Chapter 6: Bessel Functions

Special Functions of Mathematical Physics

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Outline

Where we are going

Bessel's equation of integer order m is

$$x^2 y''(x) + xy'(x) + (x^2 - m^2)y(x) = 0. \tag{6.0.1}$$

It arises when variables are separated in circular or cylindrical geometries. This chapter will establish, with all index shifts and normalization factors shown explicitly,

- a generating function for the integer-order family J_m, and the power series;
- the order and derivative recurrences;
- one carefully derived real integral representation (plus a brief contour form);
- the small- and large-argument expansions (large x by stationary phase, the real oscillatory case of steepest descent from Ch. 2);
- the second solution Y_m, a short look at Hankel functions $H^{(1)}, H^{(2)}$, and the discrete Fourier–Bessel series on the disk.

Throughout the generating-function sections, $m, n \in \mathbb{Z}$ unless stated otherwise. For small- or large-argument limits, the order is held fixed. On the infinite half-line the discrete Fourier–Bessel sum becomes a Hankel integral transform; that continuous transform is not developed here.

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Generating function for the family $\{J_m\}$ I

Why a generating function? Separation of variables in polar or cylindrical coordinates produces an infinite family of radial solutions, one for each integer angular order $m \in \mathbb{Z}$. Writing a separate series for every m works, but it is wasteful: recurrences that link neighboring orders, and integral representations that extract a single order, are far cleaner if the whole family is packaged into one closed object $g(x, t)$.

A standard packaging for a bi-infinite sequence $\{P_m(x)\}_{m\in\mathbb{Z}}$ is a *Laurent generating function*

$$g(x, t) = \sum_{m=-\infty}^{\infty} P_m(x)t^m, \qquad t \neq 0. \tag{6.1.1}$$

One closed formula for g then yields every P_m by coefficient extraction, and differentiating g with respect to t or x produces the recurrence identities automatically.

Which closed form? For integer-order Bessel functions we adopt the classical choice

$$g(x, t) = \exp\left[\frac{x}{2}\left(t - t^{-1}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(x)t^m, \qquad t \neq 0, \tag{6.1.2}$$

Generating function for the family $\{J_m\}$ II

and *define* $J_m(x)$ as the Laurent coefficient of t^m. This is not pulled from the sky: the combination $t - t^{-1}$ is precisely what appears when a plane wave $e^{ix \sin\theta}$ is expanded in angular Fourier modes (set $t = e^{i\theta}$ later). Once (6.1.2) is in place we will

1. extract the power series of J_m by the Cauchy product of two exponentials;
2. verify term by term that the series solves Bessel's equation (6.0.1);
3. derive recurrences and one real integral representation from the same g.

Two-minute review: Laurent coefficient extraction I

Step 1: ordinary power series first. If $f(t) = \sum_{n=0}^{\infty} a_n t^n$ is analytic at the origin, then

$$a_n = \left[t^n\right] f(t) = \frac{f^{(n)}(0)}{n!}.$$

The same idea works for series with both positive and negative powers.

Step 2: Laurent series on an annulus. A function analytic for $0 < r < |t| < R$ admits a unique expansion

$$f(t) = \sum_{m=-\infty}^{\infty} c_m t^m, \qquad c_m = \left[t^m\right] f(t).$$

On any positively oriented simple closed contour C lying in that annulus and winding once about 0, Cauchy's integral formula extracts the coefficient:

$$\left[t^m\right] f(t) = \frac{1}{2\pi i} \oint_C f(t)\, t^{-m-1}\, dt. \tag{6.1.3}$$

Special cases used constantly below:

- if $f(t) = \sum_m c_m t^m$, then $\left[t^m\right] f = c_m$ by uniqueness;
- for an integer power, $\left[t^m\right] t^k = \delta_{mk}$ (Kronecker delta);

Two-minute review: Laurent coefficient extraction II

- the product of two Laurent series has coefficients given by the Cauchy convolution: if $f = \sum a_j t^j$ and $g = \sum b_k t^k$, then $\left[t^m\right](fg) = \sum_{j+k=m} a_j b_k$.

Step 3: what we will do with $g(x,t)$. Write $g(x,t) = e^{xt/2}\, e^{-x/(2t)}$, expand each exponential as a power series, multiply, and collect the coefficient of t^m. By definition that coefficient is $J_m(x)$.

Reading off the integer-order series I

Step 1: expand both exponentials. For fixed x, the product representation

$$g(x,t) = e^{xt/2}e^{-x/(2t)}$$

converges locally uniformly on every compact annulus $0 < r \le |t| \le R < \infty$: on $|t| \le R$ the series $e^{xt/2} = \sum_{j\ge 0} \frac{(x/2)^j}{j!} t^j$ converges uniformly, and on $|t| \ge r > 0$ the series $e^{-x/(2t)} = \sum_{k\ge 0} \frac{(-1)^k (x/2)^k}{k!} t^{-k}$ converges uniformly. Both are absolutely convergent, so their Cauchy product is an absolutely and locally uniformly convergent Laurent series. Absolute convergence lets us regroup by the power of t and differentiate termwise in x or t on such annuli.

Explicitly,

$$e^{xt/2} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{xt}{2}\right)^j = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{x}{2}\right)^j t^j,$$

$$e^{-x/(2t)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{x}{2t}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{x}{2}\right)^k t^{-k}.$$

Reading off the integer-order series II

Multiplying gives the double series

$$g(x,t) = \left(\sum_{j=0}^{\infty} \frac{(x/2)^j}{j!} t^j\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^k}{k!} t^{-k}\right)$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{j!\,k!}\left(\frac{x}{2}\right)^{j+k} t^{j-k}.$$

Step 2: impose the power constraint. To extract the coefficient of t^m for $m \geq 0$, impose

$$j - k = m.$$

Since $j, k \geq 0$, this is equivalent to

$$j = k + m, \qquad k = 0, 1, 2, \ldots.$$

Then

$$j + k = (k+m) + k = 2k + m, \qquad j! = (k+m)!.$$

Therefore

$$[t^m]g(x,t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+m)!\,k!}\left(\frac{x}{2}\right)^{2k+m}.$$

Reading off the integer-order series III

By (6.1.2), $[t^m]g = J_m$, so

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(m+k)!} \left(\frac{x}{2}\right)^{2k+m}, \qquad m = 0, 1, 2, \ldots . \tag{6.1.4}$$

The first few series are

$$\begin{aligned} J_0(x) &= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots , \\ J_1(x) &= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18432} + \cdots , \\ J_2(x) &= \frac{x^2}{8} - \frac{x^4}{96} + \frac{x^6}{3072} + \cdots . \end{aligned}$$

The generating function contains only integer powers of t; therefore it generates integer-order functions J_m. The general-order series J_ν is stated next and is not a Laurent coefficient when $\nu \notin \mathbb{Z}$.

General order and direct verification of Bessel's equation I

Step 1: state the general-order series. For a general order ν, with Γ denoting the gamma function, the Frobenius series is

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{6.1.5}$$

Where this series comes from. It is not a new postulate: (6.1.5) is the $r=\nu$ Frobenius solution built in Chapter 4. Substituting $y = \sum_{k\geq 0} a_k x^{k+\nu}$ into Bessel's equation and using $(k+\nu)^2 - \nu^2 = k(k+2\nu)$ collapses the equation to the two-term recurrence

$$k(k+2\nu)a_k + a_{k-2} = 0, \qquad a_{-1} = a_{-2} = 0,$$

which links a_k only to a_{k-2}. Taking $a_1 = 0$ therefore removes the odd chain, while the even chain $a_{2j} = -a_{2j-2}/\big(4j(j+\nu)\big)$ unwinds to

$$a_{2j} = (-1)^j \frac{\Gamma(\nu+1)}{2^{2j}\, j!\,\Gamma(j+\nu+1)}\, a_0,$$

where the product $\prod_{\ell=1}^{j}(\nu+\ell) = \Gamma(\nu+j+1)/\Gamma(\nu+1)$ was used; that product formula is just $\Gamma(z+1) = z\,\Gamma(z)$ (Chapter 3) applied j times. The overall scale a_0 is free, and the conventional choice

$$a_0 = \frac{2^{-\nu}}{\Gamma(\nu+1)}$$

General order and direct verification of Bessel's equation II

cancels the factor $\Gamma(\nu+1)$ and leaves $a_{2j}x^{2j+\nu} = \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)}(x/2)^{2j+\nu}$, which is exactly (6.1.5). This normalization is chosen so that the general-order series agrees with the integer-order series read off the generating function, as the next paragraph checks.
For $x > 0$, $(x/2)^\nu = e^{\nu \log(x/2)}$ is unambiguous. For complex x, a branch of $\log x$ must be fixed when $\nu \notin \mathbb{Z}$. If $\nu = m \geq 0$ is an integer, then $\Gamma(k+m+1) = (k+m)!$, and (6.1.5) reduces to (6.1.4).

For fixed ν and fixed branch, let $T_k(x)$ denote the k-th term of (6.1.5). For all sufficiently large k,

$$\left|\frac{T_{k+1}(x)}{T_k(x)}\right| = \frac{|x|^2}{4(k+1)|k+\nu+1|} \longrightarrow 0.$$

The ratio tends to zero uniformly for x in a compact set. The differentiated series have this same quadratic decay in k, up to polynomial factors. Thus the series and the series obtained by differentiating once or twice with respect to x converge locally uniformly away from the chosen branch cut. This justifies the termwise substitutions below.

Step 2: verify the differential equation term by term. We now verify directly that (6.1.4) satisfies Bessel's equation. Write

$$J_m(x) = \sum_{k=0}^{\infty} a_k x^{2k+m}, \qquad a_k = \frac{(-1)^k}{2^{2k+m} k!(m+k)!}.$$

For one monomial $a_k x^{2k+m}$,

$$x^2 \frac{d^2}{dx^2}\left(a_k x^{2k+m}\right) = a_k(2k+m)(2k+m-1)x^{2k+m},$$
$$x \frac{d}{dx}\left(a_k x^{2k+m}\right) = a_k(2k+m)x^{2k+m}.$$

Adding these two expressions and subtracting $m^2 a_k x^{2k+m}$ gives

$$\begin{aligned}
&\left[(2k+m)(2k+m-1) + (2k+m) - m^2\right] a_k x^{2k+m} \\
&= \left[(2k+m)^2 - m^2\right] a_k x^{2k+m} \\
&= \left[(2k+m-m)(2k+m+m)\right] a_k x^{2k+m} \\
&= 4k(k+m) a_k x^{2k+m}.
\end{aligned}$$

General order and direct verification of Bessel's equation IV

Hence

$$\begin{aligned} &x^2 J_m'' + x J_m' - m^2 J_m \\ &\quad = \sum_{k=0}^{\infty} 4k(k+m) a_k x^{2k+m} = \sum_{k=1}^{\infty} 4k(k+m) a_k x^{2k+m}, \end{aligned}$$

because the $k = 0$ term is zero. The remaining term in Bessel's equation is

$$\begin{aligned} x^2 J_m(x) &= \sum_{k=0}^{\infty} a_k x^{2k+m+2} \\ &= \sum_{k=1}^{\infty} a_{k-1} x^{2k+m}, \end{aligned}$$

where the second line uses the index shift $k \mapsto k+1$ in reverse: writing the series as $\sum_{k=0}^{\infty} a_k x^{2k+m+2}$ and renaming the summation index $k' = k+1$ (so when $k=0$, $k'=1$, and $a_k = a_{k'-1}$), then dropping the prime, yields $\sum_{k=1}^{\infty} a_{k-1} x^{2k+m}$. Finally,

$$\begin{aligned} a_{k-1} &= \frac{(-1)^{k-1}}{2^{2k+m-2}(k-1)!(m+k-1)!} \\ &= -\frac{4k(m+k)(-1)^k}{2^{2k+m} k!(m+k)!} \\ &= -4k(m+k)a_k. \end{aligned}$$

Check the factor of $4k(m+k)$ carefully:

$$\begin{aligned} \frac{a_{k-1}}{a_k} &= \frac{(-1)^{k-1}}{(-1)^k} \cdot \frac{2^{2k+m}}{2^{2k+m-2}} \cdot \frac{k!}{(k-1)!} \cdot \frac{(m+k)!}{(m+k-1)!} \\ &= (-1) \cdot 2^2 \cdot k \cdot (m+k) = -4k(m+k). \end{aligned}$$

Thus every coefficient in

$$x^2 J_m'' + x J_m' + (x^2 - m^2) J_m = \sum_{k=1}^{\infty} \left[4k(k+m)a_k + a_{k-1}\right] x^{2k+m}$$

vanishes, proving

$$x^2 J_m''(x) + x J_m'(x) + (x^2 - m^2) J_m(x) = 0. \tag{6.1.6}$$

General order and direct verification of Bessel's equation VI

The same coefficient calculation verifies the general-order series. With

$$a_k(\nu) = \frac{(-1)^k}{2^{2k+\nu} k!\, \Gamma(k+\nu+1)},$$

the gamma recurrence $\Gamma(k+\nu+1) = (k+\nu)\Gamma(k+\nu)$ gives

$$\begin{aligned} a_{k-1}(\nu) &= \frac{(-1)^{k-1}}{2^{2k+\nu-2}(k-1)!\, \Gamma(k+\nu)} \\ &= -4k(k+\nu)a_k(\nu). \end{aligned}$$

At exceptional values for which a gamma factor has a pole, the same identity follows by continuity because $1/\Gamma(z)$ is entire and vanishes at the nonpositive integers. Replacing m by ν in the preceding coefficient identity therefore proves

$$x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) = 0$$

for the chosen branch of the series.

Step 1: read the negative coefficient off g itself. By (6.1.2) the symbol J_{-m} *means* the Laurent coefficient $[t^{-m}]g$, so the honest starting point is the double series found earlier,

$$g(x,t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{j!\,k!}\left(\frac{x}{2}\right)^{j+k} t^{j-k}.$$

Fix $m \in \mathbb{Z}_{\geq 0}$ and impose $j-k=-m$. Since $j,k \geq 0$, this is equivalent to

$$k = j+m, \qquad j = 0,1,2,\ldots,$$

the mirror image of the constraint $j = k+m$ used for nonnegative orders. Then

$$j+k = j+(j+m) = 2j+m, \qquad k! = (j+m)!, \qquad (-1)^k = (-1)^{j+m} = (-1)^m(-1)^j,$$

so

$$\begin{aligned} J_{-m}(x) = [t^{-m}]g(x,t) &= \sum_{j=0}^{\infty} \frac{(-1)^{j+m}}{j!\,(j+m)!}\left(\frac{x}{2}\right)^{2j+m} \\ &= (-1)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(m+j)!}\left(\frac{x}{2}\right)^{2j+m} \\ &= (-1)^m J_m(x), \end{aligned}$$

the last equality by (6.1.4). This settles (6.1.7) for the definition actually in force, and it closes a second loop as well: the Laurent coefficient $[t^{-m}]g$ is now known, so comparing it with the computation below shows that the general-order series (6.1.5) evaluated at $\nu = -m$ returns the same function. From here on the two names for J_{-m} may be used interchangeably.

Step 2: the same result from the general-order series.

Let $m \in \mathbb{Z}_{\geq 0}$. For $x \neq 0$, substitute $\nu = -m$ into the general-order series (6.1.5):

$$J_{-m}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-m+1)} \left(\frac{x}{2}\right)^{2k-m}.$$

For $k = 0, 1, \ldots, m-1$, the argument $k - m + 1$ is a nonpositive integer. The gamma function has a pole there, so its reciprocal is zero:

$$\frac{1}{\Gamma(k-m+1)} = 0, \qquad 0 \leq k \leq m-1.$$

Consequently the first nonzero term is $k = m$:

$$J_{-m}(x) = \sum_{k=m}^{\infty} \frac{(-1)^k}{k!\,(k-m)!} \left(\frac{x}{2}\right)^{2k-m}.$$

Negative integer orders: $J_{-m} = (-1)^m J_m$ III

Set $k = \ell + m$. Then $\ell = 0, 1, 2, \ldots$, and

$$(-1)^k = (-1)^{\ell+m} = (-1)^m (-1)^\ell, \qquad 2k - m = 2(\ell + m) - m = 2\ell + m.$$

Also $k! = (\ell + m)!$ and $(k - m)! = \ell!$. Therefore

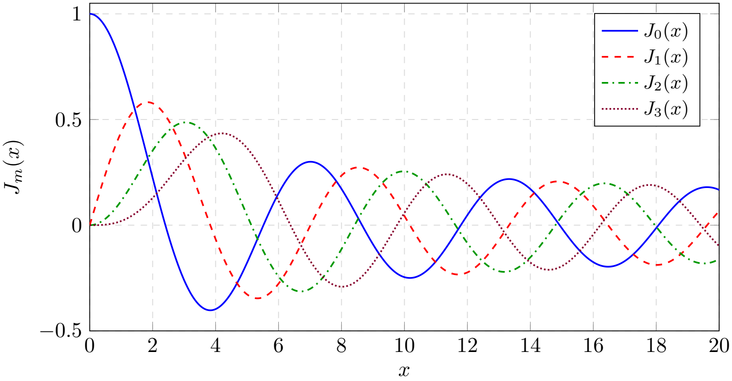
$$\begin{aligned} J_{-m}(x) &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+m}}{(\ell+m)!\,\ell!} \left(\frac{x}{2}\right)^{2\ell+m} \\ &= (-1)^m \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\,(\ell+m)!} \left(\frac{x}{2}\right)^{2\ell+m} \\ &= (-1)^m J_m(x). \end{aligned}$$

Thus, first for $x \neq 0$ and then at $x = 0$ by continuity,

$$J_{-m}(x) = (-1)^m J_m(x), \qquad m \in \mathbb{Z}. \tag{6.1.7}$$

In particular, $J_{-1} = -J_1$, $J_{-2} = J_2$, and negative integer orders do not provide a second linearly independent solution.

Bessel functions of the first kind



For $m \geq 0$, the leading term in (6.1.4) is $J_m(x) = (x/2)^m/m! + O(x^{m+2})$. Thus $J_0(0) = 1$ and $J_m(0) = 0$ for $m \geq 1$.

Jacobi–Anger: the generating function on $|t| = 1$ I

Set $t = e^{i\theta}$ in (6.1.2). Since $t^{-1} = e^{-i\theta}$,

$$\begin{aligned} t - t^{-1} &= e^{i\theta} - e^{-i\theta} \\ &= (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) \\ &= 2i\sin\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \exp\left[\frac{x}{2}(t - t^{-1})\right] &= \exp\left[\frac{x}{2}(2i\sin\theta)\right] \\ &= e^{ix\sin\theta}. \end{aligned}$$

On the series side, $t^m = e^{im\theta}$. Hence

$$e^{ix\sin\theta} = \sum_{m=-\infty}^{\infty} J_m(x) e^{im\theta}. \tag{6.1.8}$$

This is the Jacobi–Anger expansion: a plane wave in polar phase is the Fourier series of the integer-order Bessel family. For fixed x the series $\sum_m J_m(x)e^{im\theta}$ is the restriction of the Laurent series (6.1.2) to the circle $|t| = 1$, so it converges uniformly in $\theta \in [0, 2\pi]$, and a uniformly convergent series may be integrated term by term.

Jacobi–Anger: the generating function on $|t| = 1$ II

Multiplying by $e^{-in\theta}$ and integrating over one period therefore gives the coefficient explicitly:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta} e^{-in\theta}\, d\theta = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} J_m(x)\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta.$$

For an integer q,

$$\begin{aligned} \int_0^{2\pi} e^{iq\theta}\, d\theta &= \int_0^{2\pi} 1\, d\theta = 2\pi, && q = 0, \\ \int_0^{2\pi} e^{iq\theta}\, d\theta &= \left.\frac{e^{iq\theta}}{iq}\right|_0^{2\pi} = \frac{e^{i2\pi q} - 1}{iq} = 0, && q \neq 0. \end{aligned}$$

Equivalently, $\int_0^{2\pi} e^{iq\theta}\, d\theta = 2\pi\, \delta_{q0}$. Thus only $m = n$ survives:

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\theta - n\theta)}\, d\theta, \qquad n \in \mathbb{Z}. \tag{6.1.9}$$

This complex integral is the starting point for the real trigonometric form derived next.

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Differentiating in t: the three-term recurrence I

Step 1: differentiate the closed form. Differentiate the closed form in (6.1.2). The exponent is

$$h(t) = \frac{x}{2}\left(t - t^{-1}\right),$$

so

$$\begin{aligned} h'(t) &= \frac{x}{2}\left(\frac{d}{dt}t - \frac{d}{dt}t^{-1}\right) \\ &= \frac{x}{2}\left(1 - (-t^{-2})\right) \\ &= \frac{x}{2}(1 + t^{-2}). \end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{\partial g}{\partial t} &= h'(t)e^{h(t)} \\ &= \frac{x}{2}(1 + t^{-2})g(x,t). \end{aligned}$$

Differentiating in t: the three-term recurrence II

Insert the Laurent series for g:

$$\frac{\partial g}{\partial t} = \frac{x}{2}(1+t^{-2}) \sum_{m=-\infty}^{\infty} J_m(x) t^m$$
$$= \frac{x}{2} \sum_{m=-\infty}^{\infty} J_m(x) t^m + \frac{x}{2} \sum_{m=-\infty}^{\infty} J_m(x) t^{m-2}.$$

In the second sum, set $m = r+2$, equivalently $r = m-2$:

$$\sum_{m=-\infty}^{\infty} J_m(x) t^{m-2} = \sum_{r=-\infty}^{\infty} J_{r+2}(x) t^r.$$

Renaming r as m gives

$$\frac{\partial g}{\partial t} = \frac{x}{2} \sum_{m=-\infty}^{\infty} \left[J_m(x) + J_{m+2}(x) \right] t^m. \tag{6.2.1}$$

Step 2: differentiate the Laurent series. Differentiate the Laurent series directly:

$$\frac{\partial g}{\partial t} = \sum_{m=-\infty}^{\infty} m J_m(x) t^{m-1}.$$

Set $m = r + 1$, so that $m - 1 = r$:

$$\frac{\partial g}{\partial t} = \sum_{r=-\infty}^{\infty} (r+1) J_{r+1}(x) t^r$$
$$= \sum_{m=-\infty}^{\infty} (m+1) J_{m+1}(x) t^m.$$

Equating the coefficients of t^m in this expression and (6.2.1) yields

$$(m+1) J_{m+1}(x) = \frac{x}{2} \left[J_m(x) + J_{m+2}(x) \right].$$

Replace m by $m - 1$:

$$m J_m(x) = \frac{x}{2} \left[J_{m-1}(x) + J_{m+1}(x) \right].$$

For $x \neq 0$, multiplication by $2/x$ gives

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x). \tag{6.2.2}$$

At $x = 0$, any endpoint value is obtained by taking a limit from the power series rather than by substituting into the quotient $2mJ_m/x$.

Differentiating in x: the derivative recurrence I

Differentiate the closed form in (6.1.2) with respect to x. Since

$$\frac{\partial}{\partial x}\left[\frac{x}{2}(t - t^{-1})\right] = \frac{1}{2}(t - t^{-1}),$$

the chain rule gives

$$\begin{aligned}
\frac{\partial g}{\partial x} &= \frac{1}{2}(t - t^{-1})g(x,t) \\
&= \frac{1}{2}(t - t^{-1}) \sum_{m=-\infty}^{\infty} J_m(x)t^m \\
&= \frac{1}{2}\sum_{m=-\infty}^{\infty} J_m(x)t^{m+1} - \frac{1}{2}\sum_{m=-\infty}^{\infty} J_m(x)t^{m-1}.
\end{aligned}$$

For the first sum, set $m = r - 1$; for the second, set $m = r + 1$:

$$\sum_m J_m t^{m+1} = \sum_r J_{r-1} t^r,$$

$$\sum_m J_m t^{m-1} = \sum_r J_{r+1} t^r.$$

Differentiating in x: the derivative recurrence II

After renaming r as m,

$$\frac{\partial g}{\partial x} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \left[J_{m-1}(x) - J_{m+1}(x) \right] t^m. \tag{6.2.3}$$

On the other hand, direct differentiation of the series gives

$$\frac{\partial g}{\partial x} = \sum_{m=-\infty}^{\infty} J_m'(x) t^m.$$

Equating coefficients of t^m gives

$$2J_m'(x) = J_{m-1}(x) - J_{m+1}(x). \tag{6.2.4}$$

We can combine (6.2.2) and (6.2.4) to eliminate either neighboring order. First solve (6.2.2) for J_{m+1}:

$$J_{m+1} = \frac{2m}{x} J_m - J_{m-1}.$$

Substitute this into (6.2.4):

$$\begin{aligned} 2J_m' &= J_{m-1} - \left(\frac{2m}{x} J_m - J_{m-1} \right) \\ &= 2J_{m-1} - \frac{2m}{x} J_m. \end{aligned}$$

Differentiating in x: the derivative recurrence III

Dividing by 2 gives

$$J_m'(x) = J_{m-1}(x) - \frac{m}{x} J_m(x), \qquad x \neq 0. \tag{6.2.5}$$

Alternatively, solve (6.2.2) for J_{m-1}:

$$J_{m-1} = \frac{2m}{x} J_m - J_{m+1}.$$

Substitution into (6.2.4) gives

$$\begin{aligned} 2J_m' &= \left(\frac{2m}{x} J_m - J_{m+1} \right) - J_{m+1} \\ &= \frac{2m}{x} J_m - 2J_{m+1}, \end{aligned}$$

and therefore

$$J_m'(x) = \frac{m}{x} J_m(x) - J_{m+1}(x), \qquad x \neq 0. \tag{6.2.6}$$

These identities are exact algebraic consequences of the generating function: they hold with equality for the true values of J_m. They should not be read as a claim about *computation*. If one uses (6.2.2) in the direction of increasing order, computing $J_{m+1}(x) = \frac{2m}{x} J_m(x) - J_{m-1}(x)$ from the two lower orders (*upward recurrence*), then for fixed x and growing m any rounding error in the starting values J_0, J_1 is amplified rather than damped, so the computed sequence loses accuracy. In

Differentiating in x: the derivative recurrence IV

this numerical sense the upward recurrence is *unstable* for large m. The mechanism is worth spelling out, because it is the same for every three-term recurrence.

Read (6.2.2) as a linear difference equation in the order index,

$$u_{m+1} = \frac{2m}{x} u_m - u_{m-1}, \tag{6.2.7}$$

one solution of which is $u_m = J_m(x)$. Prescribing u_0 and u_1 determines the whole sequence, so the solution space is two-dimensional. Because (6.2.7) is linear, the deviation $e_m := \tilde{J}_m - J_m$ of the computed sequence from the exact one obeys that same equation, apart from the fresh rounding committed at each step; so what happens to e_m is decided by how solutions of (6.2.7) behave as m grows. The two possible behaviors are opposite.

On one side stands the Bessel solution itself. Its $k = 0$ term in (6.1.4) is $(x/2)^m/m!$, and at fixed x that factorial denominator makes $J_m(x)$ decay faster than any geometric sequence in m. On the other side, suppose u_m is a solution whose magnitude *increases* with m. Once $m > x$ the multiplier $2m/x$ in (6.2.7) exceeds 1 and keeps growing, so the subtracted term u_{m-1} is small beside $(2m/x)u_m$ and $u_{m+1} \approx (2m/x)u_m$. Iterating this from some index M onward,

$$u_m \approx u_M \prod_{k=M}^{m-1} \frac{2k}{x} = u_M \left(\frac{2}{x}\right)^{m-M} \frac{(m-1)!}{(M-1)!}$$

$$= C\,\Gamma(m) \left(\frac{2}{x}\right)^{m}, \qquad C = \frac{u_M}{(M-1)!} \left(\frac{x}{2}\right)^{M},$$

using $\prod_{k=M}^{m-1} k = (m-1)!/(M-1)!$ and $\Gamma(m) = (m-1)!$. Such a growing solution really is present in the solution space: it is the second Bessel solution Y_m constructed in (6.5.4) below, whose size (6.5.8) carries exactly the factor $\Gamma(m)(2/x)^m$.

Hence e_m is a combination of a factorially decaying and a factorially growing sequence. Exact starting data would put no weight on the growing one; a rounding error of relative size ε in J_0, J_1 puts weight $O(\varepsilon)$ on it, and then, up to a constant,

$$\frac{|e_m|}{|J_m(x)|} \sim |\varepsilon|\, m!\, \Gamma(m) \left(\frac{2}{x}\right)^{2m} \longrightarrow \infty.$$

At $x = 1$ the amplification factor $m!\,\Gamma(m)2^{2m}$ already exceeds 10^{18} at $m = 10$, so seeds carrying the double-precision relative error $\varepsilon \approx 10^{-16}$ leave nothing behind: started from correctly rounded $J_0(1), J_1(1)$, the upward recurrence returns

-6.91×10^{-9} for $J_{10}(1)$, whose true value is $+2.63 \times 10^{-10}$ — wrong sign, not one correct digit.

Running the recurrence in the opposite direction,

$$u_{m-1} = \frac{2m}{x} u_m - u_{m+1},$$

interchanges the two roles: now it is the $\Gamma(m)(2/x)^m$ solution that is divided by the large factors at each step while the J component grows relative to it, so an arbitrary starting vector is progressively purged of everything but J_m. The practical obstacle is that the two seeds at the top are exactly what one does not know. One therefore starts at some $M \gg m$ with the arbitrary trial values $u_{M+1} = 0$, $u_M = 1$, recurs down to $m = 0$, and fixes the one remaining unknown — the overall scale — at the end. A normalization is already available: setting $\theta = 0$ in (6.1.8) gives $1 = \sum_{m \in \mathbb{Z}} J_m(x)$, and pairing the term m with the term $-m$ through (6.1.7) gives

$$1 = J_0(x) + \sum_{m=1}^{\infty} \left[1 + (-1)^m\right] J_m(x) = J_0(x) + 2 \sum_{k=1}^{\infty} J_{2k}(x),$$

since $1 + (-1)^m$ vanishes for odd m and equals 2 for even m. The downward sweep produces $u_m = \lambda J_m(x)$, to within the accuracy sought, for some unknown λ, so

Differentiating in x: the derivative recurrence VII

$u_0 + 2\sum_{k\geq 1} u_{2k} = \lambda$, and dividing the computed sequence by that number restores the correct scale. This is how accurate high-order values are obtained in practice.

Consequences and low-order checks I

For $m = 0$, (6.2.6) gives

$$J_0'(x) = -J_1(x). \tag{6.2.8}$$

This agrees term by term with the series:

$$\begin{aligned} J_0'(x) &= \frac{d}{dx}\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) \\ &= -\frac{x}{2} + \frac{x^3}{16} - \frac{x^5}{384} + \cdots \\ &= -\left(\frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \cdots\right) \\ &= -J_1(x). \end{aligned}$$

For $m = 1$, (6.2.2) gives

$$\begin{aligned} J_0(x) + J_2(x) &= \frac{2}{x} J_1(x), \\ J_2(x) &= \frac{2}{x} J_1(x) - J_0(x). \end{aligned}$$

Consequences and low-order checks II

Using the series for J_0 and J_1,

$$\begin{aligned}\frac{2}{x}J_1(x) - J_0(x) &= \left(1 - \frac{x^2}{8} + \frac{x^4}{192} + \cdots\right) - \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + \cdots\right) \\ &= \frac{x^2}{8} - \frac{x^4}{96} + \cdots \\ &= J_2(x),\end{aligned}$$

which checks the index and sign conventions.

The one-sided derivative recurrences can also be written in product form. From (6.2.5),

$$\begin{aligned}\frac{d}{dx}\left[x^m J_m(x)\right] &= mx^{m-1}J_m + x^m J_m' \\ &= mx^{m-1}J_m + x^m\left(J_{m-1} - \frac{m}{x}J_m\right) \\ &= x^m J_{m-1}(x),\end{aligned}$$

so

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x), \qquad x \neq 0. \tag{6.2.9}$$

Consequences and low-order checks III

For the second product identity, use (6.2.6) explicitly:

$$\begin{aligned}
\frac{d}{dx}\left[x^{-m}J_m(x)\right] &= -mx^{-m-1}J_m + x^{-m}J_m' \\
&= -mx^{-m-1}J_m + x^{-m}\left(\frac{m}{x}J_m - J_{m+1}\right) \\
&= -mx^{-m-1}J_m + mx^{-m-1}J_m - x^{-m}J_{m+1} \\
&= -x^{-m}J_{m+1}(x).
\end{aligned}$$

Therefore

$$\frac{d}{dx}\left[x^{-m}J_m(x)\right] = -x^{-m}J_{m+1}(x), \qquad x \neq 0. \tag{6.2.10}$$

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Bessel's real integral representation I

Goal. Convert the complex full-period formula (6.1.9) into a real cosine integral on $[0, \pi]$.

Start from (renaming n as $m \in \mathbb{Z}$)

$$J_m(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x \sin\theta - m\theta)}\, d\theta.$$

Split $\int_0^{2\pi} = \int_0^{\pi} + \int_{\pi}^{2\pi}$. In the second integral substitute $\theta = 2\pi - u$, so $d\theta = -du$ and the limits $\theta : \pi \to 2\pi$ become $u : \pi \to 0$; the minus sign flips them back to $u : 0 \to \pi$. Use

$$\sin(2\pi - u) = -\sin u, \qquad -m(2\pi - u) = -2\pi m + mu.$$

The exponent becomes

$$\begin{aligned} i\Big[x \sin(2\pi - u) - m(2\pi - u)\Big] &= i\Big[x(-\sin u) - 2\pi m + mu\Big] \\ &= -i(x \sin u - mu) - i2\pi m. \end{aligned}$$

Bessel's real integral representation II

Hence

$$\begin{aligned}\int_{\pi}^{2\pi} e^{i(x\sin\theta - m\theta)}\,d\theta &= \int_0^{\pi} e^{i[x\sin(2\pi-u)-m(2\pi-u)]}\,du\\ &= \int_0^{\pi} e^{-i(x\sin u - mu)}e^{-i2\pi m}\,du\\ &= \int_0^{\pi} e^{-i(x\sin u - mu)}\,du,\end{aligned}$$

where $e^{-i2\pi m} = 1$ for integer m. Adding the first half-period integral $\int_0^{\pi} e^{i(x\sin u - mu)}\,du$ gives

$$\begin{aligned}\int_0^{\pi}\left[e^{i(x\sin u - mu)} + e^{-i(x\sin u - mu)}\right]du &= 2\int_0^{\pi}\cos(x\sin u - mu)\,du\\ &= 2\int_0^{\pi}\cos(mu - x\sin u)\,du,\end{aligned}$$

Bessel's real integral representation III

because cosine is even. Therefore

$$J_m(x) = \frac{1}{2\pi} \cdot 2 \int_0^\pi \cos(mu - x \sin u)\, du$$
$$= \frac{1}{\pi} \int_0^\pi \cos(mu - x \sin u)\, du.$$

For $m \geq 0$ this is the classical real representation

$$J_m(x) = \frac{1}{\pi} \int_0^\pi \cos\Big(m\theta - x \sin\theta\Big)\, d\theta, \qquad m \in \mathbb{Z}_{\geq 0}. \tag{6.3.1}$$

For $m = 0$ it reduces to

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta)\, d\theta. \tag{6.3.2}$$

(The same formula for negative integer m follows from (6.1.7), or directly from (6.1.9).)

Contour form (Schläfli / Cauchy) I

The complex integral (6.1.9) is the unit-circle case of a more general contour formula. From the generating function (6.1.2) and Laurent extraction (6.1.3),

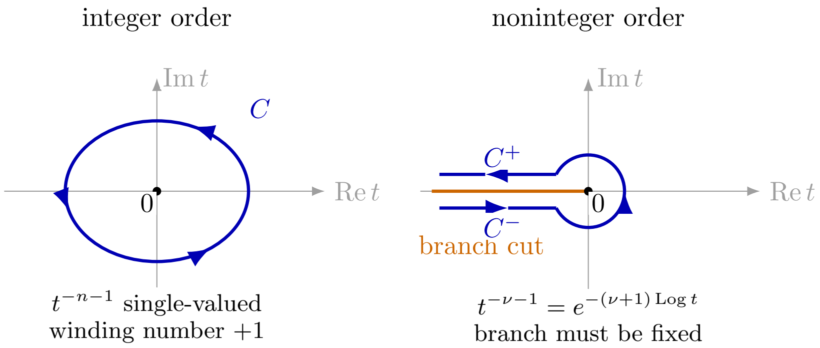
$$J_n(x) = \frac{1}{2\pi i} \oint_C e^{\frac{x}{2}(t-t^{-1})} t^{-n-1}\, dt, \qquad n \in \mathbb{Z}, \tag{6.3.3}$$

where C is any positively oriented closed contour in $\mathbb{C} \setminus \{0\}$ with winding number one about the origin. (Uniform convergence of the Laurent series on the compact contour C permits termwise integration; only the residue at $t = 0$ of the t^{-1} term survives.) Parameterizing C by $t = e^{i\theta}$ recovers (6.1.9) immediately:

$$dt = it\, d\theta, \qquad \frac{x}{2}(t - t^{-1}) = ix \sin\theta, \qquad t^{-n-1} dt = ie^{-in\theta}\, d\theta.$$

For noninteger order the factor $t^{-\nu-1}$ is multivalued and a branch-compatible (nonclosed) contour is required; we use only the integer-order closed form above.

Contours in the Schläfli representation



For integer n, t^{-n-1} is single-valued and Cauchy's coefficient formula only requires a closed contour with winding number one around 0. For noninteger order, a branch of $\mathrm{Log}\,t$ and a contour compatible with its branch cut are required; C^+ and C^- label the upper and lower sides of that cut, respectively.

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Small x: leading term and first corrections I

Step 1: factor out the leading term. For fixed $m \in \mathbb{Z}_{\geq 0}$, factor the $k = 0$ term out of the power series (6.1.4). The first three terms are

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m - \frac{1}{(m+1)!}\left(\frac{x}{2}\right)^{m+2} + \frac{1}{2!(m+2)!}\left(\frac{x}{2}\right)^{m+4} + O(x^{m+6}).$$

Divide the second term by the first:

$$\frac{-\frac{1}{(m+1)!}(x/2)^{m+2}}{\frac{1}{m!}(x/2)^m} = -\frac{m!}{(m+1)!}\left(\frac{x}{2}\right)^2 = -\frac{x^2}{4(m+1)}.$$

Small x: leading term and first corrections II

Divide the third term by the first:

$$\frac{\frac{1}{2!(m+2)!}(x/2)^{m+4}}{\frac{1}{m!}(x/2)^m} = \frac{m!}{2(m+2)!}\left(\frac{x}{2}\right)^4$$

$$= \frac{x^4}{32(m+1)(m+2)}.$$

Hence

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m \left[1 - \frac{x^2}{4(m+1)} + \frac{x^4}{32(m+1)(m+2)} + O(x^6)\right]. \qquad (6.4.1)$$

In particular, writing $m! = \Gamma(m+1)$ (which anticipates the noninteger-order case),

$$J_m(x) = \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m + O(x^{m+2}), \qquad x \to 0, \qquad (6.4.2)$$

with m fixed. This gives $J_0(0) = 1$ and $J_m(0) = 0$ for $m \geq 1$.

Large x: stationary phase on the unit circle I

For large x with fixed order, the unit-circle integral (6.1.9) is an oscillatory integral. Chapter 2 developed steepest descent for $\int g(z)e^{sf(z)}\,dz$ and recorded its real oscillatory case *stationary phase*: the leading contribution as $x \to +\infty$ comes from neighborhoods of points where the phase is stationary; away from those points, repeated integration by parts produces an arbitrarily small power of x^{-1}. Here we *apply* that tool to Bessel's integral—we do not re-derive the master Gaussian formula.

Step 1: locate the stationary points. For fixed integer n, write

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} a(\theta)e^{ix\phi(\theta)}\,d\theta, \qquad a(\theta) = e^{-in\theta}, \quad \phi(\theta) = \sin\theta.$$

The stationary points of the large phase $x\phi(\theta)$ satisfy

$$\phi'(\theta) = \cos\theta = 0,$$

so

$$\theta_+ = \frac{\pi}{2}, \qquad \theta_- = \frac{3\pi}{2}.$$

They correspond to $t = e^{i\theta} = i$ and $t = -i$ on the unit circle. Choose a smooth 2π-periodic partition of unity

$$\eta_+(\theta) + \eta_-(\theta) + \eta_0(\theta) = 1,$$

Large x: stationary phase on the unit circle II

where η_+ and η_- are supported in small neighborhoods of θ_+ and θ_-, respectively. On the support of η_0, there is a constant $c > 0$ such that $|\phi'(\theta)| \geq c$, so $\phi'(\theta) \neq 0$ there. Since $\frac{d}{d\theta} e^{ix\phi} = ix\phi'(\theta)\, e^{ix\phi}$, we may solve for the exponential,

$$e^{ix\phi(\theta)} = \frac{1}{ix\phi'(\theta)} \frac{d}{d\theta} e^{ix\phi(\theta)},$$

and one integration by parts gives

$$\int_0^{2\pi} \eta_0 a e^{ix\phi}\, d\theta = -\frac{1}{ix} \int_0^{2\pi} \frac{d}{d\theta} \left(\frac{\eta_0 a}{\phi'} \right) e^{ix\phi}\, d\theta.$$

There is no boundary term because every factor is 2π-periodic. Repeating this operation N times gives an $O(x^{-N})$ contribution for every fixed N. Hence only the two stationary neighborhoods contribute to the algebraic asymptotic expansion.

Step 2: analyze the first saddle. Near $\theta_+ = \pi/2$, set $\theta = \pi/2 + u$. Choose the local cutoff η_+ to be even in u and equal to 1 near $u = 0$. Then

$$\begin{aligned}
\sin\left(\frac{\pi}{2} + u\right) &= \cos u = 1 - \frac{u^2}{2} + O(u^4), \\
e^{ix \sin(\pi/2+u)} &= e^{ix \cos u} = e^{ix}\, e^{-ixu^2/2 + O(xu^4)}, \\
e^{-in(\pi/2+u)} &= e^{-in\pi/2} e^{-inu} = e^{-in\pi/2} \left(1 - inu - \frac{n^2u^2}{2} + O(u^3)\right).
\end{aligned}$$

Large x: stationary phase on the unit circle III

Write $\tilde{\eta}_+(u) := \eta_+\left(\frac{\pi}{2} + u\right)$. By construction $\tilde{\eta}_+$ is even, equals 1 for $|u| \le \delta/2$, and vanishes for $|u| \ge \delta$, where $\delta < \pi/2$ is the radius of the neighborhood supporting η_+. The localized contribution is then the explicit integral

$$\begin{aligned} J_n^{(+)}(x) &:= \frac{1}{2\pi}\int_0^{2\pi} \eta_+(\theta)\, e^{-in\theta} e^{ix\sin\theta}\, d\theta \\ &= \frac{e^{ix}e^{-in\pi/2}}{2\pi}\int_{-\infty}^{\infty} \tilde{\eta}_+(u)\, e^{-inu}\, e^{ix(\cos u - 1)}\, du, \end{aligned}$$

where the second line is the substitution $\theta = \frac{\pi}{2} + u$ ($d\theta = du$) combined with the two factorizations $e^{-in\theta} = e^{-in\pi/2}e^{-inu}$ and $e^{ix\sin\theta} = e^{ix\cos u} = e^{ix}e^{ix(\cos u - 1)}$ displayed above, and the range may be written as all of $\mathbb{R}$ because $\tilde{\eta}_+$ vanishes for $|u| \ge \delta$. Scale $u = v/\sqrt{x}$, so $du = dv/\sqrt{x}$. Then

$$\begin{aligned} x\left[\sin\left(\frac{\pi}{2} + u\right) - 1\right] &= -\frac{v^2}{2} + O\left(\frac{v^4}{x}\right), \\ e^{-inu} &= 1 - \frac{inv}{\sqrt{x}} + O\left(\frac{v^2}{x}\right), \\ e^{ix\sin(\pi/2+u)} &= e^{ix}\, e^{-iv^2/2 + O(v^4/x)}. \end{aligned}$$

Large x: stationary phase on the unit circle IV

The term proportional to $v/\sqrt{x}$ is odd and integrates to zero against the even local cutoff and the even quadratic phase. All first nonzero corrections are therefore of relative order x^{-1}; after the Jacobian factor $x^{-1/2}$, their absolute order is $x^{-3/2}$. With these expansions the localized integral has become

$$J_n^{(+)}(x) = \frac{e^{ix} e^{-in\pi/2}}{2\pi} \int_{-\infty}^{\infty} \tilde{\eta}_+(u)\, e^{-ixu^2/2}\, du + O\!\left(x^{-3/2}\right),$$

and it remains to delete the cutoff. On the support of $1 - \tilde{\eta}_+$ we have $|u| \geq \delta/2$, so there the quadratic phase is nonstationary and the device of Step 1 applies once more: from $\frac{d}{du} e^{-ixu^2/2} = -ixu\, e^{-ixu^2/2}$ and one integration by parts,

$$\int_{-\infty}^{\infty} \big(1 - \tilde{\eta}_+(u)\big) e^{-ixu^2/2}\, du = \frac{1}{ix} \int_{-\infty}^{\infty} \frac{d}{du} \left(\frac{1 - \tilde{\eta}_+(u)}{u} \right) e^{-ixu^2/2}\, du,$$

with no boundary term because $\big(1 - \tilde{\eta}_+(u)\big)/u \to 0$ as $|u| \to \infty$. The new amplitude is smooth (it vanishes identically near $u = 0$, so the division by u is harmless) and its derivative is $O(u^{-2})$ at infinity, hence absolutely integrable; the step may therefore be repeated, each repetition producing another factor x^{-1}. The difference between the cutoff integral and the full one is thus $O(x^{-N})$ for every fixed N, which is beyond every term we retain. (As in the damped-limit convention introduced next, whole-line oscillatory integrals are understood as $\varepsilon \downarrow 0$ limits.) Thus the leading local contribution is

$$\begin{aligned} J_n^{(+)}(x) &= \frac{e^{ix} e^{-in\pi/2}}{2\pi} \int_{-\infty}^{\infty} e^{-ixu^2/2}\, du + O(x^{-3/2}) \\ &= \frac{e^{ix} e^{-in\pi/2}}{2\pi\sqrt{x}} \int_{-\infty}^{\infty} e^{-iv^2/2}\, dv + O(x^{-3/2}). \end{aligned}$$

To define the oscillatory Gaussian unambiguously, introduce a damping factor $e^{-\varepsilon u^2}$ and let $\varepsilon \downarrow 0$. First record the Gaussian integral with a complex parameter. For real $a > 0$, the substitution $u = t/\sqrt{a}$ in Chapter 3's Gaussian integral $\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$ gives

$$\int_{-\infty}^{\infty} e^{-au^2}\, du = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\frac{\pi}{a}}.$$

This persists for complex a in the right half-plane. Indeed $\left| e^{-au^2} \right| = e^{-(\operatorname{Re} a)u^2}$, so on $\operatorname{Re} a > 0$ the integral converges absolutely and locally uniformly and therefore defines an analytic function of a there; $\sqrt{\pi/a}$, taken on the principal branch (positive for $a > 0$), is analytic there as well; and the two agree on the positive real axis. By the identity theorem (Chapter 2, used the same way in Chapter 3) they agree on the whole half-plane:

$$\int_{-\infty}^{\infty} e^{-au^2}\,du = \sqrt{\frac{\pi}{a}}, \qquad \operatorname{Re} a > 0, \tag{6.4.3}$$

with the principal square root. For $\varepsilon > 0$ the parameter $a = \varepsilon + ix/2$ has $\operatorname{Re} a = \varepsilon > 0$, so (6.4.3) applies:

$$\begin{aligned} I_\varepsilon^-(x) &:= \int_{-\infty}^{\infty} e^{-(\varepsilon + ix/2)u^2}\,du \\ &= \sqrt{\frac{\pi}{\varepsilon + ix/2}}. \end{aligned}$$

On the principal square-root branch, as $\varepsilon \downarrow 0$ with $x > 0$ fixed,

$$\varepsilon + \frac{ix}{2} = \frac{x}{2}\,e^{i\pi/2}\big(1 + o(1)\big),$$

$$\left(\varepsilon + \frac{ix}{2}\right)^{-1/2} \longrightarrow \sqrt{\frac{2}{x}}\,e^{-i\pi/4}.$$

Large x: stationary phase on the unit circle VII

Hence

$$I_\varepsilon^-(x) \longrightarrow \sqrt{\pi}\cdot\sqrt{\frac{2}{x}}\,e^{-i\pi/4} = e^{-i\pi/4}\sqrt{\frac{2\pi}{x}},$$

and the Fresnel integral, understood as this damped limit, is

$$\int_{-\infty}^{\infty} e^{-ixu^2/2}\,du = e^{-i\pi/4}\sqrt{\frac{2\pi}{x}}, \qquad x>0. \tag{6.4.4}$$

Therefore

$$\begin{aligned} J_n^{(+)}(x) &= \frac{e^{ix}e^{-in\pi/2}}{2\pi}\cdot e^{-i\pi/4}\sqrt{\frac{2\pi}{x}} + O(x^{-3/2}) \\ &= \frac{1}{\sqrt{2\pi x}}e^{i(x-n\pi/2-\pi/4)} + O(x^{-3/2}), \end{aligned}$$

where the last line uses $\sqrt{2\pi/x}/(2\pi) = 1/\sqrt{2\pi x}$.

Large x: stationary phase on the unit circle VIII

Step 3: analyze the second saddle. Near $\theta_- = 3\pi/2$, set $\theta = 3\pi/2 + u$ and choose the corresponding local cutoff even in u. Then

$$\sin\left(\frac{3\pi}{2} + u\right) = -\cos u = -1 + \frac{u^2}{2} + O(u^4),$$
$$e^{ix\sin(3\pi/2+u)} = e^{-ix}\, e^{ixu^2/2+O(xu^4)},$$
$$e^{-in(3\pi/2+u)} = e^{-i3n\pi/2} e^{-inu}.$$

For integer n,

$$e^{-i3n\pi/2} = e^{in\pi/2} e^{-i2\pi n} = e^{in\pi/2}.$$

The same scaling $u = v/\sqrt{x}$ shows that the odd amplitude correction integrates to zero and that the first nonzero omitted terms are $O(x^{-3/2})$. Thus

$$J_n^{(-)}(x) = \frac{e^{-ix} e^{in\pi/2}}{2\pi} \int_{-\infty}^{\infty} e^{ixu^2/2}\, du + O(x^{-3/2}).$$

The conjugate (damped) calculation of (6.4.4) gives

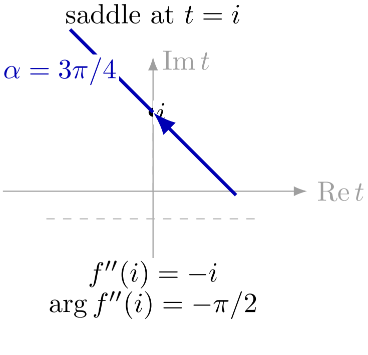
$$\int_{-\infty}^{\infty} e^{ixu^2/2}\, du = e^{i\pi/4}\sqrt{\frac{2\pi}{x}}, \qquad x > 0. \tag{6.4.5}$$

Large x: stationary phase on the unit circle IX

Hence

$$J_n^{(-)}(x) = \frac{1}{\sqrt{2\pi x}} e^{-i(x-n\pi/2-\pi/4)} + O(x^{-3/2}).$$

Steepest descent and the unit-circle saddles I



Steepest descent and the unit-circle saddles II

In the complex t-plane the same two stationary points are the saddles $t = \pm i$ of $f(t) = \frac{1}{2}(t - t^{-1})$ in the Schläfli integrand $e^{xf(t)}t^{-n-1}$. Here

$$f'(t) = \frac{1}{2}\left(1 + \frac{1}{t^2}\right), \qquad f''(t) = -\frac{1}{t^3},$$

so $f'(t) = 0$ exactly when $t^2 = -1$, i.e. $t = \pm i$, which are the same two points found on the unit circle. At those points

$$f''(i) = -\frac{1}{i^3} = -\frac{1}{-i} = \frac{1}{i} = -i = e^{-i\pi/2}, \qquad f''(-i) = -\frac{1}{(-i)^3} = -\frac{1}{i} = i = e^{i\pi/2}.$$

Chapter 2's descent rule $\varphi = (\pi - \vartheta)/2$, in which $\vartheta = \arg f''(t_0)$, therefore gives the local descent directions

$$\varphi_+ = \frac{\pi - (-\pi/2)}{2} = \frac{3\pi}{4} \quad \text{through } t = i,$$
$$\varphi_- = \frac{\pi - \pi/2}{2} = \frac{\pi}{4} \quad \text{through } t = -i,$$

each direction being determined modulo π (the two halves of the same descent line).

The quantitative evaluation above uses *stationary phase on the unit circle* instead of deforming onto those complex descent paths: both routes isolate the same two saddles; the unit-circle calculation fixes orientation and all phase factors without a second master-formula derivation.

Large x: combining the two saddle contributions I

Define the phase

$$\chi_n(x) = x - \frac{n\pi}{2} - \frac{\pi}{4}.$$

By the partition of unity $\eta_+ + \eta_- + \eta_0 = 1$ the exact splitting of Bessel's integral is $J_n = J_n^{(+)} + J_n^{(-)} + J_n^{(0)}$, where

$$J_n^{(0)}(x) = \frac{1}{2\pi} \int_0^{2\pi} \eta_0(\theta)\, e^{-in\theta} e^{ix\sin\theta}\, d\theta$$

is the nonstationary piece estimated in Step 1. Adding the two stationary-phase contributions therefore gives

$$\begin{aligned} J_n(x) &= J_n^{(+)}(x) + J_n^{(-)}(x) + J_n^{(0)}(x) \\ &= J_n^{(+)}(x) + J_n^{(-)}(x) + O\big(x^{-2}\big) \\ &= \frac{1}{\sqrt{2\pi x}} \left(e^{i\chi_n(x)} + e^{-i\chi_n(x)}\right) + O\big(x^{-3/2}\big), \end{aligned}$$

where the second line is Step 1 with $N = 2$ (any $N \geq 2$ would do, since the remainder we are about to write is only $O(x^{-3/2})$). Using

$$e^{i\chi} + e^{-i\chi} = 2\cos\chi,$$

Large x: combining the two saddle contributions II

we obtain

$$J_n(x) = \frac{2}{\sqrt{2\pi x}} \cos \chi_n(x) + O(x^{-3/2})$$
$$= \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}).$$

Thus the leading large-argument expansion is

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}), \qquad x \to +\infty, \tag{6.4.6}$$

for fixed integer n. The equality with an absolute error term is preferable to writing a ratio asymptotic $J_n \sim \sqrt{2/(\pi x)} \cos \chi_n$, because the leading cosine has infinitely many zeros.

For non-integer fixed order ν the same leading cosine law

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(x^{-3/2}\right), \qquad x \to +\infty,$$

holds, with the remainder uniform for ν in any bounded subset of the complex plane. The derivation just given does *not* cover it: the integration by parts of Step 1 discarded its boundary terms because $e^{-in\theta}$ is 2π-periodic, which fails as soon as $\nu \notin \mathbb{Z}$. The proper route is steepest descent applied to the Hankel/Schläfli-type

Large x: combining the two saddle contributions III

contour of the previous section, on which the multivalued factor $t^{-\nu-1}$ is admissible; that computation meets the same two saddles $t = \pm i$ and returns the same phase and amplitude, and it is also the source of the uniformity in ν. We quote it from the standard references on Bessel asymptotics (Watson's treatise; DLMF Chapter 10) and do not repeat the contour derivation here.

One caution, since the temptation is natural: this cannot be obtained by *continuing the integer-order result in* ν. Both $J_\nu(x)$ and $\sqrt{2/(\pi x)}\cos\chi_\nu(x)$ are entire in ν, so their difference is entire in ν for each fixed x; but a size estimate on the discrete set $\nu \in \mathbb{Z}$ says nothing in between. For instance $E(\nu, x) = \sin(\pi\nu)\sqrt{x}$ is entire in ν and vanishes identically at every integer order, yet $E(\frac{1}{2}, x) = \sqrt{x}$ grows without bound. Analytic continuation transports identities, not bounds sampled along a discrete set; the uniform bound has to come from the contour representation itself.

We will use the displayed formula, and the uniformity just stated, when deriving the large-x expansion of Y_ν below.

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Reduction of order: constructing a second solution I

Step 1: reduce order. Divide Bessel's equation by x^2 on an interval with $x > 0$:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{m^2}{x^2}\right) y = 0. \tag{6.5.1}$$

Let J_m be one solution and seek a second solution in the form

$$y_2(x) = v(x)J_m(x).$$

Differentiate:

$$\begin{aligned} y_2' &= v'J_m + vJ_m', \\ y_2'' &= v''J_m + 2v'J_m' + vJ_m''. \end{aligned}$$

Substitute into (6.5.1):

$$\begin{aligned} 0 &= v''J_m + 2v'J_m' + vJ_m'' + \frac{1}{x}(v'J_m + vJ_m') + \left(1 - \frac{m^2}{x^2}\right) vJ_m \\ &= v''J_m + v'\left(2J_m' + \frac{1}{x}J_m\right) \\ &\quad + v\left[J_m'' + \frac{1}{x}J_m' + \left(1 - \frac{m^2}{x^2}\right) J_m\right]. \end{aligned}$$

The bracket multiplying v is zero because J_m solves Bessel's equation. Thus

$$v''J_m + v'\left(2J_m' + \frac{1}{x}J_m\right) = 0.$$

Step 2: solve for the new factor. Set $w = v'$. On an interval where $J_m \neq 0$ and $w \neq 0$,

$$w' J_m + w\left(2J_m' + \frac{1}{x} J_m\right) = 0,$$

$$\frac{w'}{w} = -2\frac{J_m'}{J_m} - \frac{1}{x}.$$

Integrating gives

$$\log|w| = -2\log|J_m| - \log x + C,$$

$$w = v' = \frac{C_1}{xJ_m^2(x)}.$$

Taking $C_1 = 1$ and integrating once more,

$$v(x) = C_2 + \int_{x_*}^{x} \frac{ds}{sJ_m^2(s)},$$

$$y_2(x) = C_2 J_m(x) + J_m(x)\int_{x_*}^{x} \frac{ds}{sJ_m^2(s)}.$$

Hence the general solution on a zero-free interval is

$$y(x) = AJ_m(x) + BJ_m(x)\int_{x_*}^{x} \frac{ds}{sJ_m^2(s)}. \tag{6.5.2}$$

For functions u and v, use the Wronskian convention

$$W[u,v] = uv' - u'v.$$

The constructed second solution then has

$$\begin{aligned} W[J_m, y_2] &= J_m y_2' - J_m' y_2 \\ &= J_m(v' J_m + v J_m') - J_m'(v J_m) \\ &= v' J_m^2 \\ &= \frac{1}{x}. \end{aligned}$$

Step 3: normalize the second solution. The conventional Neumann function is normalized by

$$W[J_m, Y_m] = \frac{2}{\pi x}. \tag{6.5.3}$$

Here Y_m is the Neumann function of integer order m, built on the next two frames as the limit $\lim_{\nu \to m} Y_\nu$ of the noninteger-order function (6.5.4). That this limit again solves (6.5.1), and that it satisfies (6.5.3), are proved there; we quote the two facts now only because they are what fixes the normalization of the second solution. The constructed solution $y_2(x) = J_m(x) \int_{x_*}^{x} ds/(sJ_m^2(s))$ has $W[J_m, y_2] = 1/x$ by the

Reduction of order: constructing a second solution IV

computation above, while (6.5.3) gives $W[J_m, \frac{\pi}{2} Y_m] = \frac{\pi}{2} \cdot \frac{2}{\pi x} = \frac{1}{x}$. Subtracting, $W[J_m,\, y_2 - \frac{\pi}{2} Y_m] = 0$; two solutions of the same second-order linear equation whose Wronskian vanishes are linearly dependent, so $y_2 - \frac{\pi}{2} Y_m = cJ_m$ for some constant c. Therefore the reduction-of-order solution equals $(\pi/2)Y_m$ plus a constant multiple of J_m. The apparent singularities of the integral at zeros of J_m only limit this local formula; the resulting second solution can be continued across such points.

Noninteger order: definition and independence of Y_ν I

Bessel's equation depends on the order only through ν^2. Therefore, if $J_\nu(x)$ is a solution, then $J_{-\nu}(x)$ is also a solution. For fixed real $\nu \notin \mathbb{Z}$, define

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}. \tag{6.5.4}$$

To verify independence, first compute $W[J_\nu, J_{-\nu}]$ from their leading small-x terms. For $\nu \notin \mathbb{Z}$, retaining only the $k = 0$ term of the series (6.1.5) gives $(x/2)^{\pm\nu}/\Gamma(1 \pm \nu)$, so that

$$J_\nu(x) \sim Ax^\nu, \qquad A = \frac{2^{-\nu}}{\Gamma(\nu+1)},$$
$$J_{-\nu}(x) \sim Bx^{-\nu}, \qquad B = \frac{2^{\nu}}{\Gamma(1-\nu)}.$$

Thus

$$\begin{aligned} W[J_\nu, J_{-\nu}] &= J_\nu J_{-\nu}' - J_\nu' J_{-\nu} \\ &\sim (Ax^\nu)(-\nu Bx^{-\nu-1}) - (\nu Ax^{\nu-1})(Bx^{-\nu}) \\ &= -2\nu AB\, x^{-1}. \end{aligned}$$

Noninteger order: definition and independence of Y_ν II

This leading calculation determines the exact Wronskian constant. With the Wronskian convention introduced above, take two solutions of

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0,$$

differentiate the Wronskian:

$$\begin{aligned} W' &= \frac{d}{dx}(y_1 y_2' - y_1' y_2) \\ &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\ &= y_1 y_2'' - y_1'' y_2. \end{aligned}$$

Using

$$y_j'' = -\frac{1}{x}y_j' - \left(1 - \frac{\nu^2}{x^2}\right) y_j, \qquad j = 1, 2,$$

we obtain

$$\begin{aligned} W' &= y_1 \left[-\frac{1}{x} y_2' - \left(1 - \frac{\nu^2}{x^2} \right) y_2 \right] - \left[-\frac{1}{x} y_1' - \left(1 - \frac{\nu^2}{x^2} \right) y_1 \right] y_2 \\ &= -\frac{1}{x} (y_1 y_2' - y_1' y_2) \\ &= -\frac{1}{x} W. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dx} \big[xW(x) \big] &= W(x) + xW'(x) = W(x) - W(x) = 0, \\ xW(x) &= C_\nu, \\ W(x) &= \frac{C_\nu}{x}. \end{aligned}$$

Noninteger order: definition and independence of Y_ν IV

The small-x limit above therefore fixes $C_\nu = -2\nu AB$. Now

$$\begin{aligned} AB &= \frac{1}{\Gamma(\nu+1)\Gamma(1-\nu)} \\ &= \frac{1}{\nu\Gamma(\nu)\Gamma(1-\nu)} \\ &= \frac{\sin(\pi\nu)}{\pi\nu}, \end{aligned}$$

where the last step uses Euler's reflection formula $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin(\pi\nu)$. Hence

$$W[J_\nu, J_{-\nu}] = -\frac{2\sin(\pi\nu)}{\pi x}. \tag{6.5.5}$$

Noninteger order: definition and independence of Y_ν V

Using linearity of the Wronskian in its second argument together with $W[J_\nu, J_\nu] = 0$, the Wronskian of (6.5.4) is therefore

$$\begin{aligned} W[J_\nu, Y_\nu] &= \frac{1}{\sin(\pi\nu)} W\left[J_\nu, J_\nu \cos(\pi\nu) - J_{-\nu}\right] \\ &= \frac{1}{\sin(\pi\nu)} \left[\cos(\pi\nu)\, W[J_\nu, J_\nu] - W[J_\nu, J_{-\nu}]\right] \\ &= -\frac{1}{\sin(\pi\nu)} W[J_\nu, J_{-\nu}] \\ &= \frac{2}{\pi x}, \end{aligned}$$

which establishes independence and the normalization $W[J_\nu, Y_\nu] = 2/(\pi x)$ for every noninteger ν.

The integer case. Equation (6.5.3) was stated at integer order, and the computation just made cannot be run there, since it divides by $\sin(\pi\nu)$. We reach it by letting $\nu \to m$. Fix $x > 0$. Each term of (6.1.5) is entire in ν, because $1/\Gamma(k + \nu + 1)$ is, and the ratio bound on that frame, $|T_{k+1}/T_k| = |x|^2/\big(4(k+1)|k+\nu+1|\big)$, is uniform for ν in a compact set as well as for x in a compact set. Hence $\nu \mapsto J_{\pm\nu}(x)$ and $\nu \mapsto \partial_x J_{\pm\nu}(x)$ are analytic in ν, and so are

$$N(\nu, x) = J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x), \qquad \partial_x N(\nu, x), \qquad S(\nu) = \sin(\pi\nu).$$

Noninteger order: definition and independence of Y_ν VI

At $\nu = m$ the relations $J_{-m} = (-1)^m J_m$ and $\cos(\pi m) = (-1)^m$ give $N(m,x) = 0$ for every x, hence also $\partial_x N(m,x) = 0$, while $S(m) = 0$ with $S'(m) = \pi(-1)^m \neq 0$. Both quotients

$$Y_\nu(x) = \frac{N(\nu,x)}{S(\nu)}, \qquad \partial_x Y_\nu(x) = \frac{\partial_x N(\nu,x)}{S(\nu)}$$

therefore have removable singularities at $\nu = m$: each extends continuously in ν, the value of the first being the l'Hôpital limit (6.5.6) computed on the next frame. Letting $\nu \to m$ in the identity just proved,

$$W[J_m, Y_m] = \lim_{\nu\to m}\Big(J_\nu\,\partial_x Y_\nu - \partial_x J_\nu\ Y_\nu\Big) = \frac{2}{\pi x}, \qquad m \in \mathbb{Z},$$

which is (6.5.3) as used in Step 3. In particular the Wronskian is nonzero, so J_m and Y_m are independent at integer order too.

Integer order: the l'Hôpital limit without ambiguity I

At an integer m, both numerator and denominator of (6.5.4) vanish because

$$J_{-m} = (-1)^m J_m, \qquad \cos(\pi m) = (-1)^m, \qquad \sin(\pi m) = 0.$$

Introduce the order-derivative notation

$$D_\mu J_\mu(x) := \frac{\partial}{\partial \mu} J_\mu(x).$$

For fixed $x > 0$ and a fixed logarithm branch, the series for $J_\nu(x)$ is analytic in ν, so the order derivatives below exist. Let

$$N(\nu) = J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x), \qquad S(\nu) = \sin(\pi\nu).$$

Differentiate the numerator. The first product gives

$$\frac{d}{d\nu}\left[J_\nu \cos(\pi\nu)\right] = D_\nu J_\nu \cos(\pi\nu) - \pi J_\nu \sin(\pi\nu).$$

For the second term, the chain rule gives

$$\begin{aligned} \frac{d}{d\nu} J_{-\nu}(x) &= \left. D_\mu J_\mu(x)\right|_{\mu=-\nu} \frac{d(-\nu)}{d\nu} \\ &= - \left. D_\mu J_\mu(x)\right|_{\mu=-\nu}. \end{aligned}$$

Therefore

$$N'(\nu) = D_\nu J_\nu \cos(\pi\nu) - \pi J_\nu \sin(\pi\nu) + \left. D_\mu J_\mu\right|_{\mu=-\nu}.$$

At $\nu = m$,

$$N'(m) = (-1)^m \left. D_\nu J_\nu(x)\right|_{\nu=m} + \left. D_\nu J_\nu(x)\right|_{\nu=-m},$$
$$S'(m) = \pi \cos(\pi m) = \pi(-1)^m \neq 0.$$

L'Hôpital's rule now gives

$$\begin{aligned} Y_m(x) &= \frac{N'(m)}{S'(m)} \\ &= \frac{(-1)^m D_\nu J_\nu|_{\nu=m} + D_\nu J_\nu|_{\nu=-m}}{\pi(-1)^m} \\ &= \frac{1}{\pi}\left[\left. D_\nu J_\nu(x)\right|_{\nu=m} + (-1)^m \left. D_\nu J_\nu(x)\right|_{\nu=-m}\right]. \end{aligned}$$

Thus

$$Y_m(x) = \frac{1}{\pi}\left[\left.\frac{\partial J_\nu(x)}{\partial \nu}\right|_{\nu=m} + (-1)^m \left.\frac{\partial J_\nu(x)}{\partial \nu}\right|_{\nu=-m}\right]. \tag{6.5.6}$$

This formula explicitly distinguishes an order derivative from the total derivative of the composite function $J_{-\nu}$; omitting that chain-rule sign changes the result.

Integer order: the l'Hôpital limit without ambiguity III

The limit is still a solution. Step 3 of the reduction-of-order frame treated Y_m as a solution of Bessel's equation, so we check that. Write

$$\mathcal{L}_\nu[y] = y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y,$$

so that Bessel's equation of order ν is $\mathcal{L}_\nu[y] = 0$, and $\mathcal{L}_\nu[J_\nu] = 0$ for every ν and every $x > 0$. The check is needed because a single order derivative is *not* a solution: differentiate the identity $\mathcal{L}_\nu[J_\nu] = 0$ with respect to ν, interchanging ∂_ν with the x-derivatives (legitimate because the series and its x-derivatives converge locally uniformly in ν). The explicit ν-dependence of $\mathcal{L}_\nu$ contributes $-2\nu J_\nu/x^2$, so

$$0 = \partial_\nu\big(\mathcal{L}_\nu[J_\nu]\big) = \mathcal{L}_\nu\left[D_\nu J_\nu\right] - \frac{2\nu}{x^2}J_\nu, \qquad \text{that is} \qquad \mathcal{L}_\nu\left[D_\nu J_\nu\right] = \frac{2\nu}{x^2}J_\nu.$$

Evaluate this at $\nu = m$ and at $\nu = -m$, using $\mathcal{L}_{-m} = \mathcal{L}_m$ (the operator depends on the order only through ν^2) and $J_{-m} = (-1)^m J_m$:

$$\begin{aligned}
\mathcal{L}_m\Big[\, D_\nu J_\nu\big|_{\nu=m}\Big] &= \frac{2m}{x^2}J_m, \\
\mathcal{L}_m\Big[\, D_\nu J_\nu\big|_{\nu=-m}\Big] &= -\frac{2m}{x^2}J_{-m} = -(-1)^m\frac{2m}{x^2}J_m.
\end{aligned}$$

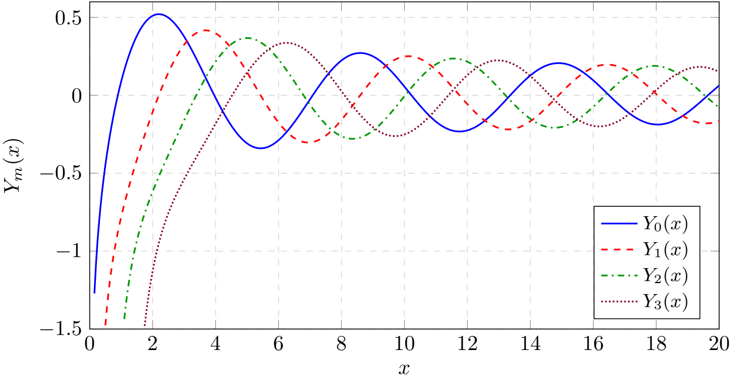
Integer order: the l'Hôpital limit without ambiguity IV

By linearity of $\mathcal{L}_m$, (6.5.6) therefore gives

$$\pi\,\mathcal{L}_m[Y_m] = \frac{2m}{x^2}J_m + (-1)^m\left(-(-1)^m\frac{2m}{x^2}J_m\right) = \frac{2m}{x^2}J_m - \frac{2m}{x^2}J_m = 0.$$

The inhomogeneities cancel exactly, so Y_m solves Bessel's equation of order m, as Step 3 assumed.

Bessel functions of the second kind



The Neumann functions are independent of J_m and are singular at $x = 0$. Away from the origin, J_m and Y_m have the same $x^{-1/2}$ envelope and phases shifted by $\pi/2$.

Small-x behavior of Y_m I

First let $\nu > 0$ be fixed and noninteger. From the first term of the series,

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}[1 + O(x^2)],$$

$$J_{-\nu}(x) = \frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu}[1 + O(x^2)].$$

Their magnitudes have orders x^ν and $x^{-\nu}$, respectively. Since $x^\nu/x^{-\nu} = x^{2\nu} \to 0$, the $J_{-\nu}$ term dominates in (6.5.4). Therefore

$$Y_\nu(x) = -\frac{1}{\Gamma(1-\nu)\sin(\pi\nu)}\left(\frac{x}{2}\right)^{-\nu}[1 + o(1)].$$

Euler's reflection formula gives

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin(\pi\nu)},$$

$$\Gamma(1-\nu)\sin(\pi\nu) = \frac{\pi}{\Gamma(\nu)},$$

$$\frac{1}{\Gamma(1-\nu)\sin(\pi\nu)} = \frac{\Gamma(\nu)}{\pi}.$$

Consequently,

$$Y_\nu(x) = -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^{\nu}[1 + o(1)], \qquad x \to 0^+, \tag{6.5.7}$$

for fixed positive noninteger ν.

We now derive the positive-integer result directly, rather than interchanging the limits $x \to 0$ and $\nu \to m$. Let $m \geq 1$. Differentiating the series termwise gives $\partial_\nu J_\nu = \sum_{k\geq 0} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}(x/2)^{2k+\nu}[L - \psi(k+\nu+1)]$, where $L = \log(x/2)$ and $\psi = \Gamma'/\Gamma$ is the digamma function. At $\nu = m$ the smallest power is the $k = 0$ term $\frac{1}{\Gamma(m+1)}(x/2)^m[L - \psi(m+1)]$; since L grows like $\log x$ and every $k \geq 1$ term carries an extra $(x/2)^{2k}$, the order-derivative formula (6.5.6) has

$$\left.\frac{\partial J_\nu(x)}{\partial \nu}\right|_{\nu=m} = O(x^m|\log x|).$$

The dominant term comes from the $k = 0$ term of the derivative at $\nu = -m$. To evaluate it, write

$$\frac{1}{\Gamma(\nu+1)} = \frac{(\nu+1)(\nu+2)\cdots(\nu+m)}{\Gamma(\nu+m+1)}.$$

At $\nu = -m$ exactly one factor of the numerator, namely $\nu + m$, vanishes. Write the right-hand side as $P(\nu)h(\nu)$ with

$$P(\nu) = (\nu+1)(\nu+2)\cdots(\nu+m), \qquad h(\nu) = \frac{1}{\Gamma(\nu+m+1)},$$

Small-x behavior of Y_m III

where h is differentiable near $\nu = -m$ because Γ is analytic and nonzero near $\Gamma(1) = 1$. The product rule gives $(Ph)' = P'h + Ph'$, and $P(-m) = 0$ kills the second term, so no derivative of Γ is needed: only $P'(-m)h(-m) = P'(-m)$ survives. For the same reason only one summand of $P'(\nu) = \sum_{i=1}^{m} \prod_{j \neq i} (\nu + j)$ survives at $\nu = -m$, the one with $i = m$, since every other summand still contains the factor $\nu + m$. Hence

$$\begin{aligned} \frac{d}{d\nu} \frac{1}{\Gamma(\nu+1)} \bigg|_{\nu=-m} &= \frac{(-m+1)(-m+2)\cdots(-1)}{\Gamma(1)} \\ &= (-1)^{m-1}(m-1)!, \end{aligned}$$

Small-x behavior of Y_m IV

the last line counting the $m-1$ negative factors, whose absolute values are $m-1, m-2, \dots, 1$. For $m=1$, the product in the first line is empty and is interpreted as 1. Let $L = \log(x/2)$. The derivative of the $k=0$ term is

$$\begin{aligned} &\frac{\partial}{\partial \nu}\left[\frac{(x/2)^{\nu}}{\Gamma(\nu+1)}\right]\Bigg|_{\nu=-m} \\ &\quad = \left(\frac{x}{2}\right)^{-m}\left[L\frac{1}{\Gamma(\nu+1)}\bigg|_{\nu=-m} + \frac{d}{d\nu}\frac{1}{\Gamma(\nu+1)}\bigg|_{\nu=-m}\right] \\ &\quad = (-1)^{m-1}(m-1)!\left(\frac{x}{2}\right)^{-m}, \end{aligned}$$

because $1/\Gamma(1-m) = 0$.

The same care is needed for the remaining terms. The ψ-form of the termwise derivative is unusable at $\nu = -m$ when $1 \le k \le m-1$: there $1/\Gamma(k-m+1) = 0$ while $\psi(k-m+1)$ has a pole, an indeterminate $0 \cdot \infty$. Repeat instead the rewrite used for $k=0$. Iterating $\Gamma(z+1) = z\Gamma(z)$ gives, for $0 \le k \le m-1$,

$$\frac{1}{\Gamma(\nu+k+1)} = \frac{(\nu+k+1)(\nu+k+2)\cdots(\nu+m)}{\Gamma(\nu+m+1)},$$

Small-x behavior of Y_m V

again with the single vanishing factor $\nu + m$ at $\nu = -m$, so the product-rule argument above applies verbatim and yields the finite value

$$\frac{d}{d\nu}\frac{1}{\Gamma(\nu+k+1)}\bigg|_{\nu=-m} = (-m+k+1)(-m+k+2)\cdots(-1) = (-1)^{m-k-1}(m-k-1)!.$$

Since $1/\Gamma(k-m+1) = 0$ removes the L contribution, the derivative of the k-th term is

$$\frac{(-1)^k(-1)^{m-k-1}(m-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-m} = O\left(x^{2k-m}\right), \qquad 0 \le k \le m-1,$$

with no logarithm. For $k \ge m$ the gamma factor is regular, $\Gamma(k-m+1) = (k-m)!$, the ψ-form applies as written, and the derivative of the k-th term is

$$\frac{(-1)^k}{k!\,(k-m)!}\left(\frac{x}{2}\right)^{2k-m}\Big[L - \psi(k-m+1)\Big] = O\left(x^{2k-m}|\log x|\right).$$

Every term with $k \ge 1$ thus carries at least two extra powers of x relative to the $k=0$ term $(-1)^{m-1}(m-1)!\,(x/2)^{-m}$, and the resulting series converges for each $x > 0$, so the whole tail $\sum_{k\ge1}$ contributes a relative $O(x^2|\log x|)$. Hence

$$\frac{\partial J_\nu(x)}{\partial\nu}\bigg|_{\nu=-m} = (-1)^{m-1}(m-1)!\left(\frac{x}{2}\right)^{-m}[1 + O(x^2|\log x|)].$$

Small-x behavior of Y_m VI

Substitution into (6.5.6) gives

$$Y_m(x) = \frac{1}{\pi}\left[O(x^m|\log x|) + (-1)^m(-1)^{m-1}(m-1)!\left(\frac{x}{2}\right)^{-m}[1+O(x^2|\log x|)]\right]$$
$$= -\frac{(m-1)!}{\pi}\left(\frac{2}{x}\right)^m[1+o(1)].$$

Since $(m-1)! = \Gamma(m)$,

$$Y_m(x) = -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m[1+o(1)], \qquad m = 1, 2, \ldots, \tag{6.5.8}$$

as $x \to 0^+$ with m fixed.

The case $m = 0$ is logarithmic. Setting $m = 0$ in (6.5.6) gives

$$Y_0(x) = \frac{2}{\pi}\left.\frac{\partial J_\nu(x)}{\partial \nu}\right|_{\nu=0}. \tag{6.5.9}$$

The series and its order derivative converge locally uniformly for ν near zero when $x > 0$ is fixed, so the general-order series may be differentiated term by term. Let

$\psi(z) = \Gamma'(z)/\Gamma(z)$ be the digamma function, let γ be the Euler–Mascheroni constant, and write $L = \log(x/2)$. Then

$$\frac{\partial}{\partial \nu}\left[\frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\right] = \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\left[L - \psi(k+\nu+1)\right].$$

At $\nu = 0$, $\Gamma(k+1) = k!$, so

$$\left.\frac{\partial J_\nu(x)}{\partial \nu}\right|_{\nu=0} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}\left[L - \psi(k+1)\right].$$

Two facts about ψ are needed, and both come from Chapter 3. First, taking logarithms in the recurrence $\Gamma(z+1) = z\Gamma(z)$ and differentiating,

$$\psi(z+1) = \frac{d}{dz}\log\Gamma(z+1) = \frac{d}{dz}\Big[\log z + \log\Gamma(z)\Big] = \frac{1}{z} + \psi(z).$$

Second, the value at $z = 1$ is where the Euler–Mascheroni constant enters. Chapter 3 established Euler's limit representation

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z\, n!}{z(z+1)\cdots(z+n)}$$

Small-x behavior of Y_m VIII

together with the definition $\gamma = \lim_{n\to\infty}\big(H_n - \log n\big)$, where $H_n = \sum_{j=1}^{n} j^{-1}$. Taking logarithms in the limit representation,

$$\log\Gamma(z) = \lim_{n\to\infty}\left[z\log n + \log n! - \sum_{k=0}^{n}\log(z+k)\right],$$

and differentiating in z — the convergence is locally uniform away from the nonpositive integers, so the limit and the derivative may be interchanged —

$$\psi(z) = \lim_{n\to\infty}\left[\log n - \sum_{k=0}^{n}\frac{1}{z+k}\right].$$

At $z=1$ the sum is $\sum_{k=0}^{n}\frac{1}{k+1} = H_{n+1} = H_n + \frac{1}{n+1}$, so

$$\psi(1) = \lim_{n\to\infty}\left[\log n - H_n - \frac{1}{n+1}\right] = -\lim_{n\to\infty}\big(H_n - \log n\big) = -\gamma.$$

Iterating the recurrence from $z=1$ now gives

$$\begin{aligned}\psi(k+1) &= \psi(1) + \sum_{j=1}^{k}\frac{1}{j}\\ &= -\gamma + H_k,\end{aligned}$$

where $H_0 = 0$ and $H_k = \sum_{j=1}^{k} j^{-1}$. Therefore

$$\begin{aligned}\left.\frac{\partial J_\nu(x)}{\partial \nu}\right|_{\nu=0} &= (L+\gamma)\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} \\ &\quad - \sum_{k=1}^{\infty}\frac{(-1)^k H_k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} \\ &= (L+\gamma)J_0(x) - \sum_{k=1}^{\infty}\frac{(-1)^k H_k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}.\end{aligned}$$

As $x \to 0^+$,

$$J_0(x) = 1 + O(x^2), \qquad \sum_{k=1}^{\infty}\frac{(-1)^k H_k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} = O(x^2).$$

Using (6.5.9),

$$\begin{aligned}Y_0(x) &= \frac{2}{\pi}\left[(L+\gamma)(1+O(x^2)) + O(x^2)\right] \\ &= \frac{2}{\pi}(L+\gamma) + O(x^2|\log x|).\end{aligned}$$

Small-x behavior of Y_m X

Thus

$$Y_0(x) = \frac{2}{\pi}\left[\log\left(\frac{x}{2}\right) + \gamma\right] + O(x^2|\log x|), \qquad x \to 0^+. \tag{6.5.10}$$

Large-x behavior of J_ν and Y_ν I

For fixed order ν, define

$$\chi_\nu(x) = x - \frac{\nu\pi}{2} - \frac{\pi}{4}, \qquad A(x) = \sqrt{\frac{2}{\pi x}}.$$

The large-x expansion of J_ν has the leading form

$$J_\nu(x) = A(x) \cos \chi_\nu(x) + O(x^{-3/2}). \tag{6.5.11}$$

This is (6.4.6) for integer order (stationary phase), extended to any fixed real order as noted after that derivation; in particular both ν and $-\nu$ are covered, which is all that is needed below. For $J_{-\nu}$, the phase is

$$\begin{aligned} \chi_{-\nu}(x) &= x + \frac{\nu\pi}{2} - \frac{\pi}{4} \\ &= \chi_\nu(x) + \pi\nu. \end{aligned}$$

Hence

$$J_{-\nu}(x) = A(x) \cos\big(\chi_\nu(x) + \pi\nu\big) + O(x^{-3/2}).$$

Large-x behavior of J_ν and Y_ν II

Recall the noninteger-order definition $Y_\nu(x) = \dfrac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}$ from (6.5.4).

For fixed noninteger ν, substituting $J_\nu = A\cos\chi_\nu$ and $J_{-\nu} = A\cos(\chi_\nu + \pi\nu)$ and factoring $A(x)$ out of the numerator gives

$$Y_\nu(x) = A(x)\frac{\cos(\pi\nu)\cos\chi_\nu - \cos(\chi_\nu + \pi\nu)}{\sin(\pi\nu)} + O(x^{-3/2}).$$

Expand the cosine of a sum:

$$\cos(\chi_\nu + \pi\nu) = \cos\chi_\nu\cos(\pi\nu) - \sin\chi_\nu\sin(\pi\nu).$$

Therefore the numerator simplifies term by term:

$$\begin{aligned} &\cos(\pi\nu)\cos\chi_\nu - \cos(\chi_\nu + \pi\nu) \\ &\quad = \cos(\pi\nu)\cos\chi_\nu - \cos\chi_\nu\cos(\pi\nu) + \sin\chi_\nu\sin(\pi\nu) \\ &\quad = \sin\chi_\nu\sin(\pi\nu). \end{aligned}$$

After division by $\sin(\pi\nu)$,

$$Y_\nu(x) = A(x)\sin\chi_\nu(x) + O(x^{-3/2})$$

for fixed noninteger ν.

For integer order, use the order-derivative formula rather than divide by a vanishing sine. At $\nu = m$ the quotient (6.5.4) is of the form $0/0$: $\sin(\pi\nu) \to 0$, and since $J_{-m} = (-1)^m J_m$ and $\cos(\pi m) = (-1)^m$ the numerator also vanishes, so we cannot substitute directly. Instead use the l'Hôpital result (6.5.6), $Y_m(x) = \frac{1}{\pi}\left[\partial_\nu J_\nu|_{\nu=m} + (-1)^m\, \partial_\nu J_\nu|_{\nu=-m}\right]$, and insert the large-$x$ form of $\partial_\nu J_\nu$. Differentiating an asymptotic estimate with respect to a parameter is a step that has to be earned: a bound that is uniform in ν does not by itself control the ν-derivative. For example $R_0(\nu, x) = x^{-3/2}\sin(\nu x^2)$ obeys $|R_0(\nu, x)| \le x^{-3/2}$ for every real ν, yet $\partial_\nu R_0 = x^{1/2}\cos(\nu x^2)$ grows with x. What licenses the step here is analyticity in the order — the property already used to form the order derivatives in (6.5.6) — together with the fact recorded when (6.5.11) was extended off the integers: the remainder there is uniform for ν in bounded subsets of the complex plane. Write that remainder as

$$R(\nu, x) := J_\nu(x) - A(x)\cos\chi_\nu(x).$$

For fixed $x > 0$ the series for $J_\nu(x)$ is entire in ν, and $A(x)\cos\chi_\nu(x)$ is a cosine of a linear function of ν, hence also entire; so $R(\cdot\,, x)$ is entire in ν, and on the two circles $|\zeta - m| = 1$ and $|\zeta + m| = 1$ the uniform bound supplies constants C_m and x_0 with

$$|R(\zeta, x)| \le C_m\, x^{-3/2} \qquad \text{for } |\zeta - m| = 1 \text{ or } |\zeta + m| = 1, \quad x \ge x_0.$$

Cauchy's integral formula for the first derivative, on the circle $|\zeta - m| = 1$, converts that bound on R into a bound on $\partial_\nu R$:

$$\left.\frac{\partial R}{\partial \nu}\right|_{\nu=m} = \frac{1}{2\pi i}\oint_{|\zeta-m|=1} \frac{R(\zeta, x)}{(\zeta - m)^2}\, d\zeta,$$

and bounding the integrand by its maximum modulus $C_m x^{-3/2}/1^2$ and the contour length by 2π,

$$\left| \frac{\partial R}{\partial \nu}\bigg|_{\nu=m} \right| \le \frac{1}{2\pi} \cdot 2\pi \cdot C_m\, x^{-3/2} = C_m\, x^{-3/2},$$

with the identical estimate on $|\zeta + m| = 1$ bounding $\partial_\nu R|_{\nu=-m}$. The remainder in (6.5.11) may therefore be differentiated in ν at $\nu = \pm m$ without changing its order, which is exactly the term-by-term differentiation used below. Since

$$\frac{\partial \chi_\nu}{\partial \nu} = -\frac{\pi}{2},$$

we have

$$\begin{aligned} \frac{\partial}{\partial \nu}\left[A(x)\cos\chi_\nu(x)\right] &= A(x)\left[-\sin\chi_\nu(x)\right]\left(-\frac{\pi}{2}\right) \\ &= \frac{\pi}{2}A(x)\sin\chi_\nu(x). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial J_\nu(x)}{\partial \nu}\bigg|_{\nu=m} &= \frac{\pi}{2}A(x)\sin\chi_m(x) + O(x^{-3/2}), \\ \frac{\partial J_\nu(x)}{\partial \nu}\bigg|_{\nu=-m} &= \frac{\pi}{2}A(x)\sin\chi_{-m}(x) + O(x^{-3/2}). \end{aligned}$$

Because

$$\chi_{-m}(x) = \chi_m(x) + m\pi,$$
$$\sin\chi_{-m}(x) = (-1)^m \sin\chi_m(x),$$

substitution into (6.5.6) gives

$$\begin{aligned} Y_m(x) &= \frac{1}{\pi}\left[\frac{\pi}{2}A\sin\chi_m + (-1)^m\frac{\pi}{2}A\sin\chi_{-m}\right] + O(x^{-3/2}) \\ &= \frac{1}{\pi}\left[\frac{\pi}{2}A\sin\chi_m + \frac{\pi}{2}A\sin\chi_m\right] + O(x^{-3/2}) \\ &= A(x)\sin\chi_m(x) + O(x^{-3/2}). \end{aligned}$$

Consequently, for fixed real order, including integer order,

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}\sin\chi_\nu(x) + O(x^{-3/2}), \qquad x\to+\infty. \tag{6.5.12}$$

Thus J_m and Y_m have the same leading amplitude and phases separated by $\pi/2$.

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Hankel functions: Bessel functions of the third kind I

Define

$$H_m^{(1)}(x) = J_m(x) + iY_m(x), \tag{6.6.1}$$

$$H_m^{(2)}(x) = J_m(x) - iY_m(x). \tag{6.6.2}$$

Let

$$\chi_m(x) = x - \frac{m\pi}{2} - \frac{\pi}{4}.$$

Using (6.5.11) and (6.5.12),

$$\begin{aligned} H_m^{(1)}(x) &= \sqrt{\frac{2}{\pi x}}\left(\cos\chi_m + i\sin\chi_m\right) + O(x^{-3/2}) \\ &= \sqrt{\frac{2}{\pi x}} e^{i\chi_m(x)} + O(x^{-3/2}), \end{aligned}$$

$$\begin{aligned} H_m^{(2)}(x) &= \sqrt{\frac{2}{\pi x}}\left(\cos\chi_m - i\sin\chi_m\right) + O(x^{-3/2}) \\ &= \sqrt{\frac{2}{\pi x}} e^{-i\chi_m(x)} + O(x^{-3/2}). \end{aligned}$$

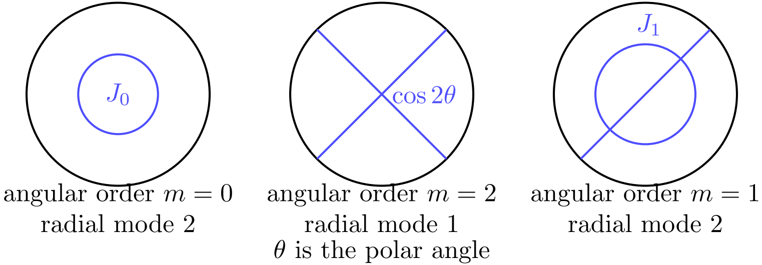
With $k > 0$ and time dependence $e^{-i\omega t}$,

$$H_m^{(1)}(kr)e^{-i\omega t} \propto r^{-1/2} e^{i(kr-\omega t)}$$

Hankel functions: Bessel functions of the third kind II

is outgoing, whereas $H_m^{(2)}$ is incoming. Since Y_m is singular at the origin, both Hankel functions are singular there; they are far-field wave functions, not replacements for regular J_m modes near $r = 0$.

Discrete radial modes on the unit disk



The boundary condition $J_m(\alpha_{m,j}) = 0$ places a radial zero at the clamped rim $r = 1$. Distinct zeros give distinct Sturm–Liouville eigenvalues $\lambda_j = \alpha_{m,j}^2$. On the infinite half-line the discrete Fourier–Bessel sum becomes a Hankel integral transform (not developed here).

Scaling Bessel's equation and obtaining Sturm–Liouville form I

Step 1: rescale Bessel's equation. Let $z = \alpha s$ and define

$$y(s) = J_m(\alpha s) = J_m(z).$$

By the chain rule,

$$y'(s) = \alpha J_m'(z), \qquad y''(s) = \alpha^2 J_m''(z).$$

Bessel's equation in the variable z is

$$z^2 J_m''(z) + z J_m'(z) + (z^2 - m^2) J_m(z) = 0.$$

Substitute $z = \alpha s$ and the chain-rule expressions $J_m' = y'/\alpha$, $J_m'' = y''/\alpha^2$:

$$\begin{aligned} 0 &= (\alpha s)^2 \frac{y''}{\alpha^2} + (\alpha s) \frac{y'}{\alpha} + \left((\alpha s)^2 - m^2\right) y \\ &= s^2 y'' + s y' + (\alpha^2 s^2 - m^2) y. \end{aligned}$$

Thus

$$s^2 y''(s) + s y'(s) + (\alpha^2 s^2 - m^2) y(s) = 0. \tag{6.6.3}$$

Divide by $s > 0$:

$$s y'' + y' + \left(\alpha^2 s - \frac{m^2}{s}\right) y = 0.$$

Scaling Bessel's equation and obtaining Sturm–Liouville form II

Since

$$\frac{d}{ds}(sy') = sy'' + y',$$

we obtain

$$\frac{d}{ds}(sy') - \frac{m^2}{s}y = -\alpha^2 sy.$$

Therefore the Sturm–Liouville form is

$$\frac{d}{ds}\left(s\frac{dy}{ds}\right) - \frac{m^2}{s}y = -\alpha^2 s\,y. \tag{6.6.4}$$

In the convention

$$(py')' + qy = -\lambda wy,$$

we identify

$$p(s) = s, \qquad q(s) = -\frac{m^2}{s}, \qquad \lambda = \alpha^2, \qquad w(s) = s.$$

Hence the natural inner product is (linear in the first slot, matching Chapter 5)

$$\langle u, v\rangle = \int_0^1 u(s)\,\overline{v(s)}\,s\,ds. \tag{6.6.5}$$

Scaling Bessel's equation and obtaining Sturm–Liouville form III

For the real Bessel modes used below, the complex conjugate may be omitted. Write $y_j(s) := J_m(\alpha_{m,j}s)$ for two disk modes y_j, y_k. Self-adjointness of the operator $\mathcal{L}y = (sy')' - \frac{m^2}{s}y$ gives Lagrange's identity

$$y_j\mathcal{L}y_k - y_k\mathcal{L}y_j = \frac{d}{ds}\Big[s(y_jy_k' - y_ky_j')\Big],$$

so integrating over $[0,1]$ leaves the boundary term $\Big[s(y_jy_k' - y_ky_j')\Big]_0^1$ (the same Wronskian term evaluated in the next frame). At the lower endpoint $s = 0$, regularity removes this term. For $m \geq 1$,

$$J_m(\alpha s) = O(s^m), \qquad \frac{d}{ds}J_m(\alpha s) = O(s^{m-1}),$$

so

$$s\Big(y_jy_k' - y_ky_j'\Big) = O(s^{2m}) \to 0.$$

For $m = 0$, $J_0(\alpha s) = 1 + O(s^2)$ and its s-derivative is $O(s)$, so the same boundary term is $O(s^2) \to 0$. Singular Y_m modes do not satisfy the regular endpoint condition.

Fourier–Bessel orthogonality: distinct eigenvalues I

Step 1: compare two eigenvalues. For positive parameters a and b, set

$$y_a(x) = J_m(ax), \qquad y_b(x) = J_m(bx).$$

They satisfy

$$(xy_a')' + \left(a^2x - \frac{m^2}{x}\right) y_a = 0,$$
$$(xy_b')' + \left(b^2x - \frac{m^2}{x}\right) y_b = 0.$$

Multiply the first equation by y_b, the second by y_a, and subtract:

$$y_b(xy_a')' - y_a(xy_b')' + (a^2 - b^2)xy_a y_b = 0.$$

The first two terms form a total derivative. Applying the product rule to every factor,

$$\begin{aligned}
\frac{d}{dx}\left[x(y_b y_a' - y_a y_b')\right] &= y_b y_a' - y_a y_b' \\
&\quad + x\left(y_b' y_a' + y_b y_a'' - y_a' y_b' - y_a y_b''\right) \\
&= y_b y_a' - y_a y_b' + x(y_b y_a'' - y_a y_b'') \\
&= y_b(y_a' + xy_a'') - y_a(y_b' + xy_b'') \\
&= y_b(xy_a')' - y_a(xy_b')'.
\end{aligned}$$

Fourier–Bessel orthogonality: distinct eigenvalues II

Therefore

$$\frac{d}{dx}\left[x(y_b y_a' - y_a y_b')\right] + (a^2 - b^2) x y_a y_b = 0.$$

Integrate from 0 to 1:

$$\left[x(y_b y_a' - y_a y_b')\right]_0^1 + (a^2 - b^2)\int_0^1 x y_a(x) y_b(x)\, dx = 0.$$

The lower boundary is zero by regularity. At $x = 1$,

$$y_a(1) = J_m(a), \quad y_a'(1) = aJ_m'(a), \qquad y_b(1) = J_m(b), \quad y_b'(1) = bJ_m'(b).$$

Hence the boundary evaluation is

$$\begin{aligned}\left[x(y_b y_a' - y_a y_b')\right]_0^1 &= 1 \cdot \left(J_m(b) \cdot aJ_m'(a) - J_m(a) \cdot bJ_m'(b)\right)\\ &= aJ_m(b)J_m'(a) - bJ_m(a)J_m'(b).\end{aligned}$$

Solving for the integral gives the cross-product formula

$$\int_0^1 xJ_m(ax)J_m(bx)\, dx = \frac{bJ_m(a)J_m'(b) - aJ_m(b)J_m'(a)}{a^2 - b^2}, \qquad a \neq b. \tag{6.6.6}$$

(The numerator is the negative of the boundary evaluation above, which cancels the overall minus sign from moving $(a^2 - b^2)\int$ to the other side.)

Fourier–Bessel orthogonality: distinct eigenvalues III

Step 2: impose the zeros at the rim. Now let $a = \alpha_{m,j}$ and $b = \alpha_{m,k}$ be distinct positive zeros of J_m. Then

$$J_m(a) = J_m(b) = 0,$$

so the numerator in (6.6.6) is zero. Since $a^2 - b^2 \neq 0$,

$$\int_0^1 x J_m(\alpha_{m,j} x) J_m(\alpha_{m,k} x)\, dx = 0, \qquad j \neq k. \tag{6.6.7}$$

Fourier–Bessel normalization: the diagonal case I

Existence of the zeros. For fixed integer $m \geq 0$, the large-x cosine law (6.4.6) oscillates with amplitude $\sim x^{-1/2}$ and phase advancing like x, so J_m changes sign infinitely often as $x \to +\infty$. Hence J_m has infinitely many positive zeros $\alpha_{m,1} < \alpha_{m,2} < \cdots$ (alternatively: Sturm comparison of Bessel's equation in normal form against $y'' + y = 0$ on large intervals yields the same conclusion). We index them increasingly and write $\alpha_{m,j}$ for the jth.

Every positive zero $a = \alpha_{m,j}$ is simple, that is, $J_m'(a) \neq 0$. Nothing new is needed to see this: with the Wronskian convention $W[u,v] = uv' - u'v$ fixed earlier, the normalization (6.5.3) reads

$$J_m(x)Y_m'(x) - J_m'(x)Y_m(x) = \frac{2}{\pi x}, \qquad x > 0,$$

and at any point $a > 0$ the second solution Y_m and its derivative are finite, since Y_m is singular only at the origin. Suppose both $J_m(a) = 0$ and $J_m'(a) = 0$. Evaluating the identity at $x = a$ makes each product vanish separately,

$$\underbrace{J_m(a)}_{=0}\, Y_m'(a) - \underbrace{J_m'(a)}_{=0}\, Y_m(a) = 0,$$

while the right-hand side equals $2/(\pi a) \neq 0$ because a is finite and positive. The two sides disagree, so J_m and J_m' cannot vanish at the same positive point:

$$J_m'(\alpha_{m,j}) \neq 0 \qquad \text{for every } j \geq 1.$$

Fourier–Bessel normalization: the diagonal case II

This is exactly the statement that a positive zero of J_m is a simple zero. It is what keeps the normalization computed at the end of this frame strictly positive, so that the coefficient formula of the next frame divides by a nonzero number.

To compute the norm at such a zero, take the limit $b \to a$ in (6.6.6). The integral passes to the limit by continuity of the integrand on the compact interval $0 \le x \le 1$. Write the numerator and denominator as functions of b with a fixed and $J_m(a) = 0$:

$$
\begin{aligned}
N(b) &:= bJ_m(a)J_m'(b) - aJ_m(b)J_m'(a) = -aJ_m(b)J_m'(a), \\
D(b) &:= a^2 - b^2.
\end{aligned}
$$

Both $N(a)$ and $D(a)$ vanish, so the limit is a $0/0$ form. Apply L'Hôpital: differentiate with respect to b,

$$
\begin{aligned}
N'(b) &= -aJ_m'(b)J_m'(a), \\
D'(b) &= -2b.
\end{aligned}
$$

Therefore

$$
\lim_{b\to a} \frac{N(b)}{D(b)} = \lim_{b\to a} \frac{N'(b)}{D'(b)} = \frac{-aJ_m'(a)J_m'(a)}{-2a} = \frac{a\left[J_m'(a)\right]^2}{2a} = \frac{1}{2}\left[J_m'(a)\right]^2,
$$

where we used $a > 0$. Equivalently, with $b = a + \varepsilon$, Taylor expansion gives the same result with every algebra line visible:

$$J_m(b) = J_m(a) + J_m'(a)\varepsilon + O(\varepsilon^2) = J_m'(a)\varepsilon + O(\varepsilon^2),$$
$$a^2 - b^2 = a^2 - (a+\varepsilon)^2 = -2a\varepsilon - \varepsilon^2,$$

so

$$\frac{N(b)}{D(b)} = \frac{-a\left[J_m'(a)\varepsilon + O(\varepsilon^2)\right]J_m'(a)}{-2a\varepsilon - \varepsilon^2} = \frac{-a\left[J_m'(a)\right]^2\varepsilon + O(\varepsilon^2)}{-2a\varepsilon - \varepsilon^2}$$
$$= \frac{-a\left[J_m'(a)\right]^2 + O(\varepsilon)}{-2a - \varepsilon} \xrightarrow{\varepsilon\to 0} \frac{a\left[J_m'(a)\right]^2}{2a} = \frac{1}{2}\left[J_m'(a)\right]^2.$$

Thus

$$\int_0^1 xJ_m^2(ax)\,dx = \frac{1}{2}\left[J_m'(a)\right]^2 \qquad \text{at a positive zero } a \text{ of } J_m. \tag{6.6.8}$$

At a zero of J_m, the derivative recurrence

$$J_m'(a) = \frac{m}{a}J_m(a) - J_{m+1}(a)$$

reduces to

$$J_m'(a) = -J_{m+1}(a).$$

Fourier–Bessel normalization: the diagonal case IV

Squaring removes the sign, and hence

$$\int_0^1 x J_m^2(\alpha_{m,j}x)\,dx = \frac{1}{2} J_{m+1}^2(\alpha_{m,j}). \tag{6.6.9}$$

Combining the diagonal and off-diagonal cases gives

$$\int_0^1 J_m(\alpha_{m,j}x) J_m(\alpha_{m,k}x) x\,dx = \frac{1}{2} J_{m+1}^2(\alpha_{m,j})\delta_{jk}. \tag{6.6.10}$$

Fourier–Bessel coefficients I

Assume a radial function of angular order m has the disk expansion

$$f(x) = \sum_{j=1}^{\infty} a_j J_m(\alpha_{m,j} x), \qquad 0 < x < 1, \tag{6.6.11}$$

where, exactly as for the sine series in Chapter 5, the equality is understood in the weighted space $L^2\big((0,1), x\,dx\big)$ attached to the inner product (6.6.5). Writing the partial sums and the weighted norm

$$S_N(x) = \sum_{j=1}^{N} a_j J_m(\alpha_{m,j} x), \qquad \|g\|^2 = \int_0^1 |g(x)|^2\, x\, dx,$$

the hypothesis is $\|f - S_N\| \to 0$ as $N \to \infty$.

Take the inner product with the kth mode. The test mode has finite norm, because (6.6.9) evaluates it exactly:

$$\big\|J_m(\alpha_{m,k}\,\cdot)\big\|^2 = \int_0^1 J_m^2(\alpha_{m,k} x)\, x\, dx = \frac{1}{2} J_{m+1}^2(\alpha_{m,k}) < \infty.$$

Fourier–Bessel coefficients II

Hence the Cauchy–Schwarz inequality in $L^2\big((0,1), x\,dx\big)$ controls the error committed by replacing f with S_N under the integral sign:

$$\left| \int_0^1 \big(f(x) - S_N(x)\big) J_m(\alpha_{m,k} x)\, x\, dx \right| \le \|f - S_N\| \, \big\| J_m(\alpha_{m,k}\, \cdot) \big\| \xrightarrow[N\to\infty]{} 0.$$

So the integral against f is the limit of the integrals against S_N, and for each fixed N the sum is finite, where only linearity of the integral is used:

$$\begin{aligned}
\int_0^1 f(x) J_m(\alpha_{m,k} x) x\, dx &= \lim_{N\to\infty} \int_0^1 S_N(x) J_m(\alpha_{m,k} x) x\, dx \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} a_j \int_0^1 J_m(\alpha_{m,j} x) J_m(\alpha_{m,k} x) x\, dx \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} a_j \frac{1}{2} J_{m+1}^2(\alpha_{m,j}) \delta_{jk} = \frac{1}{2} a_k J_{m+1}^2(\alpha_{m,k}),
\end{aligned}$$

Fourier–Bessel coefficients III

where (6.6.10) was used in the third line, and the last equality holds because as soon as $N \geq k$ the Kronecker delta leaves the single surviving term $j = k$, so the sequence of partial sums is eventually constant. Solving for a_k gives

$$a_k = \frac{2}{J_{m+1}^2(\alpha_{m,k})} \int_0^1 f(r) J_m(\alpha_{m,k} r) r \, dr. \tag{6.6.12}$$

This is the discrete Fourier–Bessel coefficient formula on the unit disk with weight r. Completeness of $\{J_m(\alpha_{m,j} x)\}_{j \geq 1}$ in the weighted space $L^2\big((0,1), x\,dx\big)$ is the standard Sturm–Liouville spectral theorem for the singular SL problem (6.6.4) with the regular endpoint condition at $s = 0$ and $y(1) = 0$; we cite it as background and do not re-prove the general SL expansion theorem here.

Table of contents

Formula sheet: J_m I

For integer $m \geq 0$,

$$x^2 J_m''(x) + x J_m'(x) + (x^2 - m^2) J_m(x) = 0,$$

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m},$$

$$J_{-m}(x) = (-1)^m J_m(x).$$

The generating function, valid for $t \neq 0$, is

$$e^{\frac{x}{2}(t - t^{-1})} = \sum_{m=-\infty}^{\infty} J_m(x) t^m.$$

The recurrences are

$$\begin{aligned} J_{m-1}(x) + J_{m+1}(x) &= \frac{2m}{x} J_m(x), \qquad x \neq 0, \\ 2J_m'(x) &= J_{m-1}(x) - J_{m+1}(x), \\ J_m'(x) &= J_{m-1}(x) - \frac{m}{x} J_m(x), \qquad x \neq 0, \\ J_m'(x) &= \frac{m}{x} J_m(x) - J_{m+1}(x), \qquad x \neq 0. \end{aligned}$$

Formula sheet: J_m II

The preferred integral representations are

$$J_m(x) = \frac{1}{\pi} \int_0^{\pi} \cos(m\theta - x \sin\theta)\, d\theta, \qquad m \geq 0,$$

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x \sin\theta - n\theta)}\, d\theta, \qquad n \in \mathbb{Z},$$

with the optional contour form

$$J_n(x) = \frac{1}{2\pi i} \oint_C e^{\frac{x}{2}(t - t^{-1})} t^{-n-1}\, dt, \qquad n \in \mathbb{Z}.$$

For fixed m,

$$J_m(x) = \frac{1}{m!} \left(\frac{x}{2}\right)^m \left[1 - \frac{x^2}{4(m+1)} + O(x^4)\right], \qquad x \to 0,$$

$$J_m(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}), \qquad x \to +\infty.$$

Formula sheet: Y_m, Hankel, Fourier–Bessel I

For noninteger ν,

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)},$$

and for integer m,

$$Y_m(x) = \frac{1}{\pi}\left[\left.\frac{\partial J_\nu(x)}{\partial\nu}\right|_{\nu=m} + (-1)^m \left.\frac{\partial J_\nu(x)}{\partial\nu}\right|_{\nu=-m} \right].$$

The normalization is

$$W[J_m, Y_m] = \frac{2}{\pi x}.$$

For fixed integer $m \geq 1$,

$$Y_m(x) = -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m [1 + o(1)], \qquad x \to 0^+,$$

whereas

$$Y_0(x) = \frac{2}{\pi}\left[\log\left(\frac{x}{2}\right) + \gamma\right] + O(x^2|\log x|).$$

For fixed m and $x \to +\infty$,

$$Y_m(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}).$$

Formula sheet: Y_m, Hankel, Fourier–Bessel II

The Hankel functions are

$$H_m^{(1)} = J_m + iY_m, \qquad H_m^{(1)}(x) = \sqrt{\frac{2}{\pi x}}e^{i\chi_m(x)} + O(x^{-3/2}),$$

$$H_m^{(2)} = J_m - iY_m, \qquad H_m^{(2)}(x) = \sqrt{\frac{2}{\pi x}}e^{-i\chi_m(x)} + O(x^{-3/2}),$$

with $\chi_m(x) = x - m\pi/2 - \pi/4$.

For the positive zeros $\alpha_{m,j}$ of J_m,

$$\int_0^1 J_m(\alpha_{m,j}x)J_m(\alpha_{m,k}x)x\,dx = \frac{1}{2}J_{m+1}^2(\alpha_{m,j})\delta_{jk},$$

If a radial function of angular order m is expanded on the disk as

$$f(r) = \sum_{j=1}^{\infty} a_j J_m(\alpha_{m,j}r), \qquad 0 < r < 1,$$

then multiplying by $J_m(\alpha_{m,k}r)\,r$, integrating over $(0,1)$ and using the orthogonality relation above (only the term $j = k$ survives) gives the coefficients

$$a_j = \frac{2}{J_{m+1}^2(\alpha_{m,j})}\int_0^1 f(r)J_m(\alpha_{m,j}r)r\,dr,$$

as derived in (6.6.11)–(6.6.12).