 (complex toolkit, Gamma/Beta, Frobenius, Sturm–Liouville, Bessel core, Legendre/Hermite/Chebyshev). Advanced side topics trimmed from earlier chapters are not tested here. Solutions are fully worked, with intermediate algebra written out.

- Problems 1–3: analytic functions, singularities, residues, Gamma poles.
- Problems 4–5: Gamma/Beta integrals and a keyhole contour verification.
- Problem 6: precise short definitions (no fractional calculus, continuous Hankel transforms, or advanced zeta machinery).
- Problem 7: essential singularity plus simple poles.
- Problems 8–10: Hermite/Chebyshev Frobenius and Hermite weighted integrals.

1. Answer the following.

(a) True or false: a function is analytic at $z = z_0$ only if it has a Taylor series representation at z_0.

(b) True or false: if a function is analytic at $z = z_0$, then its derivative must be analytic at $z = z_0$.

(c) True or false: all analytic functions satisfy the Cauchy–Riemann conditions.

(d) True or false: the function $\cos z$ has an essential singularity at infinity.

(e) Each function has a singularity at the origin. Classify it: $\sin(1/z)$, $z/\sin z$, and $1/\sin z$.

(f) Compute $\operatorname{Res}[e^z/z^5, 0]$.

2.

(a) Locate all poles of $1/\sin z$.

(b) Compute $\displaystyle\oint_{|z|=10} \frac{1}{\sin z}\, dz$, with the contour counterclockwise.

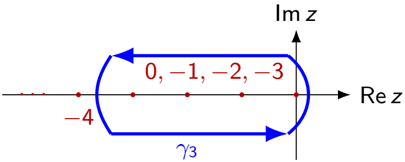
3.

Problems 1–3 II

(a) Locate all poles of the Gamma function $\Gamma(z)$.

(b) Compute the residues at those poles.

(c) Let γ_N be a counterclockwise contour enclosing $0, -1, \ldots, -N$ and no other poles of Γ. Compute $\oint_{\gamma_N} \Gamma(z)\, dz$ and then take the limit $N \to \infty$.



The loop crosses the real axis between -4 and -3, so it encloses exactly the four poles $0, -1, -2, -3$; the pole at -4, and every pole further left, stays outside.

Problems 4–7 I

4. Compute

$$\int_0^\infty e^{-x} x^{2/3}\, dx.$$

5. Compute

$$\int_0^\infty \frac{\sqrt{x}}{x^2+1}\, dx.$$

6. Definitions.

(a) Give an example of a function with a branch cut in the complex plane.

(b) Define removable singularity, pole (sometimes loosely called a "regular" singularity of a *function*), and essential singularity; give an example of each. Do not confuse a pole with a regular singular *point* of an ODE (part (d)).

(c) Explain analytic continuation.

(d) Divide by p_0 to put $p_0(x)u'' + p_1(x)u' + p_2(x)u = 0$ into the normal form $u'' + P(x)u' + Q(x)u = 0$ with $P = p_1/p_0$ and $Q = p_2/p_0$. State the condition on P and Q at a point x_0 under which Fuchs' theorem—the local existence result of Chapter 4—assures a Frobenius solution $u(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k$ with $a_0 \neq 0$, and write down the indicial equation that fixes r.

(e) For the Sturm–Liouville expression $Lu = \dfrac{d}{dx}\left(p(x)\dfrac{du}{dx}\right) + q(x)u$, state what is needed for self-adjointness (formal identity plus boundary conditions). Keep the answer at the level of Chapter 5.

(f) Describe $J_m(x)$ as $x \to 0$ for nonnegative integer m.

7. Consider

$$f(z) = \frac{e^{-1/z^2}}{1 + z^2}.$$

(a) Determine and classify all singularities.

(b) Determine the residue at $z = 0$.

(c) Determine $\displaystyle\int_0^{\infty} f(x)\,dx$.

Problems 8–10

8. Use the method of Frobenius to solve the Hermite equation
$$H''(x) - 2xH'(x) + 2nH(x) = 0.$$

9. Use the method of Frobenius to solve the Chebyshev equation
$$(1 - x^2)T_n'' - xT_n' + n^2 T_n = 0.$$

10. Let $H_n(x)$ be a Hermite polynomial and let m, n be nonnegative integers. Compute
$$\int_{-\infty}^{\infty} xe^{-x^2} H_n(x)\, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x)\, dx.$$

Solution 1: analytic functions and singularities I

(a) **True. Step 1: expand the Cauchy kernel.** Suppose that f is analytic in a neighborhood of z_0. Choose $\rho > 0$ so that the closed disk $|\zeta - z_0| \le \rho$ lies in that neighborhood. For $|z - z_0| < \rho$, Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Rewrite the Cauchy kernel as

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\
&= \frac{1}{\zeta - z_0} \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^k \\
&= \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}.
\end{aligned}$$

Solution 1: analytic functions and singularities II

Since $|z - z_0|/|\zeta - z_0| < 1$, the geometric series converges uniformly on the contour. Therefore it may be integrated term by term:

$$f(z) = \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta \right] (z - z_0)^k$$
$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

Thus analyticity implies a convergent Taylor representation about z_0.

(b) **True. Step 1: differentiate the Taylor series.** From part (a),

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

in some disk about z_0. A power series may be differentiated term by term inside its radius of convergence, so

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}.$$

This is another convergent power series in the same open disk. Hence f' is analytic there.

Solution 1: analytic functions and singularities III

(c) **True.** Write $f = u + iv$. Taking the difference quotient along a real increment h gives

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = u_x + iv_x.$$

Taking it along a purely imaginary increment ih gives

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+ih) - f(z)}{ih} \\ &= \frac{u_y + iv_y}{i} = v_y - iu_y. \end{aligned}$$

Equality of these two directional limits implies

$$u_x = v_y, \qquad v_x = -u_y,$$

which are the Cauchy–Riemann equations. They are necessary for analyticity; by themselves they are not sufficient without additional regularity assumptions on the partial derivatives.

Solution 1: analytic functions and singularities IV

(d) **True.** To classify the point at infinity, put $w = 1/z$ and examine $\cos(1/w)$ at $w = 0$:

$$\cos\left(\frac{1}{w}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} w^{-2k} = 1 - \frac{1}{2!w^2} + \frac{1}{4!w^4} - \cdots .$$

The Laurent series contains infinitely many negative powers. Therefore $w = 0$ is an essential singularity of $\cos(1/w)$, and $z = \infty$ is an essential singularity of $\cos z$.

(e) Expand each function at $z = 0$. First,

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots ,$$

which has infinitely many negative powers; hence $z = 0$ is essential.

Next,

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\ &= z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots\right), \end{aligned}$$

We write $O(z^k)$ for a remainder whose magnitude is at most a constant times $|z|^k$ near $z = 0$. Write the denominator as $1 - u$ with $u = \frac{z^2}{6} - \frac{z^4}{120} + O(z^6)$ and expand $\frac{1}{1-u} = 1 + u + u^2 + \cdots$, keeping terms through z^4:

$$\begin{aligned} \frac{z}{\sin z} &= \frac{1}{1 - \left(\frac{z^2}{6} - \frac{z^4}{120} + \cdots \right)} \\ &= 1 + \left(\frac{z^2}{6} - \frac{z^4}{120} \right) + \left(\frac{z^2}{6} \right)^2 + O(z^6) \\ &= 1 + \frac{z^2}{6} + \left(\frac{1}{36} - \frac{1}{120} \right) z^4 + O(z^6) \\ &= 1 + \frac{z^2}{6} + \frac{7z^4}{360} + O(z^6), \end{aligned}$$

where $\frac{1}{36} - \frac{1}{120} = \frac{10-3}{360} = \frac{7}{360}$. No negative powers occur, so the singularity is removable and the analytic extension has value 1 at $z = 0$.

Finally,

$$\frac{1}{\sin z} = \frac{1}{z}\frac{z}{\sin z}$$
$$= \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + O(z^5).$$

Thus $z = 0$ is a simple pole, with residue 1.

(f) Use the exponential series and display the terms through z^{-1}:

$$\frac{e^z}{z^5} = \frac{1}{z^5}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots\right)$$
$$= z^{-5} + z^{-4} + \frac{1}{2}z^{-3} + \frac{1}{6}z^{-2} + \frac{1}{24}z^{-1} + \frac{1}{120} + \cdots .$$

The residue is the coefficient of z^{-1}:

$$\mathrm{Res}\left[\frac{e^z}{z^5}, 0\right] = \frac{1}{4!} = \frac{1}{24}.$$

Solution 2: poles and residues of $\csc z$ I

Step 1: locate the poles. The zeros of $\sin z$ satisfy

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 0.$$

Hence $e^{iz} = e^{-iz}$, so $e^{2iz} = 1$. Since $e^{w} = 1$ exactly when $w = 2\pi i n$ with $n \in \mathbb{Z}$,

$$2iz = 2\pi i n \quad \Longrightarrow \quad z = n\pi, \qquad n \in \mathbb{Z}.$$

Thus all poles of $1/\sin z$ occur at $z = n\pi$.

Step 2: expand near one zero. To determine their order and residues, set $h = z - n\pi$. Then

$$\begin{aligned} \sin z &= \sin(n\pi + h) \\ &= \sin(n\pi)\cos h + \cos(n\pi)\sin h \\ &= (-1)^n \left(h - \frac{h^3}{3!} + O(h^5) \right). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{\sin z} &= \frac{(-1)^n}{h} \frac{1}{1 - \dfrac{h^2}{6} + O(h^4)} \\ &= \frac{(-1)^n}{z - n\pi} + O(z - n\pi). \end{aligned}$$

Solution 2: poles and residues of $\csc z$ II

Every pole is simple, and

$$\operatorname{Res}\left[\frac{1}{\sin z}, n\pi\right] = (-1)^n.$$

Equivalently, since $(\sin z)' = \cos z$ and $\cos(n\pi) = (-1)^n$,

$$\operatorname{Res}\left[\frac{1}{\sin z}, n\pi\right] = \frac{1}{\cos(n\pi)} = \frac{1}{(-1)^n} = (-1)^n.$$

Step 3: sum the enclosed residues. The pole $n\pi$ is inside $|z| = 10$ precisely when

$$|n|\pi < 10 \quad \iff \quad |n| < \frac{10}{\pi} \approx 3.183.$$

Hence the enclosed indices are

$$n = -3, -2, -1, 0, 1, 2, 3.$$

No pole lies on the circle because $10/\pi$ is not an integer. The residue sum is

$$\begin{aligned}\sum_{n=-3}^{3} (-1)^n &= (-1)^{-3} + (-1)^{-2} + (-1)^{-1} + 1 + (-1)^1 + (-1)^2 + (-1)^3 \\ &= -1 + 1 - 1 + 1 - 1 + 1 - 1 \\ &= -1.\end{aligned}$$

Solution 2: poles and residues of $\csc z$ III

For counterclockwise orientation, the residue theorem gives

$$\begin{aligned} \oint_{|z|=10} \frac{1}{\sin z}\, dz &= 2\pi i \sum_{n=-3}^{3} \operatorname{Res}\left[\frac{1}{\sin z}, n\pi\right] \\ &= 2\pi i(-1) \\ &= -2\pi i. \end{aligned} \tag{8.2.1}$$

Clockwise orientation would multiply the result by -1.

Solution 3: poles and residues of Γ I

Step 1: start from the defining integral. For $\operatorname{Re} z > 0$, the Gamma function is defined by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt.$$

It is analytic there and satisfies

$$\Gamma(z+1) = z\Gamma(z).$$

Applying this recurrence $n+1$ times gives, for every integer $n \geq 0$,

$$\begin{aligned}\Gamma(z+n+1) &= (z+n)\Gamma(z+n)\\ &= (z+n)(z+n-1)\Gamma(z+n-1)\\ &\ \ \vdots\\ &= z(z+1)\cdots(z+n)\Gamma(z).\end{aligned}$$

Consequently,

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}.$$

This formula meromorphically continues Γ beyond $\operatorname{Re} z > 0$. Indeed, the numerator $\Gamma(z+n+1)$ is given by the convergent defining integral whenever $\operatorname{Re}(z+n+1) > 0$, i.e. $\operatorname{Re} z > -(n+1)$, so the right-hand side is analytic on that strictly larger

Solution 3: poles and residues of Γ II

half-plane except where the denominator $z(z+1)\cdots(z+n)$ vanishes. Since $n \geq 0$ may be taken as large as we like, the identity defines Γ as a single meromorphic function on all of $\mathbb{C}$. Its only possible poles are

$$z = 0, -1, -2, \ldots.$$

Fix $n \geq 0$ and let $z \to -n$. The factor $z+n$ is the only vanishing denominator factor, while

$$\Gamma(z+n+1) \longrightarrow \Gamma(1) = 1.$$

For $n \geq 1$, the product of all remaining denominator factors is

$$\begin{aligned}\prod_{k=0}^{n-1}(-n+k) &= (-n)(-(n-1))\cdots(-2)(-1)\\ &= (-1)^n n!.\end{aligned}$$

For $n = 0$, the corresponding product is empty and is defined to be 1, which agrees with $(-1)^0 0! = 1$. Therefore

$$
\begin{aligned}
\operatorname{Res}[\Gamma(z), -n] &= \lim_{z \to -n} (z+n)\Gamma(z) \\
&= \lim_{z \to -n} \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)} \\
&= \frac{1}{(-1)^n n!} \\
&= \frac{(-1)^n}{n!}.
\end{aligned}
\tag{8.2.2}
$$

The limit is finite and nonzero, so every pole is simple.

Let γ_N be counterclockwise and enclose exactly the poles $0, -1, \ldots, -N$. The residue theorem gives

$$
\begin{aligned}
\oint_{\gamma_N} \Gamma(z)\, dz &= 2\pi i \sum_{n=0}^{N} \operatorname{Res}[\Gamma(z), -n] \\
&= 2\pi i \sum_{n=0}^{N} \frac{(-1)^n}{n!}.
\end{aligned}
$$

Solution 3: poles and residues of Γ IV

Now use the exponential Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

At $x = -1$,

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}.$$

Hence the limit of the sequence of contour integrals is

$$\begin{aligned}
\lim_{N\to\infty} \oint_{\gamma_N} \Gamma(z)\, dz &= 2\pi i \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-1)^n}{n!} \\
&= 2\pi i \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \\
&= \frac{2\pi i}{e}.
\end{aligned}$$

This is a limit of a sequence of finite-contour integrals; it is not an assertion that one may replace the sequence by a single closed contour enclosing infinitely many poles.

Solutions 4–5: two real integrals I

Step 1: match the Gamma exponent. Problem 4. The Gamma integral is

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx, \qquad \operatorname{Re} s > 0.$$

Match the exponent of x:

$$s - 1 = \frac{2}{3} \quad \Longrightarrow \quad s = 1 + \frac{2}{3} = \frac{5}{3}.$$

Therefore

$$\begin{aligned} \int_0^\infty e^{-x} x^{2/3}\,dx &= \Gamma\left(\frac{5}{3}\right) \\ &= \left(\frac{5}{3} - 1\right)\Gamma\left(\frac{2}{3}\right) \\ &= \frac{2}{3}\Gamma\left(\frac{2}{3}\right). \end{aligned} \tag{8.2.3}$$

The second equality uses $\Gamma(s+1) = s\Gamma(s)$ with $s = 2/3$.

Solutions 4–5: two real integrals II

Step 1: convert to a beta integral. Problem 5: beta-function evaluation. Let

$$I = \int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx.$$

Set $t = x^2$. Then

$$x = t^{1/2}, \qquad x^{1/2} = t^{1/4}, \qquad dx = \frac{1}{2}t^{-1/2}\,dt.$$

Chapter 3 defined the beta function as an integral over the unit interval,

$$B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\,du, \qquad \operatorname{Re} p > 0, \quad \operatorname{Re} q > 0.$$

Our integral runs over $(0,\infty)$, so we first move that definition onto the half-line. Take

$$u = \frac{t}{1+t}, \qquad \frac{du}{dt} = \frac{(1+t)-t}{(1+t)^2} = \frac{1}{(1+t)^2} > 0.$$

Because the derivative is positive for $t > 0$ the map is strictly increasing, and since $u \to 0$ as $t \to 0^+$ and $u \to 1$ as $t \to \infty$ it carries $(0,\infty)$ onto $(0,1)$; it is therefore an admissible change of variable. Using

$$1-u = 1-\frac{t}{1+t} = \frac{(1+t)-t}{1+t} = \frac{1}{1+t}, \qquad du = \frac{dt}{(1+t)^2},$$

Solutions 4–5: two real integrals III

the integrand transforms as

$$\begin{aligned} u^{p-1}(1-u)^{q-1}\,du &= \frac{t^{p-1}}{(1+t)^{p-1}} \cdot \frac{1}{(1+t)^{q-1}} \cdot \frac{dt}{(1+t)^2} \\ &= \frac{t^{p-1}}{(1+t)^{(p-1)+(q-1)+2}}\,dt \\ &= \frac{t^{p-1}}{(1+t)^{p+q}}\,dt, \end{aligned}$$

since $(p-1)+(q-1)+2 = p+q$. Hence the very same beta function also has the half-line form

$$B(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,dt, \qquad \operatorname{Re} p > 0, \quad \operatorname{Re} q > 0,$$

Solutions 4–5: two real integrals IV

which is exactly the shape our integral has, and the beta–Gamma relation proved in Chapter 3 applies to it unchanged. Thus

$$\begin{aligned} I &= \int_0^\infty \frac{t^{1/4}}{1+t}\left(\frac{1}{2}t^{-1/2}\right)dt \\ &= \frac{1}{2}\int_0^\infty \frac{t^{-1/4}}{1+t}\,dt \\ &= \frac{1}{2}\int_0^\infty \frac{t^{3/4-1}}{(1+t)^{3/4+1/4}}\,dt \\ &= \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right). \end{aligned}$$

Using the beta–Gamma relation

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

we obtain

$$\begin{aligned} I &= \frac{1}{2}\frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)} \\ &= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right). \end{aligned}$$

Solutions 4–5: two real integrals V

Euler's reflection formula,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

with $z = 3/4$ gives

$$\begin{aligned}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) &= \frac{\pi}{\sin(3\pi/4)} \\ &= \frac{\pi}{\sqrt{2}/2} \\ &= \pi\sqrt{2}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx &= \frac{1}{2}\pi\sqrt{2} \\ &= \frac{\pi}{\sqrt{2}}.\end{aligned} \tag{8.2.4}$$

Solutions 4–5: two real integrals VI

Step 2: check by residues on a keyhole contour. Problem 5: independent keyhole-contour check. Define

$$f(z) = \frac{z^{1/2}}{1+z^2}$$

using the branch

$$z^{1/2} = r^{1/2} e^{i\theta/2}, \qquad 0 < \theta < 2\pi.$$

The positive real axis is the branch cut. On the slit annulus $\varepsilon < |z| < R$, the poles are $z = i$ and $z = -i$. Their branch arguments are $\arg i = \pi/2$ and $\arg(-i) = 3\pi/2$, so

$$i^{1/2} = e^{i\pi/4}, \qquad (-i)^{1/2} = e^{3\pi i/4}.$$

Let C be the positively oriented boundary of this slit annulus, decomposed into the outer circle C_R, the lower bank γ_-, the inner circle C_ε, and the upper bank γ_+. Since $(1+z^2)' = 2z$,

$$\mathrm{Res}[f, i] = \frac{i^{1/2}}{2i} = \frac{e^{i\pi/4}}{2i},$$

$$\mathrm{Res}[f, -i] = \frac{(-i)^{1/2}}{-2i} = -\frac{e^{3\pi i/4}}{2i}.$$

Solutions 4–5: two real integrals VII

The residue sum is

$$\begin{aligned}\operatorname{Res}[f, i] + \operatorname{Res}[f, -i] &= \frac{e^{i\pi/4} - e^{3\pi i/4}}{2i} \\ &= \frac{(1+i)/\sqrt{2} - (-1+i)/\sqrt{2}}{2i} \\ &= \frac{\sqrt{2}}{2i} \\ &= -\frac{i}{\sqrt{2}}.\end{aligned}$$

Hence

$$\begin{aligned}\oint_C f(z)\,dz &= 2\pi i\left(-\frac{i}{\sqrt{2}}\right) \\ &= \pi\sqrt{2}.\end{aligned}$$

On the outer circle $|z| = R > 1$,

$$|1 + z^2| \geq \left||z|^2 - 1\right| = R^2 - 1,$$

Solutions 4–5: two real integrals VIII

so

$$\begin{aligned}\left|\int_{C_R} f(z)\,dz\right| &\le (2\pi R)\frac{R^{1/2}}{R^2-1} \\ &= \frac{2\pi R^{3/2}}{R^2-1} \\ &\longrightarrow 0 \qquad (R \to \infty).\end{aligned}$$

On the inner circle $|z| = \varepsilon < 1$,

$$|1+z^2| \ge 1-\varepsilon^2,$$

so

$$\begin{aligned}\left|\int_{C_\varepsilon} f(z)\,dz\right| &\le (2\pi\varepsilon)\frac{\varepsilon^{1/2}}{1-\varepsilon^2} \\ &= \frac{2\pi\varepsilon^{3/2}}{1-\varepsilon^2} \\ &\longrightarrow 0 \qquad (\varepsilon \to 0^+).\end{aligned}$$

Solutions 4–5: two real integrals IX

On the upper bank $\theta \to 0^+$, so $z^{1/2} = r^{1/2} e^{i0} = \sqrt{r}$ and $1 + z^2 = 1 + r^2$. With $z = r$ and orientation $r : \varepsilon \to R$,

$$\int_{\gamma_+} f(z)\,dz = \int_\varepsilon^R \frac{\sqrt{r}}{1 + r^2}\,dr.$$

On the lower bank, $z = re^{2\pi i}$ and the orientation is $r : R \to \varepsilon$. Since

$$z^{1/2} = r^{1/2} e^{\pi i} = -\sqrt{r}, \qquad dz = e^{2\pi i} dr = dr,$$

we have

$$\int_{\gamma_-} f(z)\,dz = \int_R^\varepsilon \frac{-\sqrt{r}}{1 + r^2}\,dr$$

$$= \int_\varepsilon^R \frac{\sqrt{r}}{1 + r^2}\,dr.$$

Therefore the two banks contribute

$$2 \int_\varepsilon^R \frac{\sqrt{r}}{1 + r^2}\,dr.$$

Taking $R \to \infty$ and $\varepsilon \to 0^+$ gives

$$2I = \pi\sqrt{2}, \qquad I = \frac{\pi}{\sqrt{2}},$$

Solutions 4–5: two real integrals X

in agreement with the beta-function calculation.

Solution 6: definitions I

(a) A standard example is $\log z$. Write

$$\log z = \log |z| + i \arg z.$$

Because $\arg z$ changes by 2π after one circuit around the origin, $\log z$ is multivalued on $\mathbb{C} \setminus \{0\}$. The principal branch is made single-valued by deleting the negative real axis and choosing

$$-\pi < \arg z < \pi.$$

The deleted ray is the branch cut, and $z = 0$ is the branch point.

(b) For an isolated singularity z_0, write the Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$

The standard complex-analysis classification is as follows.

Solution 6: definitions II

- The singularity is *removable* when $a_k = 0$ for every $k < 0$. Equivalently, f is bounded near z_0 and can be assigned a value at z_0 that makes it analytic. Example:
$$\frac{z}{\sin z} = 1 + \frac{z^2}{6} + O(z^4)$$
at $z_0 = 0$.
- A *pole of order m* occurs when
$$a_{-m} \neq 0, \qquad a_k = 0 \quad (k < -m).$$
Example: $1/z^2$ has a pole of order 2 at 0. Some sources use "regular singularity" informally for a pole, but the standard term for a singularity of a function is *pole*. A *regular singular point* in differential-equation theory is a different concept, defined in part (d).
- The singularity is *essential* when infinitely many coefficients with $k < 0$ are nonzero. Example:
$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$
at $z_0 = 0$.

(c) **Analytic continuation and splicing.**

The idea: Two local formulas can be joined when they agree wherever their domains overlap. The identity theorem explains when that joined formula is unique.

Let f_1 be analytic on D_1 and f_2 analytic on D_2, with $D_1 \cap D_2 \neq \varnothing$. Assume they agree on the overlap:

$$f_1(z) = f_2(z), \qquad z \in D_1 \cap D_2.$$

Define one function by choosing the formula from the domain containing the point:

$$F(z) = \begin{cases} f_1(z), & z \in D_1, \\ f_2(z), & z \in D_2. \end{cases}$$

Why this works.

1. On $D_1 \cap D_2$, the two formulas give the same value, so F is well-defined.
2. Near a point of D_1, $F = f_1$; near a point of D_2, $F = f_2$. Thus F is locally analytic everywhere on $D_1 \cup D_2$.

Solution 6: definitions IV

3. If $D_1 \cup D_2$ is connected, any other analytic function agreeing with F on a nonempty open patch agrees with F everywhere by the identity theorem.

Repeating this construction along a chain of overlapping domains extends the function. Different paths around a branch point may reach different branches; this is monodromy, not a failure of the local gluing argument.

(d) Divide

$$p_0(x)u'' + p_1(x)u' + p_2(x)u = 0$$

by $p_0(x)$ on a punctured neighborhood of x_0 and define

$$P(x) = \frac{p_1(x)}{p_0(x)}, \qquad Q(x) = \frac{p_2(x)}{p_0(x)}.$$

Then

$$u'' + P(x)u' + Q(x)u = 0.$$

The point x_0 is a regular singular point when

$$(x - x_0)P(x) \quad \text{and} \quad (x - x_0)^2 Q(x)$$

Solution 6: definitions V

extend analytically to $x = x_0$. Fuchs' theorem then guarantees at least one solution of Frobenius form

$$u(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k, \qquad a_0 \neq 0,$$

in a punctured neighborhood. If

$$p_* = \lim_{x \to x_0} (x - x_0) P(x), \qquad q_* = \lim_{x \to x_0} (x - x_0)^2 Q(x),$$

then the indicial equation is

$$r(r-1) + p_* r + q_* = 0.$$

Depending on the difference between the indicial roots, the second independent solution may contain a logarithm.

Solution 6: definitions VI

(e) **Sturm–Liouville self-adjointness.** Assume p and q are real and sufficiently smooth on a finite interval $[a, b]$, and use the Chapter 5 convention (linear in the first slot)

$$(u, v) = \int_a^b u(x)\,\overline{v(x)}\,dx.$$

For $Lu = (pu')' + qu$, one integration by parts (Green's identity) yields

$$(u, Lv) - (Lu, v) = \left[p\left(u\,\overline{v'} - u'\,\overline{v}\right)\right]_a^b.$$

The right-hand side is the *boundary form*. Two facts follow.

1. The differential expression itself is *formally* self-adjoint: the interior terms cancel and only a boundary remainder remains.
2. A concrete operator built from L is self-adjoint when the domain is restricted by boundary conditions that make the boundary form vanish for every admissible pair u, v.

Solution 6: definitions VII

Standard choices on a regular interval include real separated Robin conditions

$$\alpha_a u(a) + \beta_a p(a)u'(a) = 0,$$
$$\alpha_b u(b) + \beta_b p(b)u'(b) = 0,$$

with each real pair (α, β) not both zero. Dirichlet ($u = 0$) and Neumann ($pu' = 0$) are special cases. Periodic conditions are self-adjoint when both the function and the flux match:

$$u(a) = u(b), \qquad p(a)u'(a) = p(b)u'(b).$$

(We do not need Sobolev-domain language for the self-test; the Chapter 5 criterion is: formal SL form plus boundary conditions that kill the boundary form.)

Solution 6: definitions VIII

(f) For a nonnegative integer m, the Bessel function has the convergent series

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2k+m}.$$

Since $\Gamma(k+m+1) = (k+m)!$, the first two terms are

$$\begin{aligned} J_m(x) &= \frac{1}{m!}\left(\frac{x}{2}\right)^m - \frac{1}{(m+1)!}\left(\frac{x}{2}\right)^{m+2} + O(x^{m+4}) \\ &= \frac{1}{m!}\left(\frac{x}{2}\right)^m \left[1 - \frac{x^2}{4(m+1)} + O(x^4)\right]. \end{aligned}$$

Hence, as $x \to 0$,

$$J_m(x) \sim \frac{1}{m!}\left(\frac{x}{2}\right)^m.$$

Solution 7: $e^{-1/z^2}/(1+z^2)$ I

Let

$$f(z) = \frac{e^{-1/z^2}}{1+z^2}.$$

(a) Singularities. At $z = 0$,

$$e^{-1/z^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^{-2k} = 1 - z^{-2} + \frac{z^{-4}}{2!} - \frac{z^{-6}}{3!} + \cdots.$$

This expansion contains infinitely many negative powers. Since $1/(1+z^2)$ is analytic and nonzero at $z = 0$, multiplication by it cannot remove the essential singularity. Therefore $z = 0$ is essential.

The denominator factors as

$$1 + z^2 = (z - i)(z + i).$$

The numerator is analytic and nonzero at both zeros of the denominator because

$$e^{-1/i^2} = e^1 = e, \qquad e^{-1/(-i)^2} = e^1 = e.$$

Solution 7: $e^{-1/z^2}/(1+z^2)$ II

Thus $z=i$ and $z=-i$ are simple poles. Their residues, although not required in part (a), follow from the simple-pole formula: writing $f=g/h$ with $g(z)=e^{-1/z^2}$ and $h(z)=1+z^2$, so that $h'(z)=2z$, a simple zero of h gives

$\operatorname{Res}[f,z_0]=\dfrac{g(z_0)}{h'(z_0)}=\dfrac{e^{-1/z_0^2}}{2z_0}$. Since $e^{-1/z_0^2}=e$ at $z_0=\pm i$,

$$\operatorname{Res}[f,i]=\frac{e}{2i},$$
$$\operatorname{Res}[f,-i]=\frac{e}{-2i}.$$

If the extended complex plane is being considered, put $w=1/z$. Then

$$f(1/w)=\frac{e^{-w^2}}{1+w^{-2}}=\frac{w^2e^{-w^2}}{1+w^2},$$

which is analytic at $w=0$ and vanishes there to order 2. Hence $z=\infty$ is a removable singularity of f; after extension, $f(\infty)=0$.

Solution 7: $e^{-1/z^2}/(1+z^2)$ III

(b) Residue at the essential singularity. For $0 < |z| < 1$,

$$e^{-1/z^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^{-2k}, \qquad \frac{1}{1+z^2} = \sum_{j=0}^{\infty} (-1)^j z^{2j}.$$

Both series converge absolutely at each point of the punctured disk, so their product may be collected by powers:

$$f(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k}}{k!} z^{2j-2k}.$$

Every exponent has the form

$$2j - 2k = 2(j-k),$$

which is even. The equation $2(j-k) = -1$ has no integer solutions, so there is no z^{-1} term. Therefore

$$\operatorname{Res}[f, 0] = 0. \tag{8.2.5}$$

The same conclusion follows immediately from the symmetry $f(-z) = f(z)$: the Laurent series of an even function contains only even powers.

Solution 7: $e^{-1/z^2}/(1+z^2)$ IV

(c) Real integral. Define

$$I = \int_0^\infty \frac{e^{-1/x^2}}{1+x^2}\,dx.$$

The integral converges: near $x = 0^+$ the exponential tends to zero faster than every power, and as $x \to \infty$ the integrand is asymptotic to x^{-2}.

Set

$$u = \frac{1}{x}, \qquad x = \frac{1}{u}, \qquad dx = -\frac{du}{u^2}.$$

Also,

$$1 + x^2 = 1 + \frac{1}{u^2} = \frac{1+u^2}{u^2}.$$

As x runs from 0^+ to ∞, u runs from ∞ to 0^+. Therefore

$$\begin{aligned} I &= \int_\infty^0 \frac{e^{-u^2}}{(1+u^2)/u^2}\left(-\frac{du}{u^2}\right) \\ &= \int_\infty^0 e^{-u^2}\frac{u^2}{1+u^2}\left(-\frac{du}{u^2}\right) \\ &= \int_0^\infty \frac{e^{-u^2}}{1+u^2}\,du. \end{aligned}$$

The integral just obtained is the member $a = 1$ of a one-parameter family; we evaluate that family in closed form and then set $a = 1$. The auxiliary parameter a is what lets the substitution $t = a\sqrt{1+s}$ (used shortly) collapse the answer to a clean erfc. For $a > 0$, introduce

$$I(a) = \int_0^\infty \frac{e^{-u^2}}{u^2 + a^2}\, du,$$

so that $I = I(1)$. Since $u^2 + a^2 > 0$,

$$\begin{aligned}\int_0^\infty e^{-(u^2+a^2)s}\, ds &= \left[-\frac{1}{u^2+a^2} e^{-(u^2+a^2)s}\right]_{s=0}^{s=\infty}\\ &= \frac{1}{u^2+a^2}.\end{aligned}$$

Thus

$$I(a) = \int_0^\infty e^{-u^2} \left[\int_0^\infty e^{-(u^2+a^2)s}\, ds\right] du.$$

The integrand is nonnegative, so Tonelli's theorem permits interchange of the integrals:

$$I(a) = \int_0^\infty e^{-a^2 s} \left[\int_0^\infty e^{-(1+s)u^2}\, du\right] ds.$$

Solution 7: $e^{-1/z^2}/(1+z^2)$ VI

In the inner integral, put $v = \sqrt{1+s}\,u$. Then

$$du = \frac{dv}{\sqrt{1+s}},$$

and

$$\begin{aligned}\int_0^\infty e^{-(1+s)u^2}\,du &= \frac{1}{\sqrt{1+s}}\int_0^\infty e^{-v^2}\,dv \\ &= \frac{\sqrt{\pi}}{2\sqrt{1+s}}.\end{aligned}$$

Consequently,

$$I(a) = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-a^2 s}(1+s)^{-1/2}\,ds.$$

Solution 7: $e^{-1/z^2}/(1+z^2)$ VII

Now set

$$t = a\sqrt{1+s}.$$

Then

$$1+s = \frac{t^2}{a^2}, \qquad s = \frac{t^2}{a^2} - 1, \qquad ds = \frac{2t}{a^2}\,dt, \qquad (1+s)^{-1/2} = \frac{a}{t}.$$

The limits $s = 0$ and $s \to \infty$ correspond to $t = a$ and $t \to \infty$. Hence

$$\begin{aligned} I(a) &= \frac{\sqrt{\pi}}{2} \int_a^\infty e^{-a^2(t^2/a^2-1)}\,\frac{a}{t}\,\frac{2t}{a^2}\,dt \\ &= \frac{\sqrt{\pi}}{2} \int_a^\infty e^{-t^2+a^2}\,\frac{2}{a}\,dt \\ &= \frac{\sqrt{\pi}e^{a^2}}{a} \int_a^\infty e^{-t^2}\,dt. \end{aligned}$$

Using the complementary error function, defined by

$$\operatorname{erfc}(a) = \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-t^2}\,dt,$$

Solution 7: $e^{-1/z^2}/(1+z^2)$ VIII

we obtain

$$I(a) = \frac{\sqrt{\pi}e^{a^2}}{a}\left(\frac{\sqrt{\pi}}{2}\operatorname{erfc}(a)\right) = \frac{\pi}{2a}e^{a^2}\operatorname{erfc}(a).$$

Taking $a = 1$ gives

$$\int_0^\infty f(x)\,dx = I(1) = \frac{\pi e}{2}\operatorname{erfc}(1). \tag{8.2.6}$$

Solution 7: $e^{-1/z^2}/(1+z^2)$ IX

Why the naive upper-semicircle shortcut fails. Along a small semicircle $z=\varepsilon e^{i\theta}$ we have $z^2=\varepsilon^2 e^{2i\theta}$, so

$$-\frac{1}{z^2}=-\varepsilon^{-2}e^{-2i\theta}=-\varepsilon^{-2}\big(\cos 2\theta - i\sin 2\theta\big), \qquad \operatorname{Re}\Big(-\frac{1}{z^2}\Big)=-\frac{\cos(2\theta)}{\varepsilon^2}.$$

Since $|e^w|=e^{\operatorname{Re} w}$,

$$\left|e^{-1/z^2}\right|=\exp\left(-\frac{\cos(2\theta)}{\varepsilon^2}\right).$$

At $\theta=\pi/2$, this equals e^{1/ε^2}, so the usual ML (maximum-times-length) estimate cannot show that the indentation vanishes. In fact, it does not vanish.

Let A_ε be the integral over the clockwise upper indentation from $-\varepsilon$ to $+\varepsilon$. Close the contour with C_R: the real axis from $-R$ to $-\varepsilon$, the clockwise upper semicircle $|z|=\varepsilon$ from $-\varepsilon$ to $+\varepsilon$ (indenting past the essential singularity at 0), the real axis from $+\varepsilon$ to R, and the large upper semicircle $|z|=R$. The large arc is harmless, and it is worth seeing why the exponential factor that spoiled the small arc causes no trouble here. On $|z|=R$ the computation above, with ε replaced by R, gives

$$\left|e^{-1/z^2}\right|=\exp\left(-\frac{\cos(2\theta)}{R^2}\right)\le e^{1/R^2},$$

since $-\cos(2\theta) \le 1$. The denominator obeys the reverse triangle inequality $|1+z^2| \ge |z|^2 - 1 = R^2 - 1$, so for $R > 1$

$$M_R = \max_{|z|=R} |f(z)| \le \frac{e^{1/R^2}}{R^2-1}, \qquad RM_R \le \frac{R\,e^{1/R^2}}{R^2-1} \xrightarrow[R\to\infty]{} 0,$$

because the numerator grows like R while the denominator grows like R^2. The Chapter 2 arc criterion $RM_R \to 0$ therefore applies and the outer arc contributes nothing in the limit. The contrast with the indentation is exactly that the dangerous exponent is $-\cos(2\theta)/R^2 \to 0$ on the large circle, whereas it is $-\cos(2\theta)/\varepsilon^2 \to \pm\infty$ on the small one.

Because f is even, the two real segments combine, as $\varepsilon \to 0^+$ and $R \to \infty$, to

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx = 2I.$$

The only singularity of f inside C_R is the simple pole at $z = i$: the pole at $-i$ lies in the lower half-plane, and the origin is excluded by the indentation. Since

$$\mathrm{Res}[f,i] = \frac{e}{2i},$$

Solution 7: $e^{-1/z^2}/(1+z^2)$ XI

the residue theorem gives

$$\begin{aligned} 2I + \lim_{\varepsilon \to 0^+} A_\varepsilon &= 2\pi i \frac{e}{2i} \\ &= \pi e. \end{aligned}$$

Substituting $2I = \pi e \operatorname{erfc}(1)$ and using the error function $\operatorname{erf}(a) := 1 - \operatorname{erfc}(a)$ yields

$$\begin{aligned} \lim_{\varepsilon \to 0^+} A_\varepsilon &= \pi e \left[1 - \operatorname{erfc}(1)\right] \\ &= \pi e \operatorname{erf}(1) \\ &\neq 0. \end{aligned}$$

Therefore an argument that keeps only the residue at i while dropping the indentation is incorrect.

Solution 8: Hermite equation by Frobenius I

Consider

$$H''(x) - 2xH'(x) + 2nH(x) = 0.$$

After dividing an ODE by its leading coefficient, a point is *ordinary* when the remaining coefficient functions are analytic there. Here the leading coefficient is 1 and the other coefficients are polynomials, so $x = 0$ is ordinary. A Frobenius ansatz is still valid:

$$H(x) = \sum_{k=0}^{\infty} a_k x^{k+r}, \qquad a_0 \neq 0.$$

Differentiate term by term:

$$H'(x) = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1},$$

$$H''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}.$$

Step 2: substitute and align powers. Substitution gives

$$\begin{aligned}0 &= \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} \\ &\quad -2\sum_{k=0}^{\infty}(k+r)a_k x^{k+r} + 2n\sum_{k=0}^{\infty} a_k x^{k+r} \\ &= \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} + 2\sum_{k=0}^{\infty}(n-k-r)a_k x^{k+r}.\end{aligned}$$

In the first sum, replace k by $k+2$:

$$\begin{aligned}0 &= r(r-1)a_0 x^{r-2} + r(r+1)a_1 x^{r-1} \\ &\quad + \sum_{k=0}^{\infty}\left[(k+r+2)(k+r+1)a_{k+2} + 2(n-k-r)a_k\right] x^{k+r}.\end{aligned}$$

Step 3: read the indicial equation and recurrence. The lowest power gives the indicial equation

$$r(r-1)a_0 = 0.$$

Solution 8: Hermite equation by Frobenius III

Because $a_0 \neq 0$,

$$r(r-1) = 0, \qquad r = 0 \text{ or } r = 1.$$

The next-lowest coefficient is

$$r(r+1)a_1 = 0.$$

For $r = 0$, this imposes no condition, so both a_0 and a_1 are free. For $r = 1$, it forces $a_1 = 0$; with $r = 1$ and $a_1 = 0$ the surviving coefficients multiply $x^1, x^3, x^5, \ldots$, so the $r = 1$ solution is an odd function of x. It therefore coincides, up to an overall constant, with the odd-power branch of the $r = 0$ solution (the one with $a_0 = 0$, $a_1 \neq 0$), and nothing new is gained by keeping $r = 1$. Thus, because the expansion point is ordinary, taking $r = 0$ gives the complete two-parameter local solution. After the recurrence has been derived, linearity permits either initial coefficient to be zero; the condition $a_0 \neq 0$ was used only to identify the Frobenius exponent.

With $r = 0$, the coefficient of every x^k gives

$$(k+2)(k+1)a_{k+2} + 2(n-k)a_k = 0.$$

Solving for a_{k+2},

$$a_{k+2} = \frac{2(k-n)}{(k+2)(k+1)}a_k = -\frac{2(n-k)}{(k+2)(k+1)}a_k. \tag{8.2.7}$$

The recurrence changes the index by two, so the even and odd coefficients form independent chains.

Solution 8: Hermite equation by Frobenius IV

Even chain. Starting from a_0,

$$\begin{aligned}
a_2 &= \frac{2(0-n)}{2\cdot 1}a_0 = -na_0,\\
a_4 &= \frac{2(2-n)}{4\cdot 3}a_2\\
&= \frac{2(2-n)}{12}(-na_0)\\
&= \frac{n(n-2)}{6}a_0 = \frac{2^2 n(n-2)}{4!}a_0,\\
a_6 &= \frac{2(4-n)}{6\cdot 5}a_4\\
&= \frac{2(4-n)}{30}\frac{n(n-2)}{6}a_0\\
&= -\frac{n(n-2)(n-4)}{90}a_0 = -\frac{2^3 n(n-2)(n-4)}{6!}a_0.
\end{aligned}$$

By iteration,

$$a_{2k} = (-1)^k \frac{2^k}{(2k)!}\prod_{j=0}^{k-1}(n-2j)\,a_0, \qquad k \geq 0,$$

where the empty product for $k=0$ is 1.

Odd chain. Starting from a_1,

$$\begin{aligned} a_3 &= \frac{2(1-n)}{3\cdot 2}a_1 = -\frac{n-1}{3}a_1, \\ a_5 &= \frac{2(3-n)}{5\cdot 4}a_3 \\ &= \frac{(n-1)(n-3)}{30}a_1, \\ a_7 &= \frac{2(5-n)}{7\cdot 6}a_5 \\ &= -\frac{(n-1)(n-3)(n-5)}{630}a_1. \end{aligned}$$

In general,

$$a_{2k+1} = (-1)^k \frac{2^k}{(2k+1)!}\prod_{j=0}^{k-1}\bigl(n-(2j+1)\bigr)\, a_1.$$

For either parity, the ratio of successive nonzero terms satisfies

$$\left|\frac{a_{k+2}x^{k+2}}{a_k x^k}\right| = \left|\frac{2(k-n)x^2}{(k+2)(k+1)}\right| \longrightarrow 0,$$

so both series converge for every finite x. Hence the complete entire solution is

$$H(x) = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{2^k}{(2k)!} \prod_{j=0}^{k-1} (n-2j) x^{2k}$$
$$+ a_1 \sum_{k=0}^{\infty} (-1)^k \frac{2^k}{(2k+1)!} \prod_{j=0}^{k-1} \big(n-(2j+1)\big) x^{2k+1}.$$

If $n = 2q$ is even, the even chain terminates because the recurrence at $k = n$ gives $a_{n+2} = 0$, while the odd chain remains infinite. If $n = 2q+1$ is odd, the odd chain terminates and the even chain remains infinite. Thus exactly one parity chain produces a polynomial for each nonnegative integer n.

Solution 8: Hermite equation by Frobenius VII

Standard normalization. The equation is linear and homogeneous, so any nonzero constant multiple of a solution is again a solution; the overall scale is exactly the free parameter (a_0 for even n, a_1 for odd n). We are therefore free to fix it, and the conventional physicists' choice makes the highest-degree coefficient equal 2^n, which defines the Hermite polynomial H_n. Write

$$H_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} c_j x^{n-2j}, \qquad c_0 = 2^n.$$

Read the recurrence (8.2.7) downward: knowing the higher coefficient a_{n-2j+2} fixes the lower one a_{n-2j}. Writing $c_j = a_{n-2j}$ for the coefficient of x^{n-2j}, the neighbour just above it is $c_{j-1} = a_{n-2(j-1)} = a_{n-2j+2}$. Setting $k = n-2j$,

$$a_{n-2j+2} = \frac{2((n-2j)-n)}{(n-2j+2)(n-2j+1)} a_{n-2j}.$$

Since $2((n-2j)-n) = -4j$,

$$a_{n-2j+2} = -\frac{4j}{(n-2j+2)(n-2j+1)} a_{n-2j}.$$

Solution 8: Hermite equation by Frobenius VIII

Therefore

$$c_j = a_{n-2j} = -\frac{(n-2j+2)(n-2j+1)}{4j} a_{n-2j+2}$$
$$= -\frac{(n-2j+2)(n-2j+1)}{4j} c_{j-1}.$$

Iterating this relation gives

$$c_j = (-1)^j \frac{n(n-1)(n-2)\cdots(n-2j+1)}{4^j j!} c_0$$
$$= (-1)^j \frac{n!}{4^j j!(n-2j)!} 2^n$$
$$= (-1)^j \frac{2^{n-2j} n!}{j!(n-2j)!}.$$

Thus

$$\begin{aligned} H_n(x) &= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{2^{n-2j} n!}{j!(n-2j)!} x^{n-2j} \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j (2x)^{n-2j} \frac{n!}{j!(n-2j)!}. \end{aligned} \tag{8.2.8}$$

Solution 8: Hermite equation by Frobenius IX

The first cases are

$$\begin{aligned}
H_0(x) &= 1, \\
H_1(x) &= 2x, \\
H_2(x) &= 4x^2 - 2, \\
H_3(x) &= 8x^3 - 12x, \\
H_4(x) &= 16x^4 - 48x^2 + 12.
\end{aligned}$$

Direct substitution of the recurrence into the differential equation verifies every term, and the displayed polynomials satisfy $H_n'' - 2xH_n' + 2nH_n = 0$.

Solution 9: Chebyshev equation by Frobenius

Step 1: insert the Frobenius ansatz. Consider

$$(1-x^2)T'' - xT' + n^2 T = 0.$$

Use the Frobenius ansatz

$$T(x) = \sum_{k=0}^{\infty} a_k x^{k+r}, \qquad a_0 \neq 0.$$

Then

$$T'(x) = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1},$$

$$T''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}.$$

Solution 9: substitution

Substitution gives

$$
\begin{aligned}
0 &= \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} \\
&\quad - \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r} \\
&\quad - \sum_{k=0}^{\infty}(k+r)a_k x^{k+r} + n^2 \sum_{k=0}^{\infty} a_k x^{k+r} \\
&= \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} \\
&\quad + \sum_{k=0}^{\infty} \left[n^2 - (k+r)^2\right] a_k x^{k+r}.
\end{aligned}
$$

Solution 9: index shift

Shift the first sum by two indices:

$$
\begin{aligned}
0 &= r(r-1)a_0 x^{r-2} + r(r+1)a_1 x^{r-1} \\
&\quad + \sum_{k=0}^{\infty} \left[(k+r+2)(k+r+1)a_{k+2} + \left(n^2 - (k+r)^2 \right) a_k \right] x^{k+r}.
\end{aligned}
$$

Solution 9: indicial equation and recurrence

The indicial equation is

$$r(r-1)=0, \qquad r=0 \text{ or } r=1.$$

As in the Hermite problem, $x=0$ is an ordinary point: the leading coefficient $1-x^2$ equals 1 there, and division by it leaves coefficients analytic near 0. From the shifted equation the coefficient of x^{r-2} is $r(r-1)a_0$ and that of x^{r-1} is $r(r+1)a_1$. At $r=0$ both prefactors vanish, so a_0 and a_1 are free. For $r=1$, the coefficient of $x^{r-1}=x^0$ is $r(r+1)a_1=2a_1$, which must vanish, forcing $a_1=0$; the resulting series then contains only $x^1, x^3, \dots$ and duplicates the odd $r=0$ chain. Thus $r=0$ supplies the complete local solution. As in the Hermite case, after deriving the recurrence one may set either a_0 or a_1 to zero; the initial condition $a_0 \neq 0$ only identifies the Frobenius exponent.

For $r=0$,

$$(k+2)(k+1)a_{k+2}+(n^2-k^2)a_k=0,$$

so

$$\begin{aligned} a_{k+2} &= \frac{k^2-n^2}{(k+2)(k+1)}a_k \\ &= -\frac{(n-k)(n+k)}{(k+2)(k+1)}a_k. \end{aligned} \tag{8.2.9}$$

Again the recurrence preserves parity.

Solution 9: coefficient chains I

Even and odd coefficient chains. From a_0,

$$\begin{aligned} a_2 &= \frac{0 - n^2}{2 \cdot 1} a_0 = -\frac{n^2}{2!} a_0, \\ a_4 &= \frac{2^2 - n^2}{4 \cdot 3} a_2 \\ &= \frac{n^2(n^2 - 2^2)}{4!} a_0, \\ a_6 &= \frac{4^2 - n^2}{6 \cdot 5} a_4 \\ &= -\frac{n^2(n^2 - 2^2)(n^2 - 4^2)}{6!} a_0. \end{aligned}$$

In general,

$$a_{2k} = \frac{(-1)^k}{(2k)!} \prod_{j=0}^{k-1} \left(n^2 - (2j)^2\right) a_0.$$

Solution 9: coefficient chains II

From a_1,

$$a_3 = \frac{1-n^2}{3\cdot 2}a_1 = -\frac{n^2-1}{3!}a_1,$$
$$a_5 = \frac{3^2-n^2}{5\cdot 4}a_3 = \frac{(n^2-1)(n^2-3^2)}{5!}a_1,$$
$$a_7 = \frac{5^2-n^2}{7\cdot 6}a_5 = -\frac{(n^2-1)(n^2-3^2)(n^2-5^2)}{7!}a_1.$$

Thus

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!}\prod_{j=0}^{k-1}\left(n^2-(2j+1)^2\right)a_1.$$

If n is even, the even chain terminates at degree n; if n is odd, the odd chain terminates at degree n. The opposite-parity chain remains nonpolynomial. Indeed, for even $n = 2m$ the factor $n^2-(2j)^2$ in a_{2k} vanishes at $j = m$, so a_{n+2} and every higher even coefficient are zero and the even series stops at x^n; likewise, when n is odd the factor $n^2-(2j+1)^2$ in a_{2k+1} vanishes and the odd series stops at x^n. For a nonterminating chain,

$$\left|\frac{a_{k+2}x^{k+2}}{a_k x^k}\right| = \left|\frac{(k^2-n^2)x^2}{(k+2)(k+1)}\right| \longrightarrow |x|^2.$$

Solution 9: coefficient chains III

Thus its power series has radius of convergence 1, consistent with the nearest singular points of the differential equation at $x = \pm 1$. The terminating polynomial chain, of course, is defined for all x.

Solution 9: trigonometric form I

Trigonometric form. First take $-1 < x < 1$ and put

$$x = \cos\theta, \qquad Y(\theta) = T(\cos\theta).$$

with $0 < \theta < \pi$. The trigonometric formulas below are first identities on $-1 < x < 1$ (and at the endpoints by continuity); the resulting recurrence defines the same polynomial for every real or complex x. By the chain rule,

$$Y'(\theta) = -\sin\theta\, T'(x),$$
$$Y''(\theta) = -\cos\theta\, T'(x) + \sin^2\theta\, T''(x).$$

Since $x = \cos\theta$ and $1 - x^2 = \sin^2\theta$,

$$Y''(\theta) = (1 - x^2)T''(x) - xT'(x).$$

Therefore the Chebyshev equation becomes

$$Y'' + n^2 Y = 0.$$

For $n \geq 1$, its general solution is

$$Y(\theta) = A\cos(n\theta) + B\sin(n\theta).$$

For $n = 0$, the transformed equation is $Y'' = 0$, with solution $Y = A + B\theta$; the polynomial branch is the constant $T_0 = 1$. For $n \geq 1$, the polynomial branch normalized by $T_n(1) = 1$ is obtained with $A = 1$ and $B = 0$:

$$T_n(x) = \cos(n \arccos x), \qquad -1 \leq x \leq 1. \tag{8.2.10}$$

Solution 9: trigonometric form II

The sine branch is not a polynomial in x. For integer $n \geq 1$, let U_{n-1} denote the Chebyshev polynomial of the second kind, characterized by $U_{n-1}(\cos\theta) = \sin(n\theta)/\sin\theta$ (with endpoint values by continuity). Then

$$\sin(n \arccos x) = \sqrt{1-x^2}\, U_{n-1}(x).$$

The cosine addition identity gives

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta),$$

so

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0 = 1, \quad T_1 = x.$$

The leading coefficient of T_n is 2^{n-1} for $n \geq 1$, by induction: the base case $T_1 = x$ has leading coefficient $1 = 2^0$, and if T_n has leading term $2^{n-1}x^n$, then in $T_{n+1} = 2xT_n - T_{n-1}$ the term $2xT_n$ contributes $2 \cdot 2^{n-1}x^{n+1} = 2^n x^{n+1}$ while T_{n-1} has degree $n-1$, so the leading coefficient of T_{n+1} is 2^n.

Solution 9: explicit polynomial coefficients I

Explicit polynomial coefficients. For $n \geq 1$, write

$$T_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} c_j x^{n-2j}, \qquad c_0 = 2^{n-1}.$$

Because T_n is itself a solution of the Chebyshev equation, its coefficients obey the recurrence (8.2.9). Writing a_k for the coefficient of x^k in T_n, the nonzero coefficients are $a_n, a_{n-2}, \dots$, so we solve the recurrence *downward* from the known top coefficient $a_n = c_0 = 2^{n-1}$ rather than upward from a_0. Setting $k = n - 2j$ turns (8.2.9) into

$$\begin{aligned} a_{n-2j+2} &= \frac{(n-2j)^2 - n^2}{(n-2j+2)(n-2j+1)} a_{n-2j} \\ &= \frac{-4j(n-j)}{(n-2j+2)(n-2j+1)} a_{n-2j}. \end{aligned}$$

Therefore

$$c_j = a_{n-2j} = -\frac{(n-2j+2)(n-2j+1)}{4j(n-j)} c_{j-1}.$$

Telescoping the product over the j iterations—the numerators $(n-2i+2)(n-2i+1)$, $i = 1, \dots, j$, run through $n, n-1, \dots, n-2j+1$, so their

Solution 9: explicit polynomial coefficients II

product is $n!/(n-2j)!$, while $\prod_{i=1}^{j} 4i = 4^j j!$ and $\prod_{i=1}^{j}(n-i) = (n-1)!/(n-j-1)!$—after j iterations,

$$c_j = (-1)^j \frac{n!/(n-2j)!}{4^j j!\,[(n-1)!/(n-j-1)!]}\,2^{n-1}$$
$$= (-1)^j \frac{2^{n-2j-1} n(n-j-1)!}{j!(n-2j)!}.$$

Consequently,

$$\begin{aligned} T_n(x) &= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{2^{n-2j-1} n(n-j-1)!}{j!(n-2j)!} x^{n-2j} \\ &= \frac{n}{2} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j (n-j-1)!}{j!(n-2j)!} (2x)^{n-2j}. \end{aligned} \tag{8.2.11}$$

The case $n=0$ is handled separately: $T_0(x) = 1$.

Solution 9: examples

For example,

$$\begin{aligned} T_2(x) &= \frac{2}{2}\left[\frac{1!}{0!\,2!}(2x)^2 - \frac{0!}{1!\,0!}(2x)^0\right] \\ &= 2x^2 - 1, \end{aligned}$$

and

$$\begin{aligned} T_3(x) &= \frac{3}{2}\left[\frac{2!}{0!\,3!}(2x)^3 - \frac{1!}{1!\,1!}(2x)\right] \\ &= \frac{3}{2}\left(\frac{8}{3}x^3 - 2x\right) \\ &= 4x^3 - 3x. \end{aligned}$$

Thus

$$T_0 = 1, \qquad T_1 = x, \qquad T_2 = 2x^2 - 1, \qquad T_3 = 4x^3 - 3x, \qquad T_4 = 8x^4 - 8x^2 + 1.$$

Solution 10: Hermite weighted integrals I

Step 1: three-term recurrence. The physicists' Hermite polynomials (normalized as in Solution 8, leading coefficient 2^n) satisfy the standard recurrence

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \qquad (n \geq 1),$$

with $H_0 = 1$ and $H_1 = 2x$. (This is the Chapter 7 three-term relation; it also follows by matching coefficients in the power series of Solution 8.) Solve for the factor xH_n:

$$\begin{aligned} 2xH_n(x) &= H_{n+1}(x) + 2nH_{n-1}(x), \\ xH_n(x) &= \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x). \end{aligned}$$

Thus

$$xH_n = \frac{1}{2}H_{n+1} + nH_{n-1}. \tag{8.2.12}$$

For $n = 0$, the term nH_{n-1} is interpreted as zero, and (8.2.12) reduces to $xH_0 = x = \frac{1}{2}H_1$, which matches $H_1 = 2x$.

Let δ_{jk} denote the Kronecker delta: it is 1 when $j = k$ and 0 otherwise. The Hermite orthogonality and norm are

$$\int_{-\infty}^{\infty} e^{-x^2} H_j(x)H_k(x)\,dx = 2^k k! \sqrt{\pi}\,\delta_{jk}.$$

Because the delta forces $j = k$, the same norm could equivalently be written with $2^j j!$.

Solution 10: Hermite weighted integrals II

(a) One Hermite factor. Since $H_0(x) = 1$,

$$\begin{aligned}\int_{-\infty}^{\infty} xe^{-x^2} H_n(x)\, dx &= \int_{-\infty}^{\infty} e^{-x^2} \left[xH_n(x)\right] H_0(x)\, dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_{n+1}(x) H_0(x)\, dx \\ &\quad + n\int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x) H_0(x)\, dx.\end{aligned}$$

Apply orthogonality term by term:

$$\begin{aligned}\frac{1}{2}\int e^{-x^2} H_{n+1} H_0\, dx &= \frac{1}{2}\sqrt{\pi}\,\delta_{n+1,0} = 0 \qquad (n \geq 0), \\ n\int e^{-x^2} H_{n-1} H_0\, dx &= n\sqrt{\pi}\,\delta_{n-1,0} \\ &= \sqrt{\pi}\,\delta_{n1}.\end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} xe^{-x^2} H_n(x)\, dx = \sqrt{\pi}\,\delta_{n1}. \tag{8.2.13}$$

Solution 10: Hermite weighted integrals III

As a direct check, $H_1(x) = 2x$, so

$$\int_{-\infty}^{\infty} xe^{-x^2} H_1(x)\, dx = 2\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}.$$

For $n \neq 1$, orthogonality gives zero; parity gives the same conclusion for every even n.

Solution 10: Hermite weighted integrals IV

(b) Two Hermite factors. Multiply (8.2.12) by $H_m(x)e^{-x^2}$ and integrate:

$$\begin{aligned}
&\int_{-\infty}^{\infty} xe^{-x^2} H_n(x)H_m(x)\,dx \\
&\quad = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_{n+1}(x)H_m(x)\,dx \\
&\qquad + n\int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x)H_m(x)\,dx.
\end{aligned}$$

The first integral is

$$\begin{aligned}
\frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_{n+1}H_m\,dx &= \frac{1}{2}\,2^{n+1}(n+1)!\sqrt{\pi}\,\delta_{m,n+1} \\
&= 2^n(n+1)!\sqrt{\pi}\,\delta_{m,n+1}.
\end{aligned}$$

For $n \geq 1$, the second is

$$\begin{aligned}
n\int_{-\infty}^{\infty} e^{-x^2} H_{n-1}H_m\,dx &= n\,2^{n-1}(n-1)!\sqrt{\pi}\,\delta_{m,n-1} \\
&= 2^{n-1}n!\sqrt{\pi}\,\delta_{m,n-1}.
\end{aligned}$$

Solution 10: Hermite weighted integrals V

For $n = 0$, this second term is absent. Hence

$$\int_{-\infty}^{\infty} xe^{-x^2} H_n H_m \, dx = 2^n (n+1)! \sqrt{\pi} \, \delta_{m,n+1} + 2^{n-1} n! \sqrt{\pi} \, \delta_{m,n-1},$$

where the second delta term is understood to be zero when $n = 0$. Equivalently,

$$\int_{-\infty}^{\infty} xe^{-x^2} H_n H_m \, dx = \begin{cases} 2^n (n+1)! \sqrt{\pi}, & m = n+1, \\ 2^{n-1} n! \sqrt{\pi}, & m = n-1 \text{ and } n \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.2.14}$$

The result is symmetric under interchange of m and n. For example, if $m = n + 1$, then interchanging the indices uses the second case and gives

$$2^{m-1} m! \sqrt{\pi} = 2^n (n+1)! \sqrt{\pi},$$

which is the same value.

Summary of results I

Core self-test (Problems 1–10): analytic toolkit, Gamma/Beta, definitions, essential-singularity bookkeeping, Hermite/Chebyshev Frobenius, Hermite integrals.

- **P1.** At $z = 0$: $\sin(1/z)$ essential; $z/\sin z$ removable; $1/\sin z$ simple pole. $\operatorname{Res}[e^z/z^5, 0] = 1/24$.
- **P2.** Simple poles of $\csc z$ at $n\pi$ with residue $(-1)^n$;
$$\oint_{|z|=10} \csc z \, dz = -2\pi i.$$
- **P3.** Simple poles of Γ at $z = -n$ ($n = 0, 1, 2, \ldots$):
$$\operatorname{Res}[\Gamma(z), -n] = \frac{(-1)^n}{n!}, \qquad \lim_{N\to\infty} \oint_{\gamma_N} \Gamma(z)\, dz = \frac{2\pi i}{e}.$$
- **P4–5,7.** Real integrals
$$\int_0^\infty e^{-x} x^{2/3}\, dx = \Gamma\left(\frac{5}{3}\right) = \frac{2}{3}\Gamma\left(\frac{2}{3}\right),$$
$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\, dx = \frac{\pi}{\sqrt{2}},$$
$$\int_0^\infty \frac{e^{-1/x^2}}{1+x^2}\, dx = \frac{\pi e}{2} \operatorname{erfc}(1).$$
For $f(z) = e^{-1/z^2}/(1+z^2)$: essential at 0 with residue 0; simple poles at $\pm i$.

Summary of results II

- **P8–9.** Frobenius recurrences at the ordinary point $x = 0$:
$$a_{k+2}^{(H)} = \frac{2(k-n)}{(k+2)(k+1)} a_k^{(H)},$$
$$a_{k+2}^{(T)} = \frac{k^2 - n^2}{(k+2)(k+1)} a_k^{(T)}.$$
Integer n terminates one parity chain (polynomial).
- **P10.** Weighted Hermite integrals
$$\int_{-\infty}^{\infty} xe^{-x^2} H_n(x)\, dx = \sqrt{\pi}\, \delta_{n1},$$
$$\int_{-\infty}^{\infty} xe^{-x^2} H_n H_m\, dx = 2^n (n+1)! \sqrt{\pi}\, \delta_{m,n+1} + 2^{n-1} n! \sqrt{\pi}\, \delta_{m,n-1},$$
with the second term absent when $n = 0$.