Chapter 7: Orthogonal Polynomials

Special Functions of Mathematical Physics

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Outline

Where we are going

This chapter studies four families of **orthogonal polynomials** that arise by separating variables in Laplace, Poisson, Helmholtz, and Schrödinger problems. "Orthogonal" means a weighted inner product of distinct-degree polynomials vanishes; the interval and weight depend on the family.

- **Legendre P_n.** From Laplace in spherical coordinates to the ordinary ($m = 0$) Legendre equation, series termination, generating function, recurrence, Rodrigues, orthogonality, and Fourier–Legendre expansions. Associated Legendre functions and spherical harmonics Y_n^m for $m \neq 0$, with the degree n the same integer that the axisymmetric problem picks out through $\lambda = n(n+1)$.
- **Hermite H_n.** The Hermite equation and series solution, quantum harmonic oscillator motivation, generating function, Rodrigues formula, and weighted orthogonality.
- **Laguerre L_n and Chebyshev T_n.** Defining ODEs, first polynomials, weights and orthogonality, recurrences, and useful formula sheets.

A comparison table at the end collects interval, weight, and ODE for each family.

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Two-dimensional Laplace equation in polar coordinates I

Step 1: separate the angular and radial parts. A function satisfying Laplace's equation $\nabla^2\psi = 0$ is called a *harmonic function*. In two dimensions,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0. \tag{7.1.1}$$

Take a separated solution $\psi(r,\theta) = u(r)v(\theta)$. Then

$$\frac{\partial\psi}{\partial r} = u'(r)v(\theta), \qquad \frac{\partial^2\psi}{\partial\theta^2} = u(r)v''(\theta),$$

so substitution into (7.1.1) gives

$$\frac{v}{r}\frac{d}{dr}\left(ru'\right) + \frac{u}{r^2}v'' = 0.$$

Multiplying by $r^2/(uv)$ yields

$$\frac{r(ru')'}{u} + \frac{v''}{v} = 0,$$

and therefore

$$-\frac{v''}{v} = \frac{r(ru')'}{u} = \lambda, \tag{7.1.2}$$

because the left member depends only on θ and the right member only on r.

The angular equation is

$$v'' + \lambda v = 0.$$

For $\lambda > 0$, write $\lambda = \mu^2$; then

$$v(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta).$$

The periodicity condition $v(\theta + 2\pi) = v(\theta)$ restricts μ: since $\cos\mu\theta$ and $\sin\mu\theta$ have period $2\pi/\mu$, the condition can hold for all θ only when 2π is a whole multiple of that period, i.e. only when μ is an integer. Hence $\mu = n$ with $n \in \mathbb{Z}$. Since the modes n and $-n$ span the same real sine–cosine pair, we may take $n \geq 0$ and $\lambda = n^2$. If $\lambda = 0$, then $v = A + B\theta$, and periodicity forces $B = 0$; this is the $n = 0$ mode. If $\lambda < 0$, the angular solutions are exponentials rather than periodic functions, so no nonzero 2π-periodic mode occurs.

The radial equation is

$$r(ru')' - n^2 u = 0.$$

Expanding the derivative gives a Cauchy–Euler (equidimensional) equation — one in which every term carries as many powers of r as derivatives of u —

$$r^2 u'' + ru' - n^2 u = 0, \tag{7.1.3}$$

which is therefore solved by trial powers $u = r^p$. For $n \geq 1$, try $u = r^p$. Since

$$u' = pr^{p-1}, \qquad u'' = p(p-1)r^{p-2},$$

we obtain

$$r^2 u'' + ru' - n^2 u = \left[p(p-1) + p - n^2\right] r^p = (p^2 - n^2) r^p.$$

Two-dimensional Laplace equation in polar coordinates III

Thus $p = \pm n$, and

$$u_n(r) = C_n r^n + D_n r^{-n} \qquad (n \geq 1).$$

For $n = 0$, (7.1.3) becomes $(ru')' = 0$. Integrating twice,

$$ru' = C_0, \qquad u' = \frac{C_0}{r}, \qquad u(r) = D_0 + C_0 \log r.$$

Consequently, on an annulus the general real separated expansion has the form

$$\begin{aligned}\psi(r,\theta) = {} & A_0 + B_0 \log r \\ & + \sum_{n=1}^{\infty} \Big[(A_n r^n + B_n r^{-n}) \cos(n\theta) + (C_n r^n + D_n r^{-n}) \sin(n\theta) \Big].\end{aligned}$$

Inside a disk containing $r = 0$, boundedness removes $\log r$ and all r^{-n} terms:

$$\psi(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \Big[A_n \cos(n\theta) + C_n \sin(n\theta) \Big]. \tag{7.1.4}$$

Equivalently, with $z = re^{i\theta}$ and complex coefficients $c_n = A_n - iC_n$,

$$\psi(r,\theta) = \operatorname{Re}\left(\sum_{n=0}^{\infty} c_n z^n \right), \qquad z^n = r^n e^{in\theta}.$$

Two-dimensional Laplace equation in polar coordinates IV

Thus the regular real harmonic functions are real parts of analytic power series; exterior or annular domains additionally allow negative powers and, for the angularly independent $n = 0$ mode, a logarithm.

Three-dimensional Laplace equation in spherical coordinates I

We accept the spherical Laplacian. Unlike the 2D polar case, which we derived carefully from Cartesian coordinates in earlier courses, the three-dimensional spherical Laplacian is a standard (but longer) chain-rule computation. We take it as given:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0. \tag{7.1.5}$$

Step 1: separate the radial variable. Set $\psi(r,\theta,\phi) = u(r)y(\theta,\phi)$. Because y is independent of r and u is independent of the angles,

$$\frac{\partial}{\partial r}(r^2\psi_r) = y(r^2u')',$$
$$\frac{\partial}{\partial\theta}(\sin\theta\,\psi_\theta) = u(\sin\theta\,y_\theta)_\theta.$$

Substitution into (7.1.5) and multiplication by $r^2/(uy)$ give

$$\frac{(r^2u')'}{u} = -\frac{\dfrac{1}{\sin\theta}(\sin\theta\,y_\theta)_\theta + \dfrac{1}{\sin^2\theta}y_{\phi\phi}}{y} = \lambda. \tag{7.1.6}$$

The first expression depends only on r and the second only on (θ,ϕ), so each equals a constant λ.

Three-dimensional Laplace equation in spherical coordinates II

Step 2: radial Euler equation. Expanding $(r^2u')' = r^2u'' + 2ru'$ gives

$$r^2u'' + 2ru' - \lambda u = 0.$$

Try $u = r^\sigma$. Then $u' = \sigma r^{\sigma-1}$, $u'' = \sigma(\sigma-1)r^{\sigma-2}$, and

$$\Big[\sigma(\sigma+1) - \lambda\Big] r^\sigma = 0.$$

So $\lambda = \sigma(\sigma+1)$. Once the angular problem forces $\lambda = n(n+1)$ with $n = 0, 1, 2, \ldots$ (next frames), the two exponents are

$$\sigma = n, \qquad \sigma = -n-1,$$

and

$$u_n(r) = A_n r^n + B_n r^{-n-1}. \tag{7.1.7}$$

The r^n branch is regular at the origin; the r^{-n-1} branch decays at infinity.

The remaining angular eigenproblem is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 y}{\partial\phi^2} + \lambda y = 0.$$

We first treat the **axisymmetric** case (no ϕ-dependence), which yields the ordinary Legendre equation and $\lambda = n(n+1)$. Associated Legendre modes for $m \neq 0$ come later as a short extension.

Axisymmetry: the ordinary Legendre equation I

Step 1: set $m = 0$ and change variables. If ψ is independent of ϕ (axisymmetry about the polar axis), the angular equation reduces to

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dv}{d\theta}\right) + \lambda v = 0.$$

Set $x = \cos\theta$, so $dx/d\theta = -\sin\theta$ and $d/d\theta = -\sin\theta\, d/dx$. Then

$$\sin\theta\,\frac{dv}{d\theta} = -(1-x^2)v_x,$$

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{dv}{d\theta}\right) = \sin\theta\frac{d}{dx}\left[(1-x^2)v_x\right],$$

and therefore

$$\frac{d}{dx}\left[(1-x^2)v'(x)\right] + \lambda v(x) = 0. \qquad (7.1.8)$$

Expanding the derivative gives the equivalent form

$$(1-x^2)v'' - 2xv' + \lambda v = 0. \qquad (7.1.9)$$

(The poles $\theta = 0, \pi$ correspond to the endpoints $x = \pm 1$.)

Step 2: power series and coefficient recurrence. Set

$$v(x) = \sum_{j=0}^{\infty} a_j x^j.$$

Axisymmetry: the ordinary Legendre equation II

Then

$$v' = \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j,$$

$$v'' = \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j,$$

$$-x^2 v'' = -\sum_{j=0}^{\infty} j(j-1) a_j x^j,$$

$$-2xv' = -2 \sum_{j=0}^{\infty} j\, a_j x^j.$$

Substitution into (7.1.9) gives

$$0 = \sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} - j(j-1) a_j - 2j a_j + \lambda a_j \right] x^j$$

$$= \sum_{j=0}^{\infty} \Big[(j+2)(j+1) a_{j+2} + \big(\lambda - j(j+1) \big) a_j \Big] x^j.$$

Hence the coefficient recurrence is

$$a_{j+2} = -\frac{\lambda - j(j+1)}{(j+2)(j+1)}\, a_j. \tag{7.1.10}$$

Even and odd chains are independent.

Step 3: boundedness at the poles forces $\lambda = n(n+1)$. The endpoints $x = \pm 1$ are the poles $\theta = 0, \pi$, where the sphere is perfectly smooth and nothing physical happens, so we require v to stay bounded on the *closed* interval $[-1, 1]$. Two observations turn that requirement into a condition on λ.

(a) The two chains may be tested one at a time. Put $w(x) = v(-x)$. Then $w'(x) = -v'(-x)$ and $w''(x) = v''(-x)$, so

$$(1-x^2)w'' - 2xw' + \lambda w = (1-x^2)v''(-x) + 2x\, v'(-x) + \lambda v(-x),$$

which is the left member of (7.1.9) evaluated at $-x$. Hence w solves the equation whenever v does, and so do the even and odd parts $\frac{1}{2}\left[v(x) \pm v(-x)\right]$. Those parts are exactly the even and the odd coefficient chain of $v = \sum_j a_j x^j$. If v is bounded on $[-1, 1]$, both parts are bounded there, so each chain must pass the endpoint test on its own: a divergence of one cannot be cancelled by the other.

Axisymmetry: the ordinary Legendre equation IV

(b) A chain that never terminates is unbounded at $x = \pm 1$. Suppose the numerator in (7.1.10) never vanishes along a chain; then no coefficient of that chain is zero. Since

$$(j+2)(j+1) - \Big[j(j+1) - \lambda\Big] = (j+1)\Big[(j+2) - j\Big] + \lambda = 2j + 2 + \lambda,$$

the ratio of consecutive coefficients is

$$\frac{a_{j+2}}{a_j} = \frac{j(j+1) - \lambda}{(j+2)(j+1)} = 1 - \frac{2j+2+\lambda}{(j+2)(j+1)} \xrightarrow[j\to\infty]{} 1.$$

Therefore $|a_{j+2}x^{j+2}|/|a_j x^j| \to x^2$: the chain converges for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is exactly 1 — and at $x = \pm 1$ the ratio test is silent. We must look at the size of the coefficients. Write $j = 2k + p$, with $p = 0$ for the even chain and $p = 1$ for the odd one, and set $b_k = a_{2k+p}$. Then
$2j + 2 + \lambda = 4k + (2p + 2 + \lambda)$ and
$(j+2)(j+1) = 4k^2 + 2(2p+3)k + (p+2)(p+1)$, so

$$\frac{b_{k+1}}{b_k} = 1 - \frac{4k + O(1)}{4k^2 + O(k)} = 1 - \frac{1}{k} + O\Big(\frac{1}{k^2}\Big).$$

The ratio tends to $+1$, so beyond some index K all the b_k carry one and the same sign, and

$$\log|b_{k+1}| - \log|b_k| = \log\Big(1 - \frac{1}{k} + O\Big(\frac{1}{k^2}\Big)\Big) = -\frac{1}{k} + O\Big(\frac{1}{k^2}\Big).$$

Axisymmetry: the ordinary Legendre equation V

Summing from K to $k-1$ and using $\sum_{i=K}^{k-1} 1/i = \log k + O(1)$ together with the convergence of $\sum O(i^{-2})$ gives $\log|b_k| = -\log k + O(1)$, that is

$$k\,|b_k| \longrightarrow C > 0:$$

the coefficients die off exactly like $1/k$, no faster. Now let $x \uparrow 1$. The tail $\sum_{k\geq K} b_k x^{2k+p}$ has terms of one sign that increase in magnitude with x, so by monotone convergence

$$\lim_{x\to 1^-}\Big|\sum_{k\geq K} b_k x^{2k+p}\Big| = \Big|\sum_{k\geq K} b_k\Big| = C\sum_{k\geq K}\frac{1}{k}\big(1+o(1)\big) = \infty,$$

the harmonic series. The finitely many earlier terms are bounded, so $v(x)\to\pm\infty$ as $x\to 1^-$, and by the parity of the chain also as $x\to -1^+$.

Example. Take $\lambda = 2$ and start the *even* chain at $a_0 = 1$. Then (7.1.10) gives

$$a_2 = -\frac{2-0}{2\cdot 1} = -1, \qquad a_4 = -\frac{2-6}{4\cdot 3}(-1) = -\frac{1}{3}, \qquad a_6 = -\frac{2-20}{6\cdot 5}\left(-\frac{1}{3}\right) = -\frac{1}{5},$$

and in general $a_{2k} = -1/(2k-1)$ for $k\geq 1$ — precisely the $1/k$ decay just predicted. Using $\sum_{k\geq 1} x^{2k-1}/(2k-1) = \frac{1}{2}\log\frac{1+x}{1-x}$ for $|x|<1$,

$$v(x) = 1 - \sum_{k\geq 1}\frac{x^{2k}}{2k-1} = 1 - x\sum_{k\geq 1}\frac{x^{2k-1}}{2k-1} = 1 - \frac{x}{2}\log\frac{1+x}{1-x}.$$

This is a genuine solution of (7.1.9) with $\lambda = 2$ (a one-line check by substitution), finite at every interior point, yet $v(x) \to -\infty$ as $x \to 1^-$: useless on the sphere. (It is $-Q_1$, the second Legendre function.) Meanwhile the *odd* chain with $\lambda = 2$ terminates immediately and gives the polynomial $v = x$.

Boundedness on the closed interval therefore forces each parity chain either to terminate or to be absent. For one parity chain to terminate at degree n, the numerator in (7.1.10) must vanish at $j = n$:

$$\lambda - n(n+1) = 0 \qquad \Longrightarrow \qquad \lambda = n(n+1), \qquad n = 0, 1, 2, \ldots.$$

Then, with $\lambda = n(n+1)$,

$$a_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}\, a_j. \tag{7.1.11}$$

At $j = n$ one has $a_{n+2} = 0$, so the same-parity chain is a polynomial of degree n. The opposite-parity chain never hits a zero numerator, so by (a) and (b) it would be unbounded at $x = \pm 1$ on its own; we therefore set its initial coefficient to zero, and v is a polynomial of degree n, bounded at the poles.

With $\lambda = n(n+1)$, equations (7.1.8)–(7.1.9) become the ordinary Legendre equation

$$\frac{d}{dx}\left[(1-x^2)v'(x)\right] + n(n+1)v(x) = 0, \tag{7.1.12}$$

or

$$(1 - x^2)v'' - 2xv' + n(n+1)v = 0. \tag{7.1.13}$$

By Step 3 the solutions bounded on $[-1, 1]$ form a one-parameter family: a single parity chain, determined by its one starting coefficient, with the other chain absent. They are all multiples of one polynomial of degree n, and we pick the multiple by the standard normalization $P_n(1) = 1$. The polynomial so normalized is the *Legendre polynomial* P_n.

That normalization makes sense only if the polynomial does not already vanish at $x = 1$, so let us expand about the endpoint rather than about the origin. Put $s = 1 - x$, so that $1 - x^2 = (1 - x)(1 + x) = s(2 - s)$, $x = 1 - s$ and $d/dx = -d/ds$; the two sign flips cancel in the second derivative, and (7.1.13) becomes

$$s(2 - s)\frac{d^2v}{ds^2} + 2(1 - s)\frac{dv}{ds} + n(n+1)v = 0.$$

Axisymmetry: the ordinary Legendre equation VIII

Insert $v = \sum_{k\geq 0} b_k s^k$. Shifting the summation index in each term,

$$\frac{dv}{ds} = \sum_{k\geq 0}(k+1)b_{k+1}s^k, \qquad s\frac{d^2v}{ds^2} = \sum_{k\geq 0} k(k+1)b_{k+1}s^k,$$

$$s^2\frac{d^2v}{ds^2} = \sum_{k\geq 0} k(k-1)b_k s^k, \qquad s\frac{dv}{ds} = \sum_{k\geq 0} k\, b_k s^k,$$

so the coefficient of s^k is

$$\begin{aligned} &2k(k+1)b_{k+1} - k(k-1)b_k + 2(k+1)b_{k+1} - 2k\, b_k + n(n+1)b_k \\ &\qquad = 2(k+1)^2 b_{k+1} + \Big[n(n+1) - k(k+1)\Big]\, b_k, \end{aligned}$$

using $k(k-1) + 2k = k(k+1)$. Setting it to zero gives the endpoint recurrence

$$b_{k+1} = \frac{k(k+1) - n(n+1)}{2(k+1)^2}\, b_k.$$

It is first order: every coefficient of such an expansion is a fixed multiple of $b_0 = v(1)$. So a solution of this form with $v(1) = 0$ vanishes identically, and any nontrivial one has $v(1) \neq 0$ — dividing by it is legitimate. The same recurrence confirms Step 3 from the other end: the bracket vanishes at $k = n$, so $b_{n+1} = 0$ and the expansion stops, leaving a polynomial of degree n, and the family is one-dimensional because only b_0 is free.

The power content of P_n comes not from the normalization but from Step 3: because (7.1.11) links coefficients whose indices differ by 2, and the opposite-parity chain was set to zero, P_n contains only the powers $x^n, x^{n-2}, \ldots$, ending at x^1 when n is odd and at x^0 when n is even. Replacing x by $-x$ then multiplies every surviving power by $(-1)^n$, so

$$P_n(-x) = (-1)^n P_n(x).$$

The generating function below fixes all the normalized coefficients explicitly.

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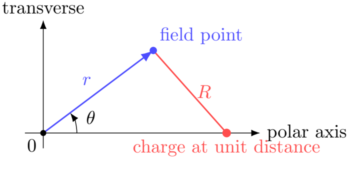
Generating function from the Coulomb potential I

Step 1: write the axisymmetric potential. Place a point charge one unit from the origin on the positive polar axis. For a field point with spherical coordinates (r, θ, ϕ), the law of cosines gives

$$R^2 = r^2 + 1^2 - 2r(1)\cos\theta = 1 - 2r\cos\theta + r^2,$$

so, for a nonzero physical scale factor C, the Coulomb potential is

$$\varphi(r,\theta) = \frac{C}{R} = \frac{C}{\sqrt{1 - 2r\cos\theta + r^2}}. \tag{7.2.1}$$



$R^2 = 1 + r^2 - 2r\cos\theta$

Generating function from the Coulomb potential II

In the picture the horizontal line is the polar axis $\theta = 0$, along which the charge sits, and θ is measured from it exactly as in (7.1.5). That axis is a genuine Cartesian direction; do not confuse it with the variable $x = \cos\theta$ used in the formulas of this chapter, which is the cosine of that angle and ranges over $[-1, 1]$.

For $0 \leq r < 1$ the charge lies outside the ball of radius r about the origin, so φ is harmonic there; moreover

$$R^2 = 1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta) \geq (1-r)^2 > 0,$$

so φ is a smooth function of r and of $x = \cos\theta$ on all of $[-1, 1]$. Separating Laplace's equation for a ϕ-independent field gave the product solutions $r^n P_n(\cos\theta)$ and $r^{-n-1} P_n(\cos\theta)$, but separation by itself does not say that *our* φ is a superposition of them, so we check it. We use two facts established later in this chapter, neither of which uses the generating function (so there is no circular reasoning): the orthogonality relation (7.2.20) and the completeness of $\{P_n\}$ in $L^2([-1, 1])$ quoted on the Fourier–Legendre frame. At each fixed $r < 1$ they let us expand

$$\varphi(r, \theta) = \sum_{n=0}^{\infty} c_n(r) P_n(x), \qquad c_n(r) = \frac{2n+1}{2} \int_{-1}^{1} \varphi \, P_n(x) \, dx.$$

Generating function from the Coulomb potential III

The coefficients are not arbitrary. Multiplying (7.1.5) by r^2 and using the change of variable $x = \cos\theta$ of the previous frame, harmonicity of φ reads

$$\frac{\partial}{\partial r}\left(r^2\varphi_r\right) + \frac{\partial}{\partial x}\left[(1-x^2)\varphi_x\right] = 0.$$

Since φ is smooth for $r < 1$ we may differentiate c_n under the integral sign, and then integrate by parts twice in x; each boundary term carries the factor $1 - x^2$, which vanishes at $x = \pm 1$:

$$\begin{aligned}\left(r^2 c_n'\right)' &= \frac{2n+1}{2}\int_{-1}^{1}\frac{\partial}{\partial r}\left(r^2\varphi_r\right)P_n\,dx = -\frac{2n+1}{2}\int_{-1}^{1}\frac{\partial}{\partial x}\left[(1-x^2)\varphi_x\right]P_n\,dx \\ &= -\frac{2n+1}{2}\int_{-1}^{1}\varphi\,\frac{d}{dx}\left[(1-x^2)P_n'\right]dx.\end{aligned}$$

By the Legendre equation (7.1.12), $\frac{d}{dx}\left[(1-x^2)P_n'\right] = -n(n+1)P_n$, so the last integral reproduces c_n and

$$\left(r^2 c_n'\right)' = n(n+1)\,c_n, \qquad \text{that is,} \qquad r^2 c_n'' + 2rc_n' - n(n+1)c_n = 0.$$

This is exactly the radial Euler equation already solved in (7.1.7), so $c_n(r) = A_n r^n + B_n r^{-n-1}$. The charge sits at distance 1 from the origin, so φ is bounded near $r = 0$; each c_n is then bounded there, which forces $B_n = 0$. Writing $c_n := A_n$,

$$\varphi(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta).$$

Dividing by C gives

$$\frac{1}{\sqrt{1 - 2r\cos\theta + r^2}} = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta), \qquad a_n = \frac{c_n}{C}. \tag{7.2.2}$$

Step 2: determine the coefficients on the axis. Set $\theta = 0$. Then $\cos\theta = 1$, $P_n(1) = 1$, and

$$1 - 2r\cos 0 + r^2 = (1 - r)^2.$$

Because $0 \le r < 1$, $|1 - r| = 1 - r$, so (7.2.2) becomes

$$\frac{1}{1 - r} = \sum_{n=0}^{\infty} a_n r^n = \sum_{n=0}^{\infty} r^n \qquad (|r| < 1).$$

Uniqueness of power-series coefficients gives $a_n = 1$. Writing $t = r$ and $x = \cos\theta$,

$$g(x, t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n, \qquad |t| < 1, \quad -1 \le x \le 1. \tag{7.2.3}$$

Step 3: expand the first three P_n from the generating function. For small t, write

$$g(x,t) = (1-u)^{-1/2}, \qquad u = 2xt - t^2,$$

and use the binomial series

$$(1-u)^{-1/2} = 1 + \tfrac{1}{2}u + \tfrac{1}{2}\cdot\tfrac{3}{2}\frac{u^2}{2!} + O(u^3) = 1 + \tfrac{1}{2}u + \tfrac{3}{8}u^2 + O(u^3).$$

Collect powers of t up to t^2:

$$\begin{aligned} u &= 2xt - t^2, \\ u^2 &= (2xt - t^2)^2 = 4x^2t^2 + O(t^3). \end{aligned}$$

Hence

$$\begin{aligned} g &= 1 + \tfrac{1}{2}(2xt - t^2) + \tfrac{3}{8}(4x^2t^2) + O(t^3) \\ &= 1 + xt + \left(-\tfrac{1}{2} + \tfrac{3}{2}x^2\right)t^2 + O(t^3) \\ &= 1 + xt + \tfrac{1}{2}(3x^2 - 1)\,t^2 + O(t^3). \end{aligned}$$

Matching $\sum P_n(x)t^n$ therefore gives the first three polynomials explicitly:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1). \tag{7.2.4}$$

Checks: $P_n(1) = 1$ and $P_n(-x) = (-1)^n P_n(x)$ hold for $n = 0, 1, 2$.

Step 1: differentiate the generating function. Let

$$D(x,t) = 1 - 2xt + t^2, \qquad g(x,t) = D(x,t)^{-1/2}.$$

Differentiating with respect to t gives

$$\frac{\partial g}{\partial t} = -\frac{1}{2} D^{-3/2} \frac{\partial D}{\partial t} = \frac{x-t}{D^{3/2}}.$$

Termwise differentiation of the power series gives

$$\frac{\partial g}{\partial t} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}. \qquad (7.2.5)$$

Multiplying $g_t = (x-t)D^{-3/2}$ by D gives $(x-t)g = Dg_t$. Substituting both series,

$$(x-t)\sum_{n=0}^{\infty} P_n t^n = (1 - 2xt + t^2)\sum_{n=1}^{\infty} nP_n t^{n-1},$$

where $P_n = P_n(x)$. Expanding,

$$\sum_{n=0}^{\infty} xP_n t^n - \sum_{n=0}^{\infty} P_n t^{n+1} = \sum_{n=1}^{\infty} nP_n t^{n-1} - 2\sum_{n=1}^{\infty} nxP_n t^n + \sum_{n=1}^{\infty} nP_n t^{n+1}. \qquad (7.2.6)$$

Three-term recurrence II

Step 2: compare coefficients of t^n. With the convention $P_{-1} = 0$, the left coefficient of t^n is $xP_n - P_{n-1}$. On the right,

$$[t^n]\sum_{j\geq 1} jP_j t^{j-1} = (n+1)P_{n+1}, \qquad [t^n]\Big(-2\sum_{j\geq 1} jxP_j t^j\Big) = -2nxP_n,$$

and

$$[t^n]\sum_{j\geq 1} jP_j t^{j+1} = (n-1)P_{n-1}$$

(using $P_{-1} = 0$). Equating coefficients,

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2nxP_n + (n-1)P_{n-1}.$$

Move P_n and P_{n-1} terms to one side:

$$(2n+1)xP_n - nP_{n-1} = (n+1)P_{n+1}.$$

Thus

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \qquad n \geq 0, \tag{7.2.7}$$

with $P_{-1} = 0$ and $P_0 = 1$. For $n = 0$, $P_1 = xP_0 = x$.

First few Legendre polynomials I

From $P_0 = 1$, $P_1 = x$ and the recurrence (7.2.7):

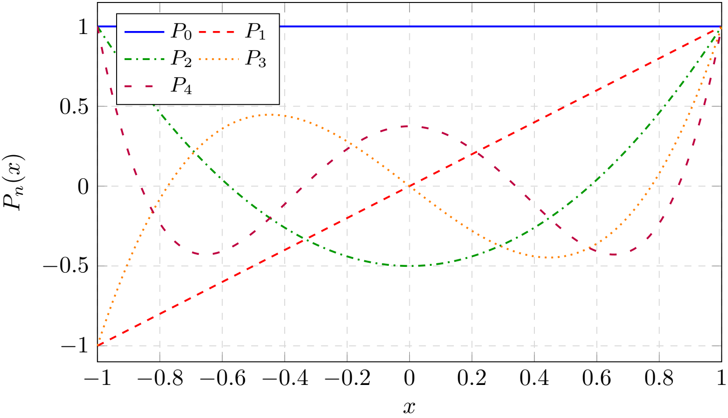
$$\begin{aligned} 2P_2 &= 3x \cdot x - 1 = 3x^2 - 1, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), \\ 3P_3 &= 5x \cdot \tfrac{1}{2}(3x^2 - 1) - 2x = \tfrac{1}{2}(15x^3 - 5x) - 2x = \tfrac{1}{2}(15x^3 - 9x), \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x). \end{aligned}$$

Thus

$$\begin{aligned} P_0(x) &= 1, & P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), & P_3(x) &= \frac{1}{2}(5x^3 - 3x). \end{aligned}$$

As checks, $P_n(1) = 1$ in all four cases, and $P_n(-x) = (-1)^n P_n(x)$.

Shapes of the first Legendre polynomials



Each P_n is bounded on $[-1,1]$ and normalized by $P_n(1)=1$. Increasing n adds interior zeros while preserving $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ for $m \neq n$.

Rodrigues' formula I

Step 1: set $w = (x^2 - 1)^n$ and record the basic identity.

$$w = (x^2 - 1)^n, \qquad w' = 2nx(x^2 - 1)^{n-1}.$$

Multiplying by $(x^2 - 1)$ gives the first-order relation

$$(x^2 - 1)w' = 2nx\, w. \tag{7.2.8}$$

Step 2: differentiate (7.2.8) **a total of n times (Leibniz).** On the left, apply the product rule to $(x^2 - 1) \cdot w'$. Only three derivatives of $x^2 - 1$ are nonzero:

$$(x^2 - 1)^{(0)} = x^2 - 1, \quad (x^2 - 1)^{(1)} = 2x, \quad (x^2 - 1)^{(2)} = 2, \quad (x^2 - 1)^{(k)} = 0 \ (k \geq 3).$$

Leibniz therefore truncates after $k = 2$:

$$\begin{aligned} \frac{d^n}{dx^n}\left[(x^2 - 1)w'\right] &= \binom{n}{0}(x^2 - 1)\, w^{(n+1)} + \binom{n}{1}(2x)\, w^{(n)} + \binom{n}{2}(2)\, w^{(n-1)} \\ &= (x^2 - 1)w^{(n+1)} + 2nx\, w^{(n)} + n(n-1)w^{(n-1)}. \end{aligned}$$

On the right, apply Leibniz to $x \cdot w$:

$$\frac{d^n}{dx^n}(xw) = x\, w^{(n)} + n\, w^{(n-1)},$$

so

$$2n\frac{d^n}{dx^n}(xw) = 2nx\, w^{(n)} + 2n^2\, w^{(n-1)}.$$

Rodrigues' formula II

Equating both sides after n differentiations of (7.2.8),

$$(x^2-1)w^{(n+1)} + 2nx\,w^{(n)} + n(n-1)w^{(n-1)} = 2nx\,w^{(n)} + 2n^2\,w^{(n-1)}.$$

Cancel $2nx\,w^{(n)}$ and rearrange:

$$(x^2-1)w^{(n+1)} = \left[2n^2 - n(n-1)\right]w^{(n-1)} = n(n+1)\,w^{(n-1)}. \tag{7.2.9}$$

Step 3: one more differentiation yields Legendre's equation. Set

$$u := w^{(n)} = \frac{d^n}{dx^n}(x^2-1)^n,$$

so $w^{(n+1)} = u'$ and $w^{(n)} = u$. Differentiate (7.2.9) once more:

$$\frac{d}{dx}\left[(x^2-1)u'\right] = n(n+1)\,w^{(n)} = n(n+1)\,u.$$

The left side expands by the product rule:

$$(x^2-1)u'' + 2xu' = n(n+1)\,u,$$

i.e.

$$(x^2-1)u'' + 2xu' - n(n+1)u = 0, \tag{7.2.10}$$

or equivalently

$$(1-x^2)u'' - 2xu' + n(n+1)u = 0.$$

Thus u solves the ordinary Legendre equation (7.1.13). Any constant multiple of u does as well. Define

$$R_n(x) := \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\,u(x). \tag{7.2.11}$$

Then R_n is a polynomial of exact degree n solving (7.1.13).

Step 4: leading coefficient of R_n (and of P_n). Expand $(x^2-1)^n$ at the top power:

$$(x^2-1)^n = x^{2n} - n\,x^{2n-2} + \cdots .$$

Only the x^{2n} term contributes to the degree-n term after n derivatives:

$$\frac{d^n}{dx^n}x^{2n} = (2n)(2n-1)\cdots(n+1)\,x^n = \frac{(2n)!}{n!}\,x^n.$$

Hence

$$u(x) = \frac{(2n)!}{n!}\,x^n + (\text{lower powers}),$$

and the leading term of R_n is

$$R_n(x) = \frac{1}{2^n n!}\cdot\frac{(2n)!}{n!}\,x^n + \cdots = \frac{(2n)!}{2^n(n!)^2}\,x^n + \cdots . \tag{7.2.12}$$

In particular the nth derivative of this monic-in-powers piece is the constant

$$R_n^{(n)}(x) = \frac{(2n)!}{2^n n!}. \tag{7.2.13}$$

Rodrigues' formula IV

(We will use (7.2.13) in the norm calculation.)

Step 5: normalization $R_n(1) = 1$ forces $R_n = P_n$. Write $(x^2-1)^n = (x-1)^n(x+1)^n$ and apply Leibniz at $x = 1$. Every term that fails to differentiate $(x-1)^n$ fully n times still carries a factor $(x-1)$ and vanishes at $x = 1$. The surviving term is

$$\binom{n}{n}\left[\frac{d^n}{dx^n}(x-1)^n\right](x+1)^n = n!\,(x+1)^n,$$

so

$$\left.\frac{d^n}{dx^n}(x^2-1)^n\right|_{x=1} = n!\,(1+1)^n = n!\,2^n, \qquad R_n(1) = \frac{1}{2^n n!}\cdot n!\,2^n = 1.$$

The degree-n polynomial solution of the Legendre equation with $P_n(1) = 1$ is unique (the terminating recurrence chain has one free scale, fixed by the endpoint condition). Therefore $R_n = P_n$, i.e.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{7.2.14}$$

In particular the leading coefficient of P_n is exactly (7.2.12):

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\,x^n + \cdots, \qquad P_n^{(n)} = \frac{(2n)!}{2^n n!}.$$

Rodrigues' formula V

Step 6: binomial form (optional bookkeeping). Expanding $(x^2-1)^n$ by the binomial theorem and differentiating n times recovers the explicit sum on the formula sheet:

$$(x^2-1)^n = \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} x^{2j}, \tag{7.2.15}$$

Rodrigues' formula (7.2.14) asks for n derivatives of (7.2.15). Differentiating one monomial n times gives

$$\frac{d^n}{dx^n} x^{2j} = \begin{cases} \dfrac{(2j)!}{(2j-n)!}\, x^{2j-n}, & 2j \geq n, \\ 0, & 2j < n, \end{cases}$$

the first line by the same falling-factorial count used in Step 4 ($2j$ choices, then $2j-1$, down to $2j-n+1$, whose product is $(2j)!/(2j-n)!$), the second because n derivatives annihilate any polynomial of degree $2j < n$. Only the terms with $j \geq n/2$ survive, so

$$P_n(x) = \frac{1}{2^n n!} \sum_{j=\lceil n/2 \rceil}^{n} (-1)^{n-j} \frac{n!}{j!\,(n-j)!} \cdot \frac{(2j)!}{(2j-n)!}\, x^{2j-n}.$$

Rodrigues' formula VI

Now reindex by $k = n - j$, i.e. $j = n - k$. Then

$$(-1)^{n-j} = (-1)^k, \quad j! = (n-k)!, \quad (n-j)! = k!,$$
$$(2j)! = (2n-2k)!, \quad (2j-n)! = (n-2k)!,$$

and $x^{2j-n} = x^{n-2k}$, while the constraint $j \geq n/2$ becomes $k \leq n/2$, i.e. $0 \leq k \leq \lfloor n/2 \rfloor$ because k is an integer. The factor $n!$ from (7.2.15) cancels the $n!$ in the Rodrigues prefactor, leaving

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!}\, x^{n-2k}.$$

Step 7: contour (Schläfli) form. Rodrigues' formula expresses P_n as an nth derivative, and Chapter 2 lets us write the nth derivative of a holomorphic function as a contour integral; combining the two costs one line. Put $f(t) = (t^2-1)^n$, a polynomial in t and therefore entire, so that (7.2.14) reads $P_n(x) = f^{(n)}(x)/(2^n n!)$. Cauchy's derivative formula, applied to f at the point x on any positively oriented simple closed contour C enclosing $t = x$, gives

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_C \frac{f(t)}{(t-x)^{n+1}}\, dt = \frac{n!}{2\pi i} \oint_C \frac{(t^2-1)^n}{(t-x)^{n+1}}\, dt.$$

Rodrigues' formula VII

Dividing by $2^n n!$ cancels the factor $n!$, and $2^n \cdot 2\pi = 2^{n+1}\pi$, so

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{n!}{2\pi i} \oint_C \frac{(t^2-1)^n}{(t-x)^{n+1}}\,dt = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(t^2-1)^n}{(t-x)^{n+1}}\,dt, \qquad (7.2.16)$$

which is the Schläfli form recorded on the formula sheet. Since f is entire, the only singularity of the integrand is the pole of order $n+1$ at $t = x$, so every contour winding once around $t = x$ returns the same value; this is why no particular curve is specified. (The generating function gives a *different* representation: reading off the t^n coefficient of (7.2.3) yields $P_n(x) = \frac{1}{2\pi i} \oint_{|t|=\rho} \frac{dt}{t^{n+1}\sqrt{1-2xt+t^2}}$ with $\rho < 1$, whose integrand carries a square root rather than the power $(t-x)^{-n-1}$.)

Orthogonality and norm of Legendre polynomials I

The Legendre equation is

$$\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] + n(n+1)P_n(x) = 0. \tag{7.2.17}$$

Write the equations for P_n and P_m, multiply the first by P_m and the second by P_n, and subtract:

$$P_m\left[(1-x^2)P_n'\right]' - P_n\left[(1-x^2)P_m'\right]' + \left[n(n+1) - m(m+1)\right]P_nP_m = 0.$$

The first two terms are a total derivative:

$$\frac{d}{dx}\left[(1-x^2)(P_mP_n' - P_nP_m')\right] = P_m\left[(1-x^2)P_n'\right]' - P_n\left[(1-x^2)P_m'\right]'.$$

Integrating from -1 to 1, the boundary term vanishes because $1-x^2=0$ at $x=\pm 1$ while P_n, P_m and their derivatives stay finite. If $n \neq m$, then $n(n+1) - m(m+1) = (n-m)(n+m+1) \neq 0$, so

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \qquad (n \neq m). \tag{7.2.18}$$

Orthogonality and norm of Legendre polynomials II

Norm via Rodrigues and integration by parts. Set

$$I_n := \int_{-1}^{1} P_n(x)^2\,dx = \frac{1}{2^n n!}\int_{-1}^{1} P_n(x)\frac{d^n}{dx^n}(x^2-1)^n\,dx.$$

Integrate by parts once:

$$I_n = \frac{1}{2^n n!}\left[P_n\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} - \frac{1}{2^n n!}\int_{-1}^{1} P_n'\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx.$$

Because $(x^2-1)^n = (x-1)^n(x+1)^n$ has a zero of order n at each endpoint,

$$\frac{d^j}{dx^j}(x^2-1)^n\bigg|_{x=\pm1} = 0, \qquad 0 \le j \le n-1.$$

Thus every boundary term in n successive integrations by parts vanishes, and

$$I_n = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} P_n^{(n)}(x)(x^2-1)^n\,dx. \tag{7.2.19}$$

Orthogonality and norm of Legendre polynomials III

From the leading-coefficient calculation (7.2.12)–(7.2.13),

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\, x^n + \cdots \qquad \Longrightarrow \qquad P_n^{(n)}(x) = \frac{(2n)!}{2^n n!}.$$

(The nth derivative kills every lower power, so only the top term survives; this constant is positive.) Insert into (7.2.19):

$$I_n = \frac{(-1)^n}{2^n n!} \cdot \frac{(2n)!}{2^n n!} \int_{-1}^{1} (x^2 - 1)^n\, dx = \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (-1)^n (x^2 - 1)^n\, dx.$$

The algebraic identity

$$(x^2 - 1)^n = (-1)^n (1 - x^2)^n \qquad \Longrightarrow \qquad (-1)^n (x^2 - 1)^n = (1 - x^2)^n$$

removes the sign and leaves a nonnegative integrand, so $I_n > 0$:

$$I_n = \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (1 - x^2)^n\, dx.$$

Orthogonality and norm of Legendre polynomials IV

To evaluate the remaining integral, set $x = 2u - 1$. Then $dx = 2\,du$ and $1 - x^2 = 4u(1-u)$, so with the beta function $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du = \Gamma(a)\Gamma(b)/\Gamma(a+b)$,

$$\int_{-1}^{1}(1-x^2)^n\,dx = 2\cdot 4^n\int_0^1 u^n(1-u)^n\,du = 2^{2n+1}B(n+1,n+1)$$
$$= 2^{2n+1}\frac{(n!)^2}{(2n+1)!}.$$

Therefore

$$I_n = \frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{2^{2n+1}(n!)^2}{(2n+1)!} = \frac{2}{2n+1}.$$

Combining orthogonality and norm,

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\,\delta_{nm}. \tag{7.2.20}$$

Fourier–Legendre series I

The Legendre polynomials are orthogonal and complete in $L^2([-1,1])$: no nonzero function in $L^2([-1,1])$ is orthogonal to every P_n, so partial sums $\sum_{n=0}^{N} a_n P_n$ converge to f in the L^2 norm as $N \to \infty$. Thus, for $f \in L^2([-1,1])$,

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

in the L^2 sense. Multiply by $P_m(x)$ and integrate:

$$\int_{-1}^{1} f(x) P_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x) P_m(x)\, dx = a_m \frac{2}{2m+1}.$$

Solving for a_m gives

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx. \tag{7.2.21}$$

Therefore the Fourier–Legendre expansion is

$$f(x) = \sum_{n=0}^{\infty} \left[\frac{2n+1}{2} \int_{-1}^{1} f(s) P_n(s)\, ds \right] P_n(x). \tag{7.2.22}$$

Useful formulas for the Legendre polynomial

differential equation	$(1-x^2)P_n'' - 2x\,P_n' + n(n+1)P_n = 0$
series solution	$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{2^n k!\,(n-2k)!\,(n-k)!}\, x^{n-2k}$
generating function	$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$
recurrence	$(n+1)P_{n+1} = (2n+1)x\,P_n - n\,P_{n-1}$
orthogonality	$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\,\delta_{nm}$
Rodrigues	$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$
Schläfli	$P_n(x) = \frac{1}{2^{n+1}\pi i}\oint \frac{(t^2-1)^n}{(t-x)^{n+1}}\,dt \qquad (7.2.16)$

Generating series: $|t| < 1$ on $x \in [-1, 1]$. Contour encloses $t = x$.

Table of contents

Associated Legendre equation and Y_n^m (short) I

When the field depends on ϕ, separate $y(\theta,\phi) = v(\theta)w(\phi)$. Single-valuedness under $\phi \mapsto \phi + 2\pi$ forces $w \sim e^{im\phi}$ with $m \in \mathbb{Z}$. Put $y(\theta,\phi) = v(\theta)e^{im\phi}$ into the angular eigenproblem of the previous frame,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 y}{\partial\phi^2} + \lambda y = 0.$$

Since $\dfrac{\partial^2}{\partial\phi^2}e^{im\phi} = (im)^2 e^{im\phi} = -m^2 e^{im\phi}$, every term carries the common factor $e^{im\phi}$, which is never zero and therefore cancels:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dv}{d\theta}\right) + \left[\lambda - \frac{m^2}{\sin^2\theta}\right]v = 0.$$

Now set $x = \cos\theta$. The first term was already converted in the axisymmetric frame: there $\sin\theta\, dv/d\theta = -(1-x^2)v'$ gave

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dv}{d\theta}\right) = \frac{d}{dx}\left[(1-x^2)v'(x)\right],$$

and for the new term $\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$, so $m^2/\sin^2\theta$ becomes $m^2/(1-x^2)$. The polar equation is therefore

$$\frac{d}{dx}\left[(1-x^2)v'(x)\right] + \left[\lambda - \frac{m^2}{1-x^2}\right]v(x) = 0. \tag{7.3.1}$$

Associated Legendre equation and Y_n^m (short) II

Regularity at $x = \pm 1$ again quantizes λ, but the $m = 0$ power-series argument needs one adjustment, because for $m \neq 0$ a bounded solution of (7.3.1) is no longer a power series in x: already $\sqrt{1 - x^2}$ is not a polynomial. To see the behaviour at a pole, set $t = 1 - x$ near $x = 1$, so that $1 - x^2 = t(2 - t)$ and $d/dx = -d/dt$, whence (7.3.1) reads

$$\frac{d}{dt}\left[t(2-t)\frac{dv}{dt}\right] + \left[\lambda - \frac{m^2}{t(2-t)}\right] v = 0.$$

Keep the leading behaviour as $t \to 0$ (replace $2 - t$ by 2) and try $v \sim t^s$:

$$\frac{d}{dt}\left[2t\, s\, t^{s-1}\right] - \frac{m^2}{2t}\, t^s = \left(2s^2 - \frac{m^2}{2}\right) t^{s-1} = 0 \qquad \Longrightarrow \qquad s = \pm\frac{|m|}{2},$$

the two Frobenius exponents of Chapter 4. The exponent $-|m|/2$ blows up at the pole, so a bounded solution must carry the factor $(1 - x)^{|m|/2}$; the same computation at $x = -1$ supplies $(1 + x)^{|m|/2}$. Write therefore

$$v = (1 - x^2)^{|m|/2} f(x), \qquad f \text{ bounded on } [-1, 1].$$

This is the substitution carried out in detail in Step 3 below, read in the opposite direction and with $n(n+1)$ replaced by λ; it gives

$$(1 - x^2) f'' - 2\big(|m| + 1\big) x f' + \big[\lambda - |m|(|m| + 1)\big] f = 0.$$

Associated Legendre equation and Y_n^m (short) III

Now repeat Step 2 of the axisymmetric frame with $f = \sum_{j\geq 0} a_j x^j$. The terms $(1-x^2)f''$, $-2(|m|+1)xf'$ and $[\lambda - |m|(|m|+1)]f$ contribute $(j+2)(j+1)a_{j+2} - j(j-1)a_j$, $-2(|m|+1)j\, a_j$ and $[\lambda - |m|(|m|+1)]a_j$ to the coefficient of x^j, so

$$\begin{aligned}(j+2)(j+1)a_{j+2} &= \Big[j(j-1) + 2(|m|+1)j + |m|(|m|+1) - \lambda\Big]\, a_j \\ &= \Big[(j+|m|)(j+|m|+1) - \lambda\Big]\, a_j,\end{aligned}$$

the last equality because $j(j-1)+2(|m|+1)j+|m|(|m|+1) = (j+|m|)^2+(j+|m|)$.

A chain that never terminates is again unbounded at the poles, and for $m \neq 0$ the estimate is even easier than at $m=0$. Since

$$(j+2)(j+1) - \Big[(j+|m|)(j+|m|+1) - \lambda\Big] = (2-2|m|)\, j + O(1),$$

writing $j = 2k+p$ ($p=0$ or 1 for the two parities) and $b_k = a_{2k+p}$ gives

$$\frac{b_{k+1}}{b_k} = 1 - \frac{(2-2|m|)j + O(1)}{(j+2)(j+1)} = 1 + \frac{|m|-1}{k} + O\Big(\frac{1}{k^2}\Big).$$

Summing the logarithms as in the axisymmetric frame yields $\log|b_k| = (|m|-1)\log k + O(1)$, i.e. $|b_k|$ is of order $k^{|m|-1}$, which for $|m|\geq 1$ stays

Associated Legendre equation and Y_n^m (short) IV

bounded away from 0. The ratio tends to $1 > 0$, so beyond some index K all the b_k share one sign, and letting $x \uparrow 1$ with monotone convergence,

$$\left|\sum_{k \geq K} b_k x^{2k+p}\right| \longrightarrow \sum_{k \geq K} |b_k| = \infty,$$

since the terms do not tend to zero. So f — and with it v — would be unbounded. One parity chain must therefore terminate, say at $j = j_0$, and the other has its initial coefficient set to zero. Termination requires the bracket to vanish at $j = j_0$:

$$\lambda = (j_0 + |m|)(j_0 + |m| + 1) = n(n+1), \qquad n := j_0 + |m| \ \geq \ |m|,$$

which is the quantization together with the constraint $n \geq |m|$; then f is a polynomial of degree $n - |m|$, consistent with $f \propto d^{|m|} P_n / dx^{|m|}$ in Step 3. The corresponding equation is the **associated Legendre equation**

$$\frac{d}{dx}\left[(1-x^2)v'(x)\right] + \left[n(n+1) - \frac{m^2}{1-x^2}\right] v(x) = 0, \tag{7.3.2}$$

or, expanded,

$$(1-x^2)v'' - 2xv' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] v = 0. \tag{7.3.3}$$

When $m = 0$ this reduces to the ordinary Legendre equation already developed.

Step 1: start from ordinary Legendre and differentiate once ($m = 1$ template). Write the ordinary Legendre equation for $P := P_n$:

$$(1 - x^2)P'' - 2xP' + n(n+1)P = 0. \tag{7.3.4}$$

Differentiate once (product rule on each term):

$$\begin{aligned} \frac{d}{dx}\Big[(1 - x^2)P''\Big] &= (1 - x^2)P''' - 2xP'', \\ \frac{d}{dx}\Big[-2xP'\Big] &= -2P' - 2xP'', \\ \frac{d}{dx}\Big[n(n+1)P\Big] &= n(n+1)P'. \end{aligned}$$

Summing and collecting gives

$$(1 - x^2)P''' - 4xP'' + \Big[n(n+1) - 2\Big]P' = 0.$$

Set $u := P' = P_n'$, so

$$(1 - x^2)u'' - 4xu' + \Big[n(n+1) - 2\Big]u = 0. \tag{7.3.5}$$

Step 2: full substitution $u = v(1 - x^2)^{-1/2}$. Set

$$v(x) := (1 - x^2)^{1/2}\,u(x) = \sqrt{1 - x^2}\,P_n'(x),$$

Associated Legendre equation and Y_n^m (short) VI

so

$$u = v\,(1-x^2)^{-1/2}. \tag{7.3.6}$$

Differentiate by the product rule. Write $\rho := 1 - x^2$ for brevity, so $u = v\rho^{-1/2}$ and $\rho' = -2x$:

$$\begin{aligned} u' &= v'\rho^{-1/2} + v\cdot\left(-\tfrac{1}{2}\right)\rho^{-3/2}\rho' = v'\rho^{-1/2} + v\cdot\left(-\tfrac{1}{2}\right)\rho^{-3/2}(-2x) \\ &= v'\,(1-x^2)^{-1/2} + x\,v\,(1-x^2)^{-3/2}. \end{aligned}$$

Differentiate again:

$$\begin{aligned} \frac{d}{dx}\left[v'(1-x^2)^{-1/2}\right] &= v''(1-x^2)^{-1/2} + v'\cdot x(1-x^2)^{-3/2}, \\ \frac{d}{dx}\left[xv(1-x^2)^{-3/2}\right] &= v(1-x^2)^{-3/2} + xv'(1-x^2)^{-3/2} \\ &\qquad + xv\cdot\left(-\tfrac{3}{2}\right)(1-x^2)^{-5/2}(-2x) \\ &= v(1-x^2)^{-3/2} + xv'(1-x^2)^{-3/2} + 3x^2v(1-x^2)^{-5/2}. \end{aligned}$$

Adding these,

$$u'' = v''(1-x^2)^{-1/2} + 2xv'(1-x^2)^{-3/2} + v(1-x^2)^{-3/2} + 3x^2v(1-x^2)^{-5/2}.$$

Associated Legendre equation and Y_n^m (short) VII

Combine the pure-v powers of $(1-x^2)$ over the common denominator $(1-x^2)^{5/2}$:

$$(1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2} = \frac{1-x^2+3x^2}{(1-x^2)^{5/2}} = \frac{1+2x^2}{(1-x^2)^{5/2}},$$

so

$$u'' = v''(1-x^2)^{-1/2} + 2xv'(1-x^2)^{-3/2} + (1+2x^2)v(1-x^2)^{-5/2}. \qquad (7.3.7)$$

Now substitute u, u', u'' into (7.3.5), term by term.

$$\begin{aligned}
(1-x^2)u'' &= (1-x^2)^{1/2}v'' + 2x(1-x^2)^{-1/2}v' + (1+2x^2)(1-x^2)^{-3/2}v,\\
-4xu' &= -4x(1-x^2)^{-1/2}v' - 4x^2(1-x^2)^{-3/2}v,\\
\Big[n(n+1)-2\Big]u &= \Big[n(n+1)-2\Big](1-x^2)^{-1/2}v.
\end{aligned}$$

Sum and multiply through by the positive clearing factor $(1-x^2)^{3/2}$ (for $|x|<1$):

$$\begin{aligned}
(1-x^2)\,u''\cdot(1-x^2)^{3/2} &= (1-x^2)^2v'' + 2x(1-x^2)v' + (1+2x^2)v,\\
(-4xu')\cdot(1-x^2)^{3/2} &= -4x(1-x^2)v' - 4x^2v,\\
\Big[n(n+1)-2\Big]u\cdot(1-x^2)^{3/2} &= \Big[n(n+1)-2\Big](1-x^2)v.
\end{aligned}$$

Collect coefficients:

$$
\begin{aligned}
v'' &: \quad (1-x^2)^2, \\
v' &: \quad 2x(1-x^2) - 4x(1-x^2) = -2x(1-x^2), \\
v &: \quad 1 + 2x^2 - 4x^2 + \Big[n(n+1) - 2\Big](1-x^2) = 1 - 2x^2 + \Big[n(n+1) - 2\Big](1-x^2).
\end{aligned}
$$

Expand the v coefficient:

$$
\begin{aligned}
1 - 2x^2 + \Big[n(n+1) - 2\Big](1-x^2) &= 1 - 2x^2 + n(n+1) - n(n+1)x^2 - 2 + 2x^2 \\
&= n(n+1) - 1 - n(n+1)x^2 \\
&= n(n+1)(1-x^2) - 1.
\end{aligned}
$$

Thus

$$
(1-x^2)^2 v'' - 2x(1-x^2)v' + \Big[n(n+1)(1-x^2) - 1\Big] v = 0.
$$

For $x^2 \neq 1$ divide by $1 - x^2$:

$$
(1-x^2)v'' - 2xv' + \left[n(n+1) - \frac{1}{1-x^2}\right] v = 0, \tag{7.3.8}
$$

which is exactly (7.3.3) with $m = 1$. Thus

$$
v = \sqrt{1-x^2}\, P_n'
$$

solves the associated Legendre equation of order $m = 1$.

Step 3: general m. We first show, by induction on m, that $u_m := \dfrac{d^m P_n}{dx^m}$ satisfies

$$(1 - x^2)u_m'' - 2(m+1)x\, u_m' + \Big[n(n+1) - m(m+1)\Big] u_m = 0. \tag{7.3.9}$$

For $m = 0$ this is (7.3.4), since $-2(0+1)x = -2x$ and $0 \cdot 1 = 0$. Assume (7.3.9) holds for some $m \geq 0$ and differentiate it once, term by term, exactly as in Step 1:

$$\frac{d}{dx}\Big[(1 - x^2)u_m''\Big] = (1 - x^2)u_m''' - 2x\, u_m'',$$

$$\frac{d}{dx}\Big[-2(m+1)x\, u_m'\Big] = -2(m+1)u_m' - 2(m+1)x\, u_m'',$$

$$\frac{d}{dx}\Bigg[\Big(n(n+1) - m(m+1)\Big) u_m\Bigg] = \Big(n(n+1) - m(m+1)\Big) u_m'.$$

Write $u_{m+1} = u_m'$ and add the three lines. The second-derivative terms give $(1 - x^2)u_{m+1}''$; the first-derivative terms give

$$-2x\, u_{m+1}' - 2(m+1)x\, u_{m+1}' = -2(m+2)x\, u_{m+1}';$$

and the undifferentiated terms give

$$\Big[n(n+1) - m(m+1) - 2(m+1)\Big] u_{m+1}, \qquad m(m+1) + 2(m+1) = (m+1)(m+2).$$

Hence

$$(1-x^2)u''_{m+1} - 2\Big[(m+1)+1\Big]x\,u'_{m+1} + \Big[n(n+1)-(m+1)(m+2)\Big]u_{m+1} = 0,$$

which is (7.3.9) at $m+1$. (At $m=1$ it reproduces (7.3.5).) Setting

$$v := (1-x^2)^{m/2}\,u = (1-x^2)^{m/2}\frac{d^mP_n}{dx^m}$$

inverts to $u = (1-x^2)^{-m/2}v$, and we repeat the product-rule substitution of Step 2 with the general exponent. Write again $\rho := 1-x^2$, so $\rho' = -2x$ and $u = \rho^{-m/2}v$:

$$\begin{aligned}
u' &= \rho^{-m/2}v' + v\Big(-\tfrac{m}{2}\Big)\rho^{-m/2-1}(-2x) = \rho^{-m/2}v' + m\,x\,\rho^{-m/2-1}v,\\
u'' &= \Big(\rho^{-m/2}v'' + m\,x\,\rho^{-m/2-1}v'\Big)\\
&\qquad + \Big(m\,\rho^{-m/2-1}v + m\,x\,\rho^{-m/2-1}v' + m(m+2)x^2\rho^{-m/2-2}v\Big)\\
&= \rho^{-m/2}v'' + 2m\,x\,\rho^{-m/2-1}v' + \Big[m\,\rho^{-m/2-1} + m(m+2)x^2\rho^{-m/2-2}\Big]v,
\end{aligned}$$

where the second line differentiates the two pieces of u' and uses $m\,x\cdot\left(-\frac{m}{2}-1\right)\rho^{-m/2-2}(-2x) = m(m+2)x^2\rho^{-m/2-2}$. Substitute into (7.3.9) and multiply through by the positive clearing factor $\rho^{m/2}$ (legitimate for $|x|<1$):

$$\begin{aligned}
\rho\,u''\cdot\rho^{m/2} &= \rho\,v'' + 2m\,x\,v' + \left[m + \frac{m(m+2)x^2}{\rho}\right]v,\\
-2(m+1)x\,u'\cdot\rho^{m/2} &= -2(m+1)x\,v' - \frac{2m(m+1)x^2}{\rho}v,\\
\Big[n(n+1)-m(m+1)\Big]u\cdot\rho^{m/2} &= \Big[n(n+1)-m(m+1)\Big]v.
\end{aligned}$$

Collect coefficients. For v',

$$2m\,x - 2(m+1)x = -2x.$$

For v, the terms without x^2/ρ give $m - m(m+1) + n(n+1) = n(n+1) - m^2$, and the terms with x^2/ρ give

$$m(m+2) - 2m(m+1) = m^2 + 2m - 2m^2 - 2m = -m^2,$$

so the whole v coefficient is

$$n(n+1) - m^2 - \frac{m^2x^2}{\rho} = n(n+1) - m^2\,\frac{\rho + x^2}{\rho} = n(n+1) - \frac{m^2}{1-x^2},$$

using $\rho + x^2 = (1 - x^2) + x^2 = 1$. Therefore

$$(1-x^2)v'' - 2xv' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]v = 0,$$

which is exactly (7.3.3); at $m = 1$ it reduces to (7.3.8). The Condon–Shortley phase convention defines

$$P_n^m(x) := (-1)^m (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}, \qquad m = 0, 1, \ldots, n. \tag{7.3.10}$$

In particular $P_n^0 = P_n$, and for $m = 1$ one has $P_n^1(x) = -(1-x^2)^{1/2} P_n'(x)$ (the minus sign is the Condon–Shortley factor $(-1)^1$).

Step 4: spherical harmonics (notation only). An unnormalized angular basis is

$$\mathcal{Y}_n^m(\theta, \phi) = P_n^{|m|}(\cos\theta)\, e^{im\phi}, \qquad -n \le m \le n.$$

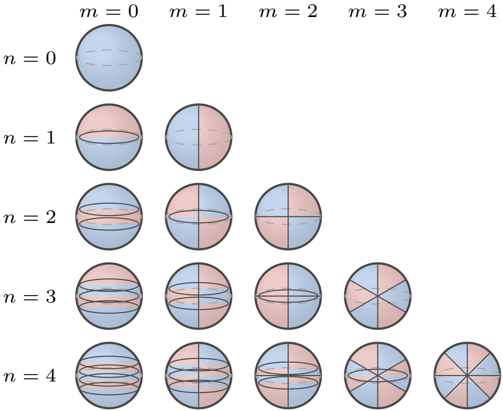
Normalized spherical harmonics Y_n^m differ from this by a constant and a fixed phase. Many texts call the degree ℓ and write the harmonics as $Y_{\ell m}$; here the degree is called n, so $Y_{\ell m}$ and Y_n^m denote the same function with $\ell = n$. Combining radial and angular factors, the regular interior expansion is

$$\psi_{\mathrm{int}}(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} r^n \mathcal{Y}_n^m(\theta, \phi), \tag{7.3.11}$$

Associated Legendre equation and Y_n^m (short) XIII

and the decaying exterior expansion uses r^{-n-1} instead of r^n.

Spherical-harmonic nodal patterns I



positive sign
negative sign

latitude circles: $n - m$
meridian planes: m

Spherical-harmonic nodal patterns II

Rows: degree n; columns: order m. $P_n^{|m|}(\cos\theta)$ has $n-|m|$ zeros in $(0,\pi)$ (nodal latitudes). A real azimuthal factor $\cos(|m|\phi)$ or $\sin(|m|\phi)$ gives $|m|$ nodal meridian planes.

Why $n-|m|$ zeros. By Rodrigues (7.2.14), $2^n n!\,P_n = g^{(n)}$ with $g(x)=(x^2-1)^n=(x-1)^n(x+1)^n$, so $g^{(k)}(\pm1)=0$ for $k=0,1,\dots,n-1$. Suppose $g^{(k)}$ with $k\le n-1$ vanishes at $x=\pm1$ and at k interior points; applying Rolle's theorem on each of the $k+1$ gaps between these $k+2$ zeros produces $k+1$ interior zeros of $g^{(k+1)}$. Iterating from $k=0$ to $k=n-1$ gives at least n zeros of $g^{(n)}$ in $(-1,1)$, and exactly n because $\deg P_n=n$. One more Rolle step per differentiation, with the same degree count, shows that d^jP_n/dx^j has exactly $n-j$ zeros in $(-1,1)$ for $0\le j\le n$. Finally (7.3.10) with $x=\cos\theta$ reads

$$P_n^{|m|}(\cos\theta)=(-1)^{|m|}(\sin\theta)^{|m|}\,\frac{d^{\,|m|}P_n}{dx^{\,|m|}}\bigg|_{x=\cos\theta},$$

and $\sin\theta\ne0$ on $(0,\pi)$, so the nodal latitudes are exactly the $n-|m|$ angles $\theta=\arccos x_j$ with x_j the zeros of $d^{\,|m|}P_n/dx^{\,|m|}$.

Table of contents

The Hermite equation and series solution I

Step 1: substitute a power series. The physicists' Hermite polynomials solve

$$y'' - 2xy' + 2ny = 0, \qquad n = 0, 1, 2, \ldots. \tag{7.4.1}$$

Let $y(x) = \sum_{j=0}^{\infty} a_j x^j$. Then

$$y' = \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j,$$

$$y'' = \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j,$$

$$-2xy' = -2 \sum_{j=0}^{\infty} j a_j x^j.$$

Substitution into (7.4.1) gives

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} + 2(n-j) a_j\right] x^j = 0.$$

Every coefficient vanishes, so

$$a_{j+2} = \frac{2(j-n)}{(j+2)(j+1)} a_j. \tag{7.4.2}$$

The Hermite equation and series solution II

Step 2: termination condition. Even and odd chains are independent. For integer $n \geq 0$, set $j = n$ in (7.4.2):

$$a_{n+2} = \frac{2(n-n)}{(n+2)(n+1)} a_n = 0.$$

Thus the same-parity chain terminates at degree n, giving a polynomial of exact degree n. The opposite-parity chain never hits a zero numerator, so we set its initial coefficient to zero (the non-polynomial second solution of (7.4.1)).

Step 3: explicit series from the downward product. Adopt the physicists' leading coefficient $a_n = 2^n$. The two-step recurrence (7.4.2) run *downward* from degree n is

$$a_j = \frac{(j+2)(j+1)}{2(j-n)}\, a_{j+2} \qquad (j = n-2, n-4, \ldots).$$

The first few products (with $a_n = 2^n$) are

$$\begin{aligned} a_{n-2} &= \frac{n(n-1)}{2(n-2-n)}\, a_n = -\frac{n(n-1)}{4}\, 2^n = -n(n-1)\, 2^{n-2}, \\ a_{n-4} &= \frac{(n-2)(n-3)}{2(n-4-n)}\, a_{n-2} = \frac{(n-2)(n-3)}{-8} \left(-n(n-1)\, 2^{n-2}\right) \\ &= \frac{n(n-1)(n-2)(n-3)}{2}\, 2^{n-4}. \end{aligned}$$

(Check: the $k = 2$ closed form below is $(+1)\,2^{n-4}\,n!/\big(2!\,(n-4)!\big) = 2^{n-4}\,n(n-1)(n-2)(n-3)/2$, matching the last line.) In general, k downward steps means applying the recurrence at $j = n-2, n-4, \ldots, n-2k$. Write the ith step as $j = n - 2i$, so that $(j+2)(j+1) = (n-2i+2)(n-2i+1)$ and $2(j-n) = 2(-2i) = -4i$. Multiplying the k steps together,

$$a_{n-2k} = a_n \prod_{i=1}^{k} \frac{(n-2i+2)(n-2i+1)}{-4i} = 2^n \frac{\displaystyle\prod_{i=1}^{k}(n-2i+2)(n-2i+1)}{\displaystyle\prod_{i=1}^{k}(-4i)}.$$

The numerator telescopes into one falling factorial: the k pairs are $(n, n-1)$, $(n-2, n-3)$, ..., $(n-2k+2, n-2k+1)$, which together run through every integer from n down to $n-2k+1$, so

$$\prod_{i=1}^{k}(n-2i+2)(n-2i+1) = n(n-1)(n-2)\cdots(n-2k+1) = \frac{n!}{(n-2k)!}.$$

The Hermite equation and series solution IV

The denominator is where both the sign and the factorial come from:

$$\prod_{i=1}^{k}(-4i) = (-4)^k \prod_{i=1}^{k} i = (-1)^k\, 2^{2k}\, k!.$$

Dividing, and using $2^n/2^{2k} = 2^{n-2k}$ together with $1/(-1)^k = (-1)^k$,

$$a_{n-2k} = (-1)^k\, 2^{n-2k}\, \frac{n!}{k!\,(n-2k)!} = (-1)^k\, 2^{n-2k}\, \frac{n(n-1)\cdots(n-2k+1)}{k!}.$$

Writing $a_{n-2k}x^{n-2k} = (-1)^k(2x)^{n-2k}\, n!/\big(k!\,(n-2k)!\big)$ and summing gives

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k (2x)^{n-2k} \frac{n!}{k!(n-2k)!}. \tag{7.4.3}$$

Low-n checks:

$$H_0 = 1, \quad H_1 = 2x, \quad H_2 = 4x^2 - 2, \quad H_3 = 8x^3 - 12x.$$

Multiplying (7.4.1) by e^{-x^2} yields the Sturm–Liouville form

$$\frac{d}{dx}\left(e^{-x^2} y'\right) + 2ne^{-x^2} y = 0. \tag{7.4.4}$$

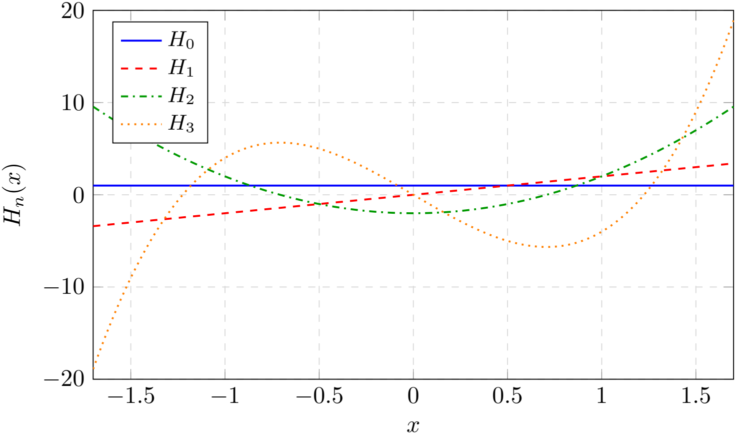
The Hermite equation and series solution V

The same Lagrange-identity argument used for Legendre shows, for $n \neq m$,

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} \, dx = 0. \tag{7.4.5}$$

(The diagonal norm is computed from the generating function below.)

Shapes of the first Hermite polynomials



Hermite polynomials grow away from the origin; the physical oscillator functions are $e^{-x^2/2}H_n(x)$.

Quantum harmonic oscillator I

For a mass-M particle in $V(x) = \frac{1}{2}M\omega^2 x^2$, the time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2M}\psi'' + \frac{1}{2}M\omega^2 x^2\psi = E\psi.$$

The dimensionless coordinate $\xi = \sqrt{M\omega/\hbar}\,x$ converts this to

$$\frac{d^2\psi}{d\xi^2} + \left(\varepsilon - \xi^2\right)\psi = 0, \qquad \varepsilon = \frac{2E}{\hbar\omega}. \tag{7.4.6}$$

(The full constant-chasing that produces (7.4.6) is a short exercise in the chain rule; we record the result.)

For large $|\xi|$, $\psi'' - \xi^2\psi \approx 0$ suggests the decaying factor $e^{-\xi^2/2}$. Set

$$\psi(\xi) = e^{-\xi^2/2}\,y(\xi).$$

Compute the derivatives by the product rule. Write $f := e^{-\xi^2/2}$, so $f' = -\xi f$ and $f'' = (-1+\xi^2)f$ (because $f'' = -(f + \xi f') = -(f + \xi(-\xi f)) = (-1+\xi^2)f$). Then

$$\begin{aligned}
\psi' &= f'y + fy' = e^{-\xi^2/2}\big(y' - \xi y\big),\\
\psi'' &= f''(y) + 2f'y' + fy''\\
&= e^{-\xi^2/2}\Big[(-1+\xi^2)y + 2(-\xi)y' + y''\Big]\\
&= e^{-\xi^2/2}\big(y'' - 2\xi y' + (\xi^2-1)y\big).
\end{aligned}$$

Substitute into (7.4.6):

$$\psi'' + (\varepsilon - \xi^2)\psi = e^{-\xi^2/2}\Big[y'' - 2\xi y' + (\xi^2-1)y + (\varepsilon-\xi^2)y\Big] = e^{-\xi^2/2}\big[y'' - 2\xi y' + (\varepsilon-1)y\big].$$

The Gaussian prefactor never vanishes, so

$$y'' - 2\xi y' + (\varepsilon - 1)y = 0. \tag{7.4.7}$$

A power series $y = \sum a_j \xi^j$ has coefficient recurrence

$$a_{j+2} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}\,a_j.$$

Why the series must terminate. So far nothing restricts ε: the recurrence produces a power-series solution of (7.4.7) for every real ε. The restriction is physical. A bound state must be normalizable, $\int_{-\infty}^{\infty} |\psi|^2 \, d\xi < \infty$, so ψ must decay as $|\xi| \to \infty$. Suppose a parity chain does *not* terminate. For large j the numerator is dominated by $2j$ and the denominator by j^2, so

$$\frac{a_{j+2}}{a_j} = \frac{2j+1-\varepsilon}{(j+2)(j+1)} = \frac{2}{j}\Big(1 + O(1/j)\Big).$$

The Taylor coefficients of $e^{\xi^2} = \sum_{m \geq 0} \xi^{2m}/m!$ have exactly this large-index ratio: writing $c_{2m} = 1/m!$ and $j = 2m$,

$$\frac{c_{j+2}}{c_j} = \frac{m!}{(m+1)!} = \frac{1}{m+1} = \frac{2}{j+2} \sim \frac{2}{j}.$$

A non-terminating chain therefore grows like the tail of e^{ξ^2}, that is $y(\xi) \sim Ce^{\xi^2}$ with $C \neq 0$, and then

$$\psi = e^{-\xi^2/2} y \sim Ce^{\xi^2/2} \longrightarrow \infty \qquad (|\xi| \to \infty),$$

which is not square integrable. Normalizability thus forces the opposite-parity chain to start with coefficient 0, and forces the surviving chain to stop.

Termination at degree n means the numerator $2j + 1 - \varepsilon$ vanishes at $j = n$, i.e.

$$\varepsilon = 2n + 1, \qquad n = 0, 1, 2, \ldots,$$

Quantum harmonic oscillator IV

and then $\varepsilon - 1 = 2n$, so (7.4.7) is exactly the Hermite equation $y'' - 2\xi y' + 2ny = 0$. Since $\varepsilon = 2E/(\hbar\omega)$, the energies are therefore

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, \ldots, \tag{7.4.8}$$

and the spatial factors are $\psi_n \propto e^{-\xi^2/2} H_n(\xi)$.

Running the same product-rule calculation in reverse (now with $y = H_n$ and $\varepsilon = 2n + 1$) shows that

$$\psi_n(x) := e^{-x^2/2} H_n(x) \quad \text{solves} \quad \psi_n'' + (2n + 1 - x^2)\psi_n = 0. \tag{7.4.9}$$

Generating function, norm, and Rodrigues I

Step 1: generating function from the series. Define

$$g(x,t) := \exp(2xt - t^2) = e^{2xt} e^{-t^2}.$$

Expand each exponential:

$$e^{2xt} = \sum_{m=0}^{\infty} \frac{(2x)^m}{m!}\, t^m, \qquad e^{-t^2} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\, t^{2j}.$$

The product is a double sum. The coefficient of t^n arises from pairs (m,j) with $m + 2j = n$, i.e. $m = n - 2j$ for $j = 0, \dots, \lfloor n/2 \rfloor$:

$$\begin{aligned}
[t^n]g(x,t) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(2x)^{n-2j}}{(n-2j)!} \cdot \frac{(-1)^j}{j!} \\
&= \frac{1}{n!} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j (2x)^{n-2j} \frac{n!}{j!\,(n-2j)!}.
\end{aligned}$$

Comparing with the series solution (7.4.3), the sum in brackets is exactly $H_n(x)$. Therefore

$$g(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \qquad (7.4.10)$$

Generating function, norm, and Rodrigues II

Step 2: Rodrigues from $g = e^{x^2} e^{-(x-t)^2}$. Rewrite the exponent:

$$2xt - t^2 = x^2 - (x-t)^2, \qquad g(x,t) = e^{x^2}\, e^{-(x-t)^2}.$$

Taylor-expand in t about $t = 0$:

$$e^{-(x-t)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\bigg|_{t=0}.$$

On the composite $e^{-(x-t)^2}$, one has $\partial/\partial t = -\partial/\partial x$, so

$$\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\bigg|_{t=0} = (-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

Hence

$$g(x,t) = e^{x^2} \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n \frac{d^n}{dx^n} e^{-x^2} = \sum_{n=0}^{\infty} \left[(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \right] \frac{t^n}{n!}.$$

Matching coefficients with (7.4.10) gives Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{7.4.11}$$

Step 3: recurrences from $\partial_t g$ and $\partial_x g$. Differentiate the closed form:

$$\frac{\partial g}{\partial t} = (2x - 2t)g, \qquad \frac{\partial g}{\partial x} = 2tg.$$

Using the series (7.4.10),

$$\begin{aligned}
\partial_t g &= \sum_{n=1}^{\infty} H_n(x) \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} H_{n+1}(x) \frac{t^n}{n!}, \\
(2x - 2t)g &= \sum_{n=0}^{\infty} 2x H_n(x) \frac{t^n}{n!} - \sum_{n=0}^{\infty} 2H_n(x) \frac{t^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} 2x H_n \frac{t^n}{n!} - \sum_{n=1}^{\infty} 2n H_{n-1} \frac{t^n}{n!}
\end{aligned}$$

(the second sum reindexes $n \mapsto n-1$ and uses $n/n! = 1/(n-1)!$). Equating $[t^n]/n!$ for $n \geq 1$ (and checking $n = 0$ separately) yields

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x). \tag{7.4.12}$$

Likewise,

$$\partial_x g = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!} = 2tg = \sum_{n=0}^{\infty} 2H_n(x) \frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} 2n H_{n-1}(x) \frac{t^n}{n!},$$

Generating function, norm, and Rodrigues IV

so comparing $[t^n]/n!$ gives

$$H_n'(x) = 2nH_{n-1}(x). \tag{7.4.13}$$

Together,

$$H_n'(x) = 2nH_{n-1}(x), \qquad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{7.4.14}$$

Step 4: norm via the double generating function. For real s, t near 0,

$$I(s,t) := \int_{-\infty}^{\infty} e^{-x^2} g(x,s)g(x,t)\,dx = \int_{-\infty}^{\infty} e^{-x^2-s^2+2sx-t^2+2tx}\,dx.$$

Complete the square in x:

$$-x^2 + 2(s+t)x - s^2 - t^2 = -\big[x-(s+t)\big]^2 + 2st,$$

so

$$I(s,t) = \sqrt{\pi}\,e^{2st} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(2st)^k}{k!} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{2^k}{k!}\,s^k t^k.$$

On the other hand, expand both generating functions:

$$g(x,s)g(x,t) = \sum_{m,n=0}^{\infty} H_m(x)H_n(x)\frac{s^m t^n}{m!\,n!},$$

Generating function, norm, and Rodrigues V

Before integrating this double series term by term, we check that the series of absolute values has an integrable sum. Replacing every term of (7.4.3) by its absolute value gives $|H_n(x)| \le \sum_{k=0}^{\lfloor n/2 \rfloor} (2|x|)^{n-2k} n!/\big(k!\,(n-2k)!\big)$, and summing that bound with $m = n - 2k$ (so that the $n!$ cancels and the double sum factors) yields

$$\sum_{n=0}^{\infty} \frac{|t|^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} (2|x|)^{n-2k} \frac{n!}{k!\,(n-2k)!} = \Bigg(\sum_{m \ge 0} \frac{(2|x|\,|t|)^m}{m!}\Bigg)\Bigg(\sum_{k \ge 0} \frac{|t|^{2k}}{k!}\Bigg) = e^{2|x||t| + t^2}.$$

Applying this once with t and once with s and multiplying,

$$\int_{-\infty}^{\infty} e^{-x^2} \sum_{m,n \ge 0} \frac{\big|H_m(x)H_n(x)\big|\,|s|^m |t|^n}{m!\,n!}\,dx \le e^{s^2+t^2} \int_{-\infty}^{\infty} e^{-x^2+2|x|(|s|+|t|)}\,dx < \infty,$$

the last integral being finite because the Gaussian $-x^2$ beats the linear exponent. Every term is therefore absolutely integrable and sum and integral may be exchanged:

$$I(s,t) = \sum_{m,n=0}^{\infty} \left(\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx \right) \frac{s^m t^n}{m!\,n!}.$$

The right-hand closed form $\sqrt{\pi}\,e^{2st}$ contains only diagonal powers $s^k t^k$. Equating coefficients of $s^m t^n$:

Generating function, norm, and Rodrigues VI

- if $m \neq n$, the right side has coefficient 0, so $\displaystyle\int H_m H_n e^{-x^2}\,dx = 0$;
- if $m = n$, compare $[s^n t^n]$:

$$\frac{1}{(n!)^2}\int_{-\infty}^{\infty} H_n^2 e^{-x^2}\,dx = \sqrt{\pi}\,\frac{2^n}{n!},$$

  hence

$$\int_{-\infty}^{\infty} H_n(x)^2 e^{-x^2}\,dx = 2^n n!\,\sqrt{\pi}.$$

Combining both cases,

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = 2^n n!\sqrt{\pi}\,\delta_{mn}. \tag{7.4.15}$$

Low-n check: $H_0 = 1$ and $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Useful formulas for the Hermite polynomial

differential equation	$H_n'' - 2x\, H_n' + 2n\, H_n = 0$
series solution	$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k (2x)^{n-2k} \dfrac{n!}{k!\,(n-2k)!}$
generating function	$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\, \dfrac{t^n}{n!}$
recurrence	$H_n' = 2n\, H_{n-1}, \quad H_{n+1} = 2x\, H_n - 2n\, H_{n-1}$
orthogonality	$\int_{-\infty}^{\infty} H_m H_n\, e^{-x^2} dx = n!\, 2^n \sqrt{\pi}\, \delta_{mn}$
Rodrigues	$H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n} e^{-x^2}$

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Laguerre equation, polynomials, and orthogonality I

Standard Laguerre polynomials solve

$$x\,y'' + (1-x)y' + n\,y = 0, \qquad n = 0, 1, 2, \ldots . \tag{7.5.1}$$

(A one-sentence context: generalized Laguerre polynomials $L_n^{(\alpha)}$ appear in the radial hydrogenic Schrödinger equation; here we develop only the standard family $L_n = L_n^{(0)}$.)

Step 1: power series and coefficient recurrence. *Why an ordinary power series is the right ansatz.* Dividing (7.5.1) by x puts it in the standard form $y'' + P(x)y' + Q(x)y = 0$ with

$$P(x) = \frac{1-x}{x}, \qquad Q(x) = \frac{n}{x},$$

so the origin is not an ordinary point. It is a regular singular point, since $xP(x) = 1 - x$ and $x^2Q(x) = nx$ are analytic there, with values $p_0 = 1$ and $q_0 = 0$ at $x = 0$. The indicial equation $r(r-1) + p_0 r + q_0 = 0$ therefore reads

$$r(r-1) + r = r^2 = 0,$$

a repeated root $r = 0$. The Frobenius ansatz $y = x^r \sum_{k\geq 0} a_k x^k$ thus collapses to the ordinary power series $\sum_{k\geq 0} a_k x^k$, and it delivers exactly one solution; the second independent solution carries a logarithm and blows up at the origin. For $n = 0$ one

Laguerre equation, polynomials, and orthogonality II

sees this explicitly: $xy'' + (1-x)y' = 0$ with $u = y'$ gives $u'/u = 1 - 1/x$, hence $u = e^x/x$ and

$$y_2(x) = \int^x \frac{e^u}{u}\,du = \log x + (\text{analytic}),$$

which is unbounded as $x \to 0^+$. Since we want solutions bounded on $[0, \infty)$ — and in fact polynomials — we keep only the $r = 0$ branch.

Set $y = \sum_{k=0}^{\infty} a_k x^k$. Then

$$
\begin{aligned}
y' &= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k, \\
xy'' &= \sum_{k=0}^{\infty} k(k+1) a_{k+1} x^k \\
&\qquad \Big(\text{from } y'' = \sum (k+2)(k+1) a_{k+2} x^k, \text{ times } x, \text{ reindex}\Big), \\
-xy' &= -\sum_{k=0}^{\infty} k a_k x^k, \\
ny &= \sum_{k=0}^{\infty} n a_k x^k.
\end{aligned}
$$

Collecting the coefficient of x^k in $xy'' + (1-x)y' + ny$ gives

$$
k(k+1)a_{k+1} + (k+1)a_{k+1} + (n-k)a_k = (k+1)^2 a_{k+1} + (n-k)a_k.
$$

Setting every coefficient to zero yields the one-step recurrence

$$
a_{k+1} = -\frac{n-k}{(k+1)^2}\, a_k \qquad \left(\text{equivalently } \frac{a_{k+1}}{a_k} = \frac{k-n}{(k+1)^2}\right). \tag{7.5.2}
$$

At $k = n$ one has $a_{n+1} = 0$, so the series terminates as a polynomial of degree n. With the conventional choice $a_0 = 1$, iterating (7.5.2) gives

$$a_k = (-1)^k \frac{n(n-1)\cdots(n-k+1)}{(k!)^2} = (-1)^k \frac{1}{k!}\binom{n}{k},$$

and therefore

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}. \tag{7.5.3}$$

The first few:

$$L_0 = 1, \quad L_1 = 1 - x, \quad L_2 = 1 - 2x + \tfrac{1}{2}x^2, \quad L_3 = 1 - 3x + \tfrac{3}{2}x^2 - \tfrac{1}{6}x^3.$$

Step 2: Sturm–Liouville form and off-diagonal orthogonality. Multiplying (7.5.1) by e^{-x} produces the SL form $\left(xe^{-x}y'\right)' + ne^{-x}y = 0$. The same boundary-vanishing argument as for Hermite (the factor xe^{-x} kills the boundary form at 0 and ∞) gives, for $n \neq m$,

$$\int_0^\infty L_n(x)L_m(x)e^{-x}\,dx = 0. \tag{7.5.4}$$

Step 3: generating function and the diagonal norm. Define

$$g(x,t) := \frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)\, t^n, \qquad |t|<1. \tag{7.5.5}$$

The first equality is the definition of g; the second is a claim about the polynomials (7.5.3), and we prove it now. Expand the exponential and absorb the prefactor:

$$\frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right) = \frac{1}{1-t}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-xt}{1-t}\right)^k = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\,\frac{t^k}{(1-t)^{k+1}}.$$

Each factor $(1-t)^{-(k+1)}$ is a differentiated geometric series: differentiating $\frac{1}{1-t} = \sum_{n\geq 0} t^n$ k times term by term (allowed inside the radius of convergence) gives $\frac{k!}{(1-t)^{k+1}} = \sum_{n\geq k} n(n-1)\cdots(n-k+1)\, t^{n-k}$, that is

$$\frac{1}{(1-t)^{k+1}} = \sum_{j=0}^{\infty}\binom{k+j}{k} t^j, \qquad |t|<1.$$

So the power t^n is produced by the pairs with $j = n - k$ and $0 \le k \le n$, each carrying the factor $\binom{k+(n-k)}{k} = \binom{n}{k}$:

$$[t^n]\, g(x,t) = \sum_{k=0}^{n} \frac{(-x)^k}{k!} \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!} = L_n(x)$$

by (7.5.3), which is (7.5.5). The rearrangement of the double series is legitimate: replacing x, t by $|x|, |t|$ makes every term nonnegative and the same computation sums them to $\frac{1}{1-|t|} \exp\big(|x||t|/(1-|t|)\big) < \infty$ for $|t| < 1$, so the convergence is absolute and the order of summation is immaterial. (Consistency check: at $t = 0$, $g = 1 = L_0$; the coefficient of t is $1 - x = L_1$; the coefficient of t^2 is $1 - 2x + \frac{1}{2}x^2 = L_2$.)

Now evaluate the double generating integral. For $|s|, |t| < 1$,

$$\begin{aligned} e^{-x} g(x,s) g(x,t) &= \frac{1}{(1-s)(1-t)} \exp\left(-x - \frac{xs}{1-s} - \frac{xt}{1-t} \right) \\ &= \frac{1}{(1-s)(1-t)} \exp\left(-x \left[1 + \frac{s}{1-s} + \frac{t}{1-t} \right] \right). \end{aligned}$$

Laguerre equation, polynomials, and orthogonality VII

The coefficient of x in the exponent simplifies as

$$1+\frac{s}{1-s}+\frac{t}{1-t}=\frac{1-s}{1-s}+\frac{s}{1-s}+\frac{t}{1-t}=\frac{1}{1-s}+\frac{t}{1-t}$$
$$=\frac{1-t+t(1-s)}{(1-s)(1-t)}=\frac{1-st}{(1-s)(1-t)}.$$

Hence

$$e^{-x}g(x,s)g(x,t)=\frac{1}{(1-s)(1-t)}\exp\left(-x\,\frac{1-st}{(1-s)(1-t)}\right).$$

Integrate in x from 0 to ∞. With $\alpha:=(1-st)/((1-s)(1-t))>0$ for small real s,t, one has $\int_0^\infty e^{-\alpha x}\,dx=1/\alpha$, so

$$\int_0^\infty e^{-x}g(x,s)g(x,t)\,dx=\frac{1}{(1-s)(1-t)}\cdot\frac{(1-s)(1-t)}{1-st}=\frac{1}{1-st}.$$

Expand both sides. The right side is

$$\frac{1}{1-st}=\sum_{n=0}^{\infty}(st)^n=\sum_{n=0}^{\infty}s^nt^n.$$

The left side is

$$\sum_{m,n=0}^{\infty} \left(\int_0^{\infty} L_m(x) L_n(x) e^{-x}\, dx \right) s^m t^n.$$

Equating coefficients of $s^m t^n$: if $m \neq n$ the integral vanishes (already known); if $m = n$ the coefficient is 1. Therefore the family is orthonormal:

$$\int_0^{\infty} L_m(x) L_n(x) e^{-x}\, dx = \delta_{mn}. \tag{7.5.6}$$

Step 4: three-term recurrence from $\partial_t g$. From the closed form (7.5.5),

$$\frac{\partial g}{\partial t} = g \cdot \frac{1 - t - x}{(1-t)^2},$$

or equivalently

$$(1-t)^2\, \partial_t g = (1 - t - x)\, g. \tag{7.5.7}$$

Expand both sides in powers of t. With $g = \sum_{n \geq 0} L_n t^n$ one has $\partial_t g = \sum_{n \geq 0} (n+1) L_{n+1} t^n$, so the left side of (7.5.7) is

$$(1 - 2t + t^2) \sum_{n \geq 0} (n+1) L_{n+1} t^n.$$

Laguerre equation, polynomials, and orthogonality IX

The coefficient of t^n on the left is

$$(n+1)L_{n+1} - 2nL_n + (n-1)L_{n-1}$$

(for $n \geq 1$, with the convention $L_{-1} := 0$). On the right, $(1-t-x)\sum L_n t^n$ contributes

$$(1-x)L_n - L_{n-1}.$$

Equating coefficients and rearranging yields

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x). \tag{7.5.8}$$

Check at $n=1$: $(2)L_2 = (3-x)L_1 - L_0$ gives $2\left(1-2x+\frac{1}{2}x^2\right) = (3-x)(1-x)-1$, which holds identically.

Step 5: Rodrigues formula. Define

$$R_n(x) := \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right). \tag{7.5.9}$$

Laguerre equation, polynomials, and orthogonality X

Low-n match to the series (7.5.3):

$$R_0 = e^x \cdot x^0 e^{-x} = 1 = L_0,$$

$$R_1 = e^x \frac{d}{dx}\left(xe^{-x}\right) = e^x(e^{-x} - xe^{-x}) = 1 - x = L_1,$$

$$R_2 = \frac{e^x}{2!}\frac{d^2}{dx^2}\left(x^2 e^{-x}\right).$$

For the last, $\frac{d}{dx}(x^2 e^{-x}) = (2x - x^2)e^{-x}$ and $\frac{d^2}{dx^2}(x^2 e^{-x}) = (2 - 4x + x^2)e^{-x}$, so

$$R_2 = \frac{e^x}{2}(2 - 4x + x^2)e^{-x} = 1 - 2x + \tfrac{1}{2}x^2 = L_2.$$

General n: by Leibniz' rule,

$$\frac{d^n}{dx^n}\left(x^n e^{-x}\right) = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j}{dx^j}(x^n)\, \frac{d^{\,n-j}}{dx^{\,n-j}}(e^{-x}).$$

Both derivatives are elementary,

$$\frac{d^j}{dx^j}x^n = \frac{n!}{(n-j)!}\, x^{\,n-j}, \qquad \frac{d^{\,n-j}}{dx^{\,n-j}} e^{-x} = (-1)^{\,n-j} e^{-x},$$

Laguerre equation, polynomials, and orthogonality XI

so, keeping every term of the sum,

$$\frac{d^n}{dx^n}\left(x^n e^{-x}\right) = e^{-x} \sum_{j=0}^{n} \binom{n}{j} \frac{n!}{(n-j)!} (-1)^{n-j} x^{n-j}.$$

Reindex by $k := n - j$, which runs over $0, \ldots, n$ as j does, and use $\binom{n}{n-k} = \binom{n}{k}$ together with $(n-j)! = k!$:

$$\frac{d^n}{dx^n}\left(x^n e^{-x}\right) = e^{-x} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} (-1)^k x^k.$$

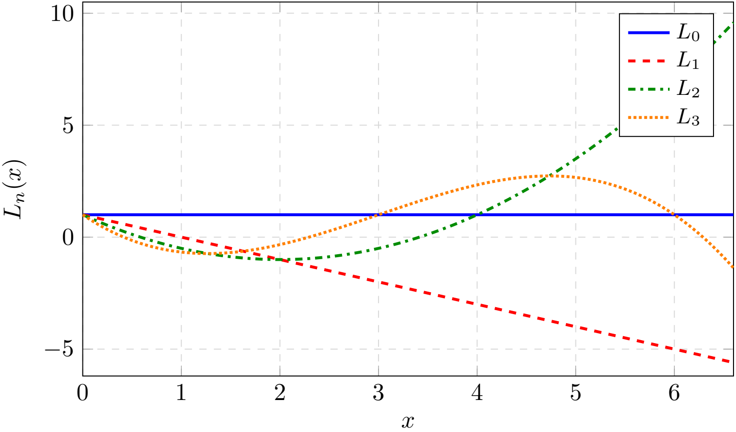
Multiplying by $e^x/n!$ cancels the exponential and the $n!$:

$$R_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!} = L_n(x)$$

by (7.5.3), for every n — no ODE or uniqueness argument needed. The two ends of the sum are the checks already made: $k = n$ gives the leading coefficient $(-1)^n/n!$, and $k = 0$ gives $R_n(0) = 1 = L_n(0)$. This is Rodrigues' formula

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}\left(x^n e^{-x}\right).$$

Shapes of the first Laguerre polynomials



All standard Laguerre polynomials satisfy $L_n(0) = 1$. The weight e^{-x} makes them natural on $[0, \infty)$.

Useful formulas for the Laguerre polynomial

differential equation	$xL_n'' + (1-x)L_n' + nL_n = 0$
series / first polynomials	$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}$
	$L_0 = 1,\ L_1 = 1 - x,\ L_2 = 1 - 2x + \frac{1}{2}x^2$
generating function	$\frac{1}{1-t} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x) t^n$
recurrence	$(n+1)L_{n+1} = (2n+1-x)L_n - nL_{n-1}$
orthonormality	$\int_0^{\infty} L_m L_n\, e^{-x}\, dx = \delta_{mn}$
Rodrigues	$L_n = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x})$

Generating series requires $|t| < 1$.

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Definition through $\cos n\theta$ I

The Chebyshev polynomial of the first kind is defined by the elementary identity

$$T_n(\cos\theta) = \cos(n\theta). \tag{7.6.1}$$

That is, set $x = \cos\theta$ and read off a polynomial in x:

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1, \qquad T_3(x) = 4x^3 - 3x.$$

(For T_2: $\cos 2\theta = 2\cos^2\theta - 1$. For T_3: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

Recurrence from angle addition. Adding
$\cos((n+1)\theta) = \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta$ and
$\cos((n-1)\theta) = \cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta$ cancels the sines:

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x).$$

Thus

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1, \tag{7.6.2}$$

with $T_0 = 1$, $T_1 = x$. The recurrence is also what guarantees that $\cos(n\theta)$ really is a polynomial in $x = \cos\theta$ for every n, not just for the four cases listed above: T_0 and T_1 are polynomials of degrees 0 and 1, and if T_{n-1} and T_n are polynomials of degrees $n-1$ and n, with T_n having leading coefficient k_n, then $2xT_n - T_{n-1}$ has degree $n+1$ and its leading term is $2x \cdot k_n x^n = 2k_n x^{n+1}$. So the degree goes up by one at each step and $k_{n+1} = 2k_n$; starting from $k_1 = 1$,

$$k_n = 2^{n-1} \qquad (n \geq 1).$$

Definition through $\cos n\theta$ II

Two further facts come straight from (7.6.1). The endpoints $x = 1$ and $x = -1$ correspond to $\theta = 0$ and $\theta = \pi$, so

$$T_n(1) = \cos 0 = 1, \qquad T_n(-1) = \cos(n\pi) = (-1)^n.$$

For the parity, note that $\theta \mapsto \pi - \theta$ sends $x = \cos\theta$ to $\cos(\pi - \theta) = -x$, so

$$T_n(-x) = \cos\big(n\pi - n\theta\big) = \cos(n\pi)\cos(n\theta) + \sin(n\pi)\sin(n\theta) = (-1)^n T_n(x),$$

because $\sin(n\pi) = 0$ for every integer n.

Generating function from the recurrence. A single power series packages all the T_n at once. Put

$$G(x,t) := \sum_{n=0}^{\infty} T_n(x)\, t^n.$$

For $x = \cos\theta \in [-1,1]$ we have $|T_n(x)| = |\cos n\theta| \le 1$, so the series is dominated by $\sum |t|^n$ and converges absolutely for $|t| < 1$; the regroupings below are therefore legitimate. Multiply (7.6.2) by $t^{\,n+1}$ and sum over $n \ge 1$, which is the full range in which the recurrence holds:

$$\sum_{n=1}^{\infty} T_{n+1} t^{\,n+1} = 2xt \sum_{n=1}^{\infty} T_n t^{\,n} - t^2 \sum_{n=1}^{\infty} T_{n-1} t^{\,n-1}.$$

Each sum is G with a few terms removed or added. Using $T_0 = 1$ and $T_1 = x$,

$$\sum_{n=1}^{\infty} T_{n+1} t^{n+1} = \sum_{k=2}^{\infty} T_k t^k = G - 1 - xt, \qquad \sum_{n=1}^{\infty} T_n t^n = G - 1,$$

$$\sum_{n=1}^{\infty} T_{n-1} t^{n-1} = \sum_{k=0}^{\infty} T_k t^k = G.$$

So $G - 1 - xt = 2xt\,(G-1) - t^2 G$; collecting the G terms on the left and the rest on the right gives $\left(1 - 2xt + t^2\right) G = 1 + xt - 2xt = 1 - xt$, that is

$$\sum_{n=0}^{\infty} T_n(x)\, t^n = \frac{1 - xt}{1 - 2xt + t^2}, \qquad |t| < 1, \quad -1 \le x \le 1. \tag{7.6.3}$$

The restriction $|t| < 1$ is sharp: writing $x = \cos\theta$ factors the denominator as $1 - 2xt + t^2 = (1 - te^{i\theta})(1 - te^{-i\theta})$, so both poles sit on $|t| = 1$. Indeed the same factorisation reproves (7.6.3) in one line, since for $|t| < 1$

$$\sum_{n=0}^{\infty} \cos(n\theta)\, t^n = \operatorname{Re} \sum_{n=0}^{\infty} \left(te^{i\theta}\right)^n = \operatorname{Re} \frac{1}{1 - te^{i\theta}} = \operatorname{Re} \frac{1 - te^{-i\theta}}{|1 - te^{i\theta}|^2} = \frac{1 - t\cos\theta}{1 - 2t\cos\theta + t^2}.$$

Definition through $\cos n\theta$ IV

Differential equation via the chain rule. Set $x = \cos\theta$ and $Y(\theta) := T_n(\cos\theta) = \cos(n\theta)$, so $Y''(\theta) = -n^2\cos(n\theta) = -n^2 Y(\theta)$. Chain rule with $dx/d\theta = -\sin\theta$ gives

$$\frac{d}{d\theta} = -\sin\theta\,\frac{d}{dx}, \qquad \frac{dY}{d\theta} = -\sin\theta\,T_n'(x).$$

Differentiate again:

$$\begin{aligned}\frac{d^2Y}{d\theta^2} &= \frac{d}{d\theta}\big(-\sin\theta\,T_n'(x)\big)\\ &= -\cos\theta\,T_n'(x) - \sin\theta\cdot\Big(-\sin\theta\,T_n''(x)\Big)\\ &= -\cos\theta\,T_n'(x) + \sin^2\theta\,T_n''(x).\end{aligned}$$

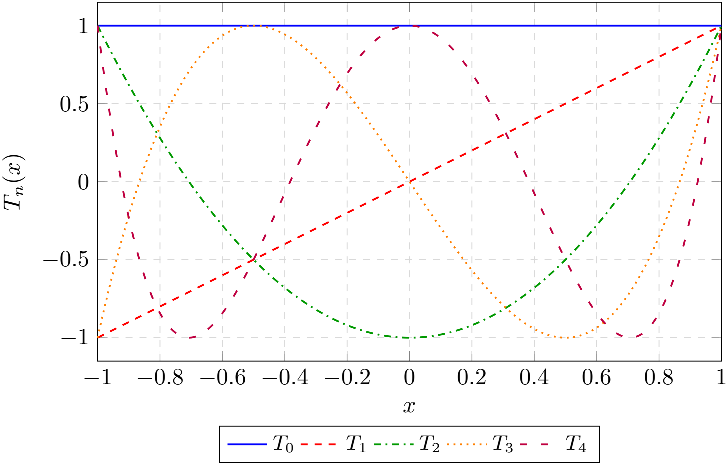
Substitute $\cos\theta = x$ and $\sin^2\theta = 1 - x^2$, and use $Y'' = -n^2 Y = -n^2 T_n$:

$$(1-x^2)T_n''(x) - xT_n'(x) = -n^2 T_n(x).$$

Rearranging yields the Chebyshev equation

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0. \tag{7.6.4}$$

Shapes of the first Chebyshev polynomials



$T_n(\cos\theta) = \cos(n\theta)$ keeps every T_n between -1 and 1 on $[-1, 1]$.

Chebyshev nodes and orthogonality I

Zeros (nodes). On $x = \cos\theta \in [-1, 1]$,

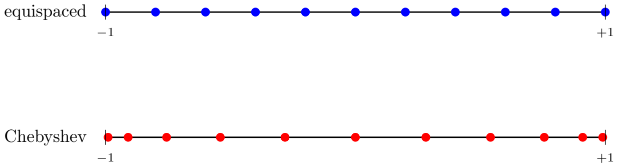
$$T_n(x) = 0 \iff \cos(n\theta) = 0 \iff \theta_k = \frac{(2k-1)\pi}{2n}, \quad k = 1, \dots, n.$$

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \qquad k = 1, \dots, n. \tag{7.6.5}$$

These nodes cluster near the endpoints. To see the rate, take the outermost node $x_1 = \cos\frac{\pi}{2n}$ and measure its distance from $x = 1$ using the half-angle identity $1 - \cos\varepsilon = 2\sin^2(\varepsilon/2)$ together with $\sin u \le u$ for $u \ge 0$:

$$1 - x_1 = 1 - \cos\frac{\pi}{2n} = 2\sin^2\frac{\pi}{4n} \ \le\ 2\left(\frac{\pi}{4n}\right)^2 = \frac{\pi^2}{8n^2}.$$

Since $\sin u/u \to 1$ as $u \to 0$, this is sharp: $1 - x_1 \sim \pi^2/(8n^2)$; and because $x_n = \cos\left(\pi - \frac{\pi}{2n}\right) = -x_1$, the node nearest -1 is the same distance away. Near the middle of the interval the picture is different: there $\theta_{k+1} - \theta_k = \pi/n$ and $|dx/d\theta| = \sin\theta$ is of order 1, so consecutive nodes are only $O(n^{-1})$ apart. This $O(n^{-2})$ crowding at the two ends is why Chebyshev interpolation is stable.



Orthogonality. The weight is not a guess: it is exactly what puts (7.6.4) into Sturm–Liouville form, just as the factor e^{-x^2} did for Hermite in (7.4.4). Since $\dfrac{d}{dx}\sqrt{1-x^2} = -\dfrac{x}{\sqrt{1-x^2}}$, the product rule gives

$$\frac{d}{dx}\left(\sqrt{1-x^2}\;T_n'\right) = \sqrt{1-x^2}\;T_n'' - \frac{x}{\sqrt{1-x^2}}\;T_n',$$

which is precisely the first two terms of (7.6.4) after division by $\sqrt{1-x^2}$. Dividing that equation through by $\sqrt{1-x^2}$ therefore yields

$$\frac{d}{dx}\left(\sqrt{1-x^2}\;T_n'\right) + \frac{n^2}{\sqrt{1-x^2}}\;T_n = 0,$$

Chebyshev nodes and orthogonality III

that is $\left(pT_n'\right)' + \lambda w T_n = 0$ with

$$p(x) = \sqrt{1-x^2}, \qquad w(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \lambda = n^2.$$

The weight is the coefficient w that the Sturm–Liouville form hands us, and no other choice works. Moreover $p(\pm 1) = 0$, so the boundary form $\left[p\left(T_n T_m' - T_n' T_m\right)\right]_{-1}^{1}$ vanishes without imposing any boundary condition, and the same Lagrange-identity argument used for Legendre and Hermite already predicts $\int_{-1}^{1} T_n T_m \, w \, dx = 0$ for $n \neq m$. The substitution below confirms this and supplies the normalisation as well.

With the weight $(1-x^2)^{-1/2}$, substitute $x = \cos\theta$. Then $dx = -\sin\theta \, d\theta$, and as x runs from -1 to 1, θ runs from π down to 0. Also $\sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$ for $\theta \in [0,\pi]$. Therefore

$$\begin{aligned}
\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx &= \int_{\pi}^{0} \frac{T_n(\cos\theta)\;T_m(\cos\theta)}{\sin\theta}(-\sin\theta)\,d\theta \\
&= \int_0^{\pi} T_n(\cos\theta)\;T_m(\cos\theta)\,d\theta \\
&= \int_0^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta,
\end{aligned}$$

Chebyshev nodes and orthogonality IV

where the last step is the defining identity $T_k(\cos\theta) = \cos(k\theta)$. The cosine integrals all follow from one product-to-sum identity. Adding $\cos(A-B) = \cos A \cos B + \sin A \sin B$ and $\cos(A+B) = \cos A \cos B - \sin A \sin B$ cancels the sines and gives, for arbitrary A, B,

$$\cos A \cos B = \tfrac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big].$$

Take $A = n\theta$ and $B = m\theta$. If $n \neq m$ then $n - m \neq 0$, and also $n + m \geq 1$ (since $n + m = 0$ would force $n = m = 0$), so both frequencies are nonzero integers and

$$\begin{aligned}
\int_0^\pi \cos(n\theta)\cos(m\theta)\,d\theta &= \frac{1}{2}\int_0^\pi \Big[\cos((n-m)\theta) + \cos((n+m)\theta)\Big]\,d\theta \\
&= \frac{1}{2}\left[\frac{\sin((n-m)\theta)}{n-m} + \frac{\sin((n+m)\theta)}{n+m}\right]_{\theta=0}^{\theta=\pi} \\
&= \frac{1}{2}\left[\frac{\sin((n-m)\pi)}{n-m} + \frac{\sin((n+m)\pi)}{n+m}\right] = 0,
\end{aligned}$$

because $n \pm m$ are integers and $\sin(k\pi) = 0$ for every integer k. If $n = m \geq 1$ the same identity with $A = B = n\theta$ gives $\cos^2(n\theta) = \tfrac{1}{2}\Big[1 + \cos(2n\theta)\Big]$, so

$$\int_0^\pi \cos^2(n\theta)\,d\theta = \left[\frac{\theta}{2} + \frac{\sin(2n\theta)}{4n}\right]_0^\pi = \frac{\pi}{2},$$

again since $\sin(2n\pi) = 0$. Finally, if $n = m = 0$ the integrand is 1 and the integral is π. Collecting the three cases,

$$\int_0^\pi \cos(n\theta)\cos(m\theta)\, d\theta = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \geq 1, \\ 0, & n \neq m. \end{cases}$$

Hence

$$\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1 - x^2}}\, dx = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \geq 1, \\ 0, & n \neq m. \end{cases} \tag{7.6.6}$$

Useful formulas for the Chebyshev polynomial

definition	$T_n(\cos\theta) = \cos(n\theta)$
differential equation	$(1-x^2)T_n'' - xT_n' + n^2 T_n = 0$
first polynomials	$T_0 = 1,\ T_1 = x,\ T_2 = 2x^2 - 1,\ T_3 = 4x^3 - 3x$
recurrence	$T_{n+1} = 2xT_n - T_{n-1}$
generating function	$\dfrac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n$, (7.6.3)
zeros	$x_k = \cos\left(\dfrac{(2k-1)\pi}{2n}\right)$
orthogonality	$\int_{-1}^{1} \dfrac{T_n T_m}{\sqrt{1-x^2}}\,dx = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \geq 1, \\ 0, & n \neq m. \end{cases}$

Generating series requires $|t| < 1$.

Table of contents

Summary: four orthogonal families

Family	**Interval**	**Weight**	**ODE (sketch)**
Legendre P_n	$[-1, 1]$	1	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$
Hermite H_n	$(-\infty, \infty)$	e^{-x^2}	$y'' - 2xy' + 2ny = 0$
Laguerre L_n	$[0, \infty)$	e^{-x}	$xy'' + (1-x)y' + ny = 0$
Chebyshev T_n	$[-1, 1]$	$(1-x^2)^{-1/2}$	$(1-x^2)y'' - xy' + n^2 y = 0$

Physical origins (one line each). P_n: axisymmetric Laplace / multipoles; H_n: quantum harmonic oscillator; L_n: hydrogenic radial equation (via $L_n^{(\alpha)}$); T_n: $\cos n\theta$ and stable interpolation.

The shared structure is a second-order Sturm–Liouville ODE, a recurrence, and an orthogonality relation on a natural interval and weight. Orthogonality alone only tells us how to *compute* the coefficients of a series; to know that the series actually reproduces f we need the separate property of *completeness*, stated for P_n on the Fourier–Legendre slide. It holds for all four families, for the reason on the next slide, and then lets us expand reasonable functions exactly as Fourier series expand in e^{inx}.

Why the expansions converge: completeness I

Completeness of a family $\{\phi_n\}$ in $L^2(I, w\,dx)$ means what it meant for P_n: the only f in that space with $\int_I f\,\phi_n\,w\,dx = 0$ for every n is $f = 0$, so that the partial sums of the coefficient series converge to f in the weighted L^2 norm. This does *not* follow from the Sturm–Liouville form — Chapter 5 gave us real eigenvalues and weighted orthogonality, and nothing more — so it is an extra input. Each of the following families is complete in the space named beside it:

$$\{P_n\} \text{ in } L^2\big([-1,1], dx\big), \qquad \{T_n\} \text{ in } L^2\Big([-1,1], (1-x^2)^{-1/2}dx\Big),$$

$$\{H_n\} \text{ in } L^2\Big(\mathbb{R}, e^{-x^2}dx\Big), \qquad \{L_n\} \text{ in } L^2\Big([0,\infty), e^{-x}dx\Big).$$

In each case the family contains exactly one polynomial of every degree $0, 1, 2, \ldots$, so its closed span is the closure of *all* polynomials; the claim is therefore that polynomials are dense in the weighted space. On $[-1,1]$ this is the Weierstrass approximation theorem plus a finite total mass: since

$$\int_{-1}^{1} dx = 2, \qquad \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \Big[\arcsin x\Big]_{-1}^{1} = \pi,$$

both measures are finite, and for a finite measure μ a uniform approximation is an L^2 approximation because $\|f\|_{L^2(\mu)} \le \mu(I)^{1/2}\|f\|_\infty$.

On $\mathbb{R}$ and on $[0,\infty)$ the interval is unbounded, so Weierstrass does not apply and a different fact is needed. We quote the standard criterion: polynomials are dense in

Why the expansions converge: completeness II

$L^2(w\,dx)$ whenever $\int e^{c|x|}w(x)\,dx < \infty$ for some $c > 0$. Our two weights satisfy it.
Completing the square, $c|x| - x^2 = \frac{c^2}{4} - \left(|x| - \frac{c}{2}\right)^2$, so

$$\int_{-\infty}^{\infty} e^{c|x|}e^{-x^2}dx = e^{c^2/4}\int_{-\infty}^{\infty} e^{-(|x|-c/2)^2}dx \le 2\sqrt{\pi}\,e^{c^2/4} < \infty,$$

and on the half line, for any $0 < c < 1$,

$$\int_0^{\infty} e^{cx}e^{-x}\,dx = \int_0^{\infty} e^{-(1-c)x}\,dx = \frac{1}{1-c} < \infty.$$

That criterion is the one statement on this slide we use without proof.