Chapter 4: Method of Frobenius

Special Functions of Mathematical Physics

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Outline

Where we are going

Separation of variables turns many partial differential equations (PDEs) into linear, second-order, homogeneous ordinary differential equations. Two recurring examples are

- the **guitar string**, which gives the harmonic-oscillator equation
$$\frac{d^2y}{dx^2} + k^2 y = 0,$$
- the **drum**, which gives Bessel's equation
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - m^2)y = 0.$$

Here k is a wave number, and m is the nonnegative integer angular order of a drum mode. On an interval where the coefficient of y'' is nonzero, a linear homogeneous second-order equation can be written as

$$y'' + P(x)y' + Q(x)y = 0.$$

Here *linear* means that y, y', y'' occur only to the first power and are not multiplied together; *homogeneous* means that the right-hand side is zero. The method of Frobenius is a local series method for this class of equations. It succeeds at ordinary points and regular singular points; later we state the precise conditions.

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Guitar string: the harmonic oscillator by Taylor series I

Step 1: insert a power series. If the physical string of length L obeys $\dfrac{d^2y}{ds^2} + k^2 y = 0$ on $0 \le s \le L$, substitute $x = ks$; by the chain rule $\dfrac{d^2y}{ds^2} = k^2 \dfrac{d^2y}{dx^2}$, so the equation becomes $y'' + y = 0$ and the physical endpoints $s = 0, L$ map to $x = 0$ and $x = kL$. The fundamental mode has wave number $k = \pi/L$, which makes the rescaled interval exactly $0 \le x \le \pi$. We therefore solve

$$y'' + y = 0, \qquad y(0) = 0, \quad y(\pi) = 0. \tag{4.1.1}$$

Assume that y is analytic at $x = 0$ and write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Term-by-term differentiation gives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots ,$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots .$$

Guitar string: the harmonic oscillator by Taylor series II

Substitution into (4.1.1) yields

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 2: align the powers. The powers do not yet match. Reindexing aligns every sum on the same power of x. In the first sum set

$$j = n-2, \qquad n = j+2.$$

When $n=2$, $j=0$, and as $n\to\infty$, $j\to\infty$. Therefore

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n,\end{aligned}$$

where the last line only renames the dummy index j as n. Hence

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0.$$

Guitar string: the harmonic oscillator by Taylor series III

Step 3: read off the recurrence. A power series is identically zero only when each coefficient is zero. Thus, for every $n \geq 0$,

$$(n+2)(n+1)a_{n+2} + a_n = 0,$$

and solving for the next coefficient gives

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, \qquad n = 0, 1, 2, \ldots. \tag{4.1.2}$$

The recurrence advances the index by two. Consequently, a_0 generates all even coefficients, while a_1 generates all odd coefficients.

Even coefficients. For the even chain, set $n = 0, 2, 4, \ldots$ in (4.1.2):

$$a_2 = -\frac{a_0}{1\cdot 2} = -\frac{a_0}{2!},$$
$$a_4 = -\frac{a_2}{3\cdot 4} = \frac{a_0}{(1\cdot 2)(3\cdot 4)} = \frac{a_0}{4!},$$
$$a_6 = -\frac{a_4}{5\cdot 6} = -\frac{a_0}{(1\cdot 2)(3\cdot 4)(5\cdot 6)} = -\frac{a_0}{6!}.$$

After j steps,

$$\begin{aligned} a_{2j} &= \frac{(-1)^j a_0}{\displaystyle\prod_{\ell=1}^{j}(2\ell-1)(2\ell)} \\ &= \frac{(-1)^j a_0}{1\cdot 2\cdot 3\cdots(2j)} \\ &= (-1)^j \frac{a_0}{(2j)!}. \end{aligned}$$

Thus the even part of the series is

$$\sum_{j=0}^{\infty} a_{2j}x^{2j} = a_0 \sum_{j=0}^{\infty}(-1)^j \frac{x^{2j}}{(2j)!} = a_0 \cos x.$$

Resolving the odd recurrence into sin x I

Odd coefficients. For the odd chain, set $n = 1, 3, 5, \ldots$ in (4.1.2):

$$a_3 = -\frac{a_1}{2 \cdot 3} = -\frac{a_1}{3!},$$
$$a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{(2 \cdot 3)(4 \cdot 5)} = \frac{a_1}{5!},$$
$$a_7 = -\frac{a_5}{6 \cdot 7} = -\frac{a_1}{7!}.$$

After j steps,

$$a_{2j+1} = \frac{(-1)^j a_1}{\prod_{\ell=1}^{j} (2\ell)(2\ell+1)} = \frac{(-1)^j a_1}{2 \cdot 3 \cdots (2j+1)} = (-1)^j \frac{a_1}{(2j+1)!}.$$

The omitted factor 1 does not change the product. Therefore

$$\sum_{j=0}^{\infty} a_{2j+1} x^{2j+1} = a_1 \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = a_1 \sin x.$$

Resolving the odd recurrence into $\sin x$ II

Combining the even and odd parts gives

$$y(x) = a_0 \cos x + a_1 \sin x.$$

Applying the endpoint conditions

The general solution is

$$y(x) = a_0 \cos x + a_1 \sin x.$$

Apply the condition at $x = 0$:

$$\begin{aligned} 0 = y(0) &= a_0 \cos 0 + a_1 \sin 0 \\ &= a_0. \end{aligned}$$

Hence $a_0 = 0$. The second endpoint condition then becomes

$$\begin{aligned} y(\pi) &= a_1 \sin \pi \\ &= 0, \end{aligned}$$

which holds for every a_1. Therefore the nonzero solutions satisfying both endpoint conditions are

$$y(x) = A \sin x, \qquad A \neq 0.$$

The amplitude A is fixed only by an additional normalization or amplitude condition.

Drum: Bessel's equation needs a generalized series I

Step 1: choose the Frobenius form. Bessel's equation left the drum carrying the wave number k,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(k^2x^2-m^2)y=0,$$

and k can be scaled away just as it was for the string. Put $t=kx$, so $x=t/k$, and let $\widetilde{y}(t)=y(x)$ be the same displacement read off against t. By the chain rule each x-derivative brings down one factor k,

$$\frac{dy}{dx}=k\,\frac{d\widetilde{y}}{dt},\qquad \frac{d^2y}{dx^2}=k^2\,\frac{d^2\widetilde{y}}{dt^2},$$

and therefore

$$x^2\frac{d^2y}{dx^2}=\frac{t^2}{k^2}\cdot k^2\frac{d^2\widetilde{y}}{dt^2}=t^2\frac{d^2\widetilde{y}}{dt^2},\qquad x\frac{dy}{dx}=\frac{t}{k}\cdot k\frac{d\widetilde{y}}{dt}=t\frac{d\widetilde{y}}{dt},\qquad k^2x^2=t^2.$$

Every power of k cancels termwise: the combinations $x^2\,d^2/dx^2$ and $x\,d/dx$ count derivatives and powers of the variable equally, so they do not feel a rescaling of that variable. What is left is

$$t^2\frac{d^2\widetilde{y}}{dt^2}+t\frac{d\widetilde{y}}{dt}+(t^2-m^2)\widetilde{y}=0.$$

Renaming t back to x and $\widetilde{y}$ back to y, Bessel's equation in its normalized form is

$$x^2y''+xy'+(x^2-m^2)y=0. \tag{4.1.3}$$

Drum: Bessel's equation needs a generalized series II

The rescaling is worth remembering: a drum of physical radius R has its rim at $x = R$, hence at $t = kR$, which is where the factor k will reappear in the eigenvalue condition $J_m(kR) = 0$. For the drum, the angular order is $m = 0, 1, 2, \ldots$. Because $x = 0$ is a singular point of the normalized equation, a Taylor series need not capture every solution. On $x > 0$, use the Frobenius ansatz

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}, \qquad a_0 \neq 0. \tag{4.1.4}$$

The first two derivatives are

$$y'(x) = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1},$$
$$y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}.$$

No lower-limit terms should be deleted merely because r is unknown: a coefficient such as $r(r-1)$ may or may not vanish after the indicial equation is solved.

Drum: Bessel's equation needs a generalized series III

Step 2: substitute and align powers. Substitution into (4.1.3) gives

$$0 = x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} + x \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1}$$
$$+ (x^2 - m^2) \sum_{k=0}^{\infty} a_k x^{k+r}.$$

Distributing the powers of x yields

$$0 = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r} + \sum_{k=0}^{\infty} (k+r) a_k x^{k+r}$$
$$+ \sum_{k=0}^{\infty} a_k x^{k+r+2} - m^2 \sum_{k=0}^{\infty} a_k x^{k+r}.$$

Only the third sum has the shifted power x^{k+r+2}. Set

$$j = k+2, \qquad k = j-2.$$

Drum: Bessel's equation needs a generalized series IV

Then

$$\sum_{k=0}^{\infty} a_k x^{k+r+2} = \sum_{j=2}^{\infty} a_{j-2} x^{j+r}$$
$$= \sum_{k=2}^{\infty} a_{k-2} x^{k+r}.$$

Reindexing aligns every sum on the same power of x. Introduce the bookkeeping convention

$$a_{-2} = a_{-1} = 0.$$

It allows the shifted sum to begin at $k = 0$ without adding nonzero terms:

$$\sum_{k=2}^{\infty} a_{k-2} x^{k+r} = \sum_{k=0}^{\infty} a_{k-2} x^{k+r}.$$

Step 3: read off the indicial equation. Therefore

$$\sum_{k=0}^{\infty} \Big[(k+r)(k+r-1)a_k + (k+r)a_k - m^2 a_k + a_{k-2}\Big] x^{k+r} = 0.$$

Since

$$(k+r)(k+r-1) + (k+r) = (k+r)\Big[(k+r-1)+1\Big] = (k+r)^2,$$

Drum: Bessel's equation needs a generalized series V

the coefficient equation is

$$\left[(k+r)^2 - m^2\right] a_k + a_{k-2} = 0, \qquad k = 0, 1, 2, \ldots. \tag{4.1.5}$$

At $k = 0$, $a_{-2} = 0$, so

$$(r^2 - m^2)a_0 = 0.$$

Because $a_0 \neq 0$, the indicial equation is

$$r^2 - m^2 = 0 \quad \Longleftrightarrow \quad (r-m)(r+m) = 0, \tag{4.1.6}$$

and hence

$$r = m \qquad \text{or} \qquad r = -m.$$

The root $r = m \geq 0$ gives the solution that is finite at the origin.

Step 1: choose the regular root. Choose the regular root $r = m$. Then

$$\begin{aligned}(k+r)^2 - m^2 &= (k+m)^2 - m^2 \\ &= (k+m-m)(k+m+m) \\ &= k(k+2m).\end{aligned}$$

Equation (4.1.5) becomes

$$k(k+2m)a_k + a_{k-2} = 0, \qquad k = 0, 1, 2, \dots, \tag{4.1.7}$$

or, for $k \geq 2$,

$$a_k = -\frac{a_{k-2}}{k(k+2m)}.$$

At $k = 1$,

$$(1+2m)a_1 + a_{-1} = 0.$$

Since $a_{-1} = 0$ and $1 + 2m \neq 0$ for $m \geq 0$,

$$a_1 = 0.$$

Step 2: eliminate the odd chain. The recurrence preserves parity. Hence a_3 depends on a_1, a_5 depends on a_3, and so forth; induction gives

$$a_{2j+1} = 0, \qquad j = 0, 1, 2, \dots.$$

The Bessel recurrence and $J_m(x)$ II

Step 3: solve the even chain. For the even coefficients, apply $a_k = -a_{k-2}/[k(k+2m)]$ three times:

$$\begin{aligned} a_2 &= -\frac{a_0}{2(2+2m)} = -\frac{a_0}{4(m+1)}, \\ a_4 &= -\frac{a_2}{4(4+2m)} = \frac{a_0}{4(m+1)\cdot 4\cdot 2(m+2)} = \frac{a_0}{4^2\, 2!\,(m+1)(m+2)}, \\ a_6 &= -\frac{a_4}{6(6+2m)} = -\frac{a_0}{4^2\, 2!\,(m+1)(m+2)\cdot 6\cdot 2(m+3)} \\ &= -\frac{a_0}{4^3\, 3!\,(m+1)(m+2)(m+3)}. \end{aligned}$$

Here the factors $4+2m = 2(m+2)$ and $6+2m = 2(m+3)$ supply the extra 2 that builds $2! = 2$ and $3! = 6$ in the denominators. The pattern is visible: each even step multiplies by a factor $-1/[4\ell(\ell+m)]$ and appends one more rising factor $(m+\ell)$. At the ℓth even step, $k = 2\ell$, so the new denominator is

$$(2\ell)(2\ell+2m) = 4\ell(\ell+m).$$

After j steps,

$$\begin{aligned}
a_{2j} &= \frac{(-1)^j a_0}{\displaystyle\prod_{\ell=1}^{j}(2\ell)(2\ell+2m)} \\
&= \frac{(-1)^j a_0}{\displaystyle\prod_{\ell=1}^{j} 4\ell(\ell+m)} \\
&= \frac{(-1)^j a_0}{4^j \left(\prod_{\ell=1}^{j}\ell\right)\left(\prod_{\ell=1}^{j}(m+\ell)\right)}.
\end{aligned}$$

Now

$$\prod_{\ell=1}^{j}\ell = j!, \qquad \prod_{\ell=1}^{j}(m+\ell) = (m+1)(m+2)\cdots(m+j) = \frac{(m+j)!}{m!}.$$

(The $j = 1, 2, 3$ specializations recover a_2, a_4, a_6 above.) Therefore

$$a_{2j} = (-1)^j \frac{m!}{2^{2j} j!(m+j)!}\, a_0, \qquad a_{2j+1} = 0. \tag{4.1.8}$$

The Bessel recurrence and $J_m(x)$ IV

Insert these coefficients into $y = \sum_{k=0}^{\infty} a_k x^{k+m}$:

$$y(x) = \sum_{j=0}^{\infty} a_{2j} x^{2j+m}$$
$$= a_0 m! \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j} j!(m+j)!} x^{2j+m}.$$

Because

$$\frac{x^{2j+m}}{2^{2j}} = 2^m \frac{x^{2j+m}}{2^{2j+m}} = 2^m \left(\frac{x}{2}\right)^{2j+m},$$

we obtain

$$y(x) = a_0\, 2^m m! \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m}.$$

The standard Bessel function of the first kind is defined by

$$J_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+j)!} \left(\frac{x}{2}\right)^{2j+m}. \tag{4.1.9}$$

Thus

$$y(x) = a_0\, 2^m m!\, J_m(x).$$

The Bessel recurrence and $J_m(x)$ V

The conventional normalization chooses

$$a_0 = \frac{1}{2^m m!},$$

so that $y = J_m$ and the leading term is

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m + O(x^{m+2}) \qquad (x \to 0).$$

What the other Bessel root is trying to say I

Write the order as ν and consider

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

The indicial roots are $r = \nu$ and $r = -\nu$. For $\nu > 0$ with $\nu \notin \mathbb{Z}$, both roots lead to standard Frobenius solutions on $x > 0$. The reason is structural: the recurrence links a_k only to a_{k-2}, so the potential obstruction, where the coefficient of a_k vanishes at $k = 2\nu$, can spoil the even chain carrying $a_0 \neq 0$ only when 2ν is a positive even integer, i.e. only when $\nu \in \mathbb{Z}$. The delicate half-integer case, where 2ν is an odd integer, is examined at the end of this frame.

For $r = \nu$, equation (4.1.5) becomes

$$k(k + 2\nu)a_k + a_{k-2} = 0.$$

The same even-chain product as in the integer case applies verbatim, with the factorials rewritten through the Gamma function. From $\Gamma(z + 1) = z\,\Gamma(z)$ applied repeatedly,

$$\prod_{\ell=1}^{j}(\nu + \ell) = (\nu + 1)(\nu + 2)\cdots(\nu + j) = \frac{\Gamma(\nu + j + 1)}{\Gamma(\nu + 1)},$$

What the other Bessel root is trying to say II

the analytic continuation of $(m+j)!/m!$. Hence the even coefficients satisfy

$$a_{2j} = (-1)^j \frac{\Gamma(\nu+1)}{2^{2j} j!\,\Gamma(j+\nu+1)} a_0.$$

Choosing

$$a_0 = \frac{2^{-\nu}}{\Gamma(\nu+1)}$$

gives

$$J_\nu(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j+\nu}.$$

For the other root, $r=-\nu$,

$$\begin{aligned}(k-\nu)^2-\nu^2 &= (k-\nu-\nu)(k-\nu+\nu)\\ &= k(k-2\nu),\end{aligned}$$

so

$$k(k-2\nu)a_k + a_{k-2} = 0. \tag{4.1.10}$$

The odd chain. Put $k=1$ in (4.1.10) and use the convention $a_{-1}=0$:

$$1\cdot(1-2\nu)a_1 + a_{-1} = (1-2\nu)a_1 = 0.$$

What the other Bessel root is trying to say III

For odd k the factor $k(k-2\nu)$ vanishes only at $k=2\nu$, that is, only when 2ν is an odd positive integer. Suppose first that it is not. Then $1-2\nu \neq 0$ forces $a_1=0$, and if $a_{k-2}=0$ for some odd k, then $a_k = -a_{k-2}/[k(k-2\nu)] = 0$ as well. By induction

$$a_{2j+1} = 0, \qquad j = 0, 1, 2, \ldots,$$

so the $r=-\nu$ solution carries even powers only, and the even chain below is the whole story.

If instead 2ν *is* an odd integer, the coefficients $a_1, a_3, \ldots, a_{2\nu-2}$ still vanish for the same reason, and the $k=2\nu$ equation reads $0 \cdot a_{2\nu} + a_{2\nu-2} = 0$, i.e. $0=0$: it frees an odd coefficient instead of forcing one. The extra series it generates begins at the power $x^{2\nu} \cdot x^{-\nu} = x^{\nu}$, and substituting $k = k' + 2\nu$ turns $k(k-2\nu)$ into $k'(k'+2\nu)$, the recurrence of the $r=+\nu$ root: the extra series is simply a multiple of the $r=+\nu$ solution. Setting that free constant to zero selects $J_{-\nu}$. This is the half-order case examined at the end of the frame. For the even chain,

$$a_{2j} = \frac{(-1)^j a_0}{\displaystyle\prod_{\ell=1}^{j} (2\ell)(2\ell-2\nu)} = \frac{(-1)^j a_0}{4^j j! \displaystyle\prod_{\ell=1}^{j} (\ell-\nu)}.$$

What the other Bessel root is trying to say IV

By $\Gamma(z+1) = z\,\Gamma(z)$,

$$\prod_{\ell=1}^{j}(\ell-\nu) = (1-\nu)(2-\nu)\cdots(j-\nu) = \frac{\Gamma(j-\nu+1)}{\Gamma(1-\nu)},$$

so

$$a_{2j} = (-1)^j \frac{\Gamma(1-\nu)}{2^{2j} j!\,\Gamma(j-\nu+1)}\,a_0.$$

The choice

What the other Bessel root is trying to say V

$$a_0 = \frac{2^\nu}{\Gamma(1-\nu)}$$

then gives

$$J_{-\nu}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j-\nu}.$$

For $\nu > 0$ and $\nu \notin \mathbb{Z}$, the leading terms are

$$J_\nu(x) \sim \frac{x^\nu}{2^\nu \Gamma(\nu+1)}, \qquad J_{-\nu}(x) \sim \frac{2^\nu x^{-\nu}}{\Gamma(1-\nu)} \qquad (x \to 0^+).$$

One tends like x^ν and the other like $x^{-\nu}$, so they cannot be constant multiples of one another. Thus they are linearly independent.

What the other Bessel root is trying to say VI

For a positive integer m, the negative root $r = -m$ is obstructed. Before the obstruction, the even coefficients are

$$a_{2j} = \frac{(-1)^j a_0}{4^j j! \prod_{\ell=1}^{j} (\ell - m)}, \qquad 0 \le j \le m-1.$$

At $j = m-1$,

$$\prod_{\ell=1}^{m-1} (\ell - m) = (-1)^{m-1}(m-1)!,$$

$$a_{2m-2} = \frac{a_0}{4^{m-1}[(m-1)!]^2} \neq 0.$$

But the $k = 2m$ equation in (4.1.10) is

$$(2m)(2m-2m)a_{2m} + a_{2m-2} = 0,$$

that is,

$$0 \cdot a_{2m} + a_{2m-2} = 0,$$

which contradicts $a_{2m-2} \neq 0$. This is a compatibility failure, not merely a formal division by zero. For $m = 0$, the two indicial roots coincide and the second solution

What the other Bessel root is trying to say VII

contains a logarithm. In both integer cases, the missing independent solution is the Bessel function of the second kind Y_m.

The half-order case illustrates why a noninteger order remains consistent even when the root difference happens to be an integer. Take $\nu = \frac{1}{2}$ and $r = -\frac{1}{2}$. Then

$$k(k-1)a_k + a_{k-2} = 0 \qquad \big(k \geq 0;\ a_{-2} = a_{-1} := 0\big),$$

which is the range inherited from (4.1.5). It holds at every index: at $k = 0$ it reads $0 \cdot (-1)a_0 + a_{-2} = 0$, i.e. $0 = 0$, which is the statement that $r = -\frac{1}{2}$ already solves the indicial equation and leaves a_0 unconstrained. The $k = 1$ equation is

$$0 \cdot a_1 + a_{-1} = 0,$$

which is the identity $0 = 0$ because $a_{-1} = 0$. Hence a_1 is free. Since $k(k-1) \neq 0$ for every $k \geq 2$, the recurrence rearranges to

$$a_k = -\frac{a_{k-2}}{k(k-1)}, \qquad k \geq 2,$$

What the other Bessel root is trying to say VIII

and this one form drives both chains below: the even chain from a_0 and the odd chain from the free a_1. The first few even coefficients are therefore

$$a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{a_0}{2!},$$
$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{2 \cdot 1 \cdot 4 \cdot 3} = \frac{a_0}{4!},$$
$$a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6!}.$$

The pattern is

$$a_{2j} = (-1)^j \frac{a_0}{(2j)!}, \qquad j = 0, 1, 2, \ldots.$$

An identical product for the odd chain (starting from free a_1) gives

$$a_{2j+1} = (-1)^j \frac{a_1}{(2j+1)!}.$$

What the other Bessel root is trying to say IX

Hence the full series is

$$\begin{aligned} y(x) &= x^{-1/2} \sum_{k=0}^{\infty} a_k x^k \\ &= x^{-1/2} \left(a_0 \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} + a_1 \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j+1} \right) \\ &= x^{-1/2} \big(a_0 \cos x + a_1 \sin x \big). \end{aligned}$$

The two constants are not free choices: they are the normalizations already fixed above, specialized to $\nu = \frac{1}{2}$ with $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and, from $\Gamma(z+1) = z\Gamma(z)$, $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$. The coefficient a_0 multiplies $x^{-1/2}$, the leading power of the $r = -\frac{1}{2}$ branch, whose normalization was $a_0 = 2^{\nu}/\Gamma(1-\nu)$:

$$a_0 = \frac{2^{1/2}}{\Gamma\left(\frac{1}{2}\right)} = \frac{\sqrt{2}}{\sqrt{\pi}} = \sqrt{\frac{2}{\pi}}.$$

What the other Bessel root is trying to say X

The coefficient a_1 multiplies $x^{-1/2} \cdot x = x^{1/2}$, the leading power of the $r = +\frac{1}{2}$ branch, so it is fixed by the other normalization, $a_0 = 2^{-\nu}/\Gamma(\nu+1)$:

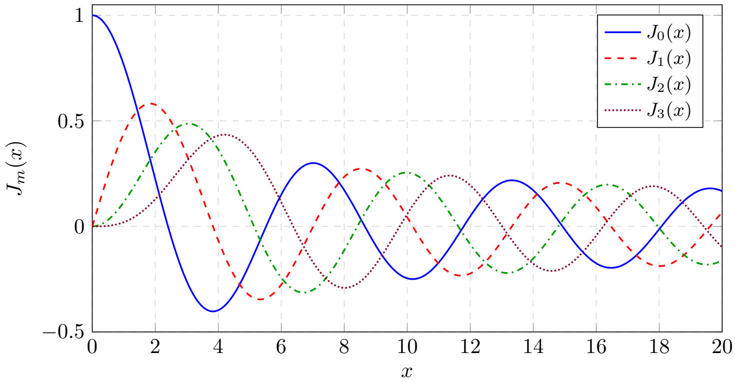
$$a_1 = \frac{2^{-1/2}}{\Gamma\left(\frac{3}{2}\right)} = \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{\pi}} = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}.$$

The two values happen to agree, and $x^{-1/2}\sqrt{2/\pi} = \sqrt{2/(\pi x)}$. Taking $(a_0, a_1) = (\sqrt{2/\pi}, 0)$ and $(a_0, a_1) = (0, \sqrt{2/\pi})$, respectively, gives

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \qquad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

In the second choice $a_0 = 0$, so the first nonzero term is $a_1 x^{1/2}$. It is the $r = 1/2$ solution reindexed inside the $r = -1/2$ recurrence, which is *resonant* in the sense that its coefficient $k(k-2\nu) = k(k-1)$ vanishes at $k = 2\nu = 1$—exactly the index where the $k = 1$ equation became the identity $0 = 0$ and left a_1 free.

The Bessel functions of the first kind $J_0, \ldots, J_3$



Near the origin, $J_0(0) = 1$, while $J_m(0) = 0$ for every integer $m \geq 1$. The next frame states the corresponding local and large-x formulas explicitly.

Bessel asymptotics and drum eigenvalues I

Near the origin the defining series (4.1.9) already contains the answer. Pull the $j = 0$ term out in front:

$$J_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(m+j)!} \left(\frac{x}{2}\right)^{2j+m} = \frac{1}{m!} \left(\frac{x}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j\, m!}{j!\,(m+j)!} \left(\frac{x}{2}\right)^{2j}.$$

In the remaining sum the $j = 0$ term is 1, and the $j = 1$ term is

$$-\frac{m!}{1!\,(m+1)!} \left(\frac{x}{2}\right)^2 = -\frac{1}{m+1} \cdot \frac{x^2}{4} = -\frac{x^2}{4(m+1)},$$

where $m!/(m+1)! = 1/(m+1)$ was used. Every later term carries $(x/2)^{2j}$ with $j \geq 2$, so the tail is $O(x^4)$; the first of them is $\dfrac{m!}{2!\,(m+2)!} \left(\dfrac{x}{2}\right)^4 = \dfrac{x^4}{32(m+1)(m+2)}$. Hence

$$J_m(x) = \frac{1}{m!} \left(\frac{x}{2}\right)^m \left[1 - \frac{x^2}{4(m+1)} + O(x^4)\right],$$

which sharpens the leading-order statement $J_m(x) = \frac{1}{m!}(x/2)^m + O(x^{m+2})$ recorded when J_m was normalized. For fixed m and large positive x,

$$J_m(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}),$$

Bessel asymptotics and drum eigenvalues II

(derived by stationary phase in Chapter 6), so the oscillation amplitude decays like $x^{-1/2}$. If a drum has radius R, the rim condition is $J_m(kR) = 0$. Writing $j_{m,n}$ for the nth positive zero of J_m, the allowed radial wave numbers are

$$k_{m,n} = \frac{j_{m,n}}{R}.$$

The method of Frobenius: the recipe I

Let $z = x - x_0$. To analyze

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$$

near $x = x_0$:

1. First divide by $p_0(x)$ on a punctured neighborhood where $p_0(x) \neq 0$, giving $y'' + P(x)y' + Q(x)y = 0$.
2. Classify x_0. Frobenius is guaranteed at an ordinary point or a regular singular point.
3. Substitute

$$y(x) = \sum_{k=0}^{\infty} a_k z^{k+r}, \qquad a_0 \neq 0.$$

4. Differentiate term by term and shift indices until every sum is expressed in one common power z^{k+r+s}. The offset s is not a free choice: it is the fixed integer the equation itself produces once the powers of x multiplying y'', y' and y have been distributed. In Bessel's equation above every sum was brought to x^{k+r}, so $s = 0$; in Airy's equation on the next frame every sum is brought to x^{k+r-2}, so $s = -2$. Factoring z^{r+s} out of the aligned sums leaves an ordinary power series in z, and a power series vanishes identically only if each of its coefficients vanishes.

The method of Frobenius: the recipe II

5. Set the coefficient of the lowest power equal to zero. This is the *indicial equation* for r.
6. For each admissible root r, set every remaining coefficient equal to zero and solve the resulting recurrence, checking any index at which its denominator vanishes as a separate compatibility equation.
7. Substitute the coefficients back into the series and verify the result in the original differential equation and against any initial or boundary conditions.

A vanishing recurrence denominator must never be crossed by formal division: it can force a coefficient, free a new coefficient, or make a branch inconsistent.

Example: Airy's equation by series I

Airy's equation is

$$y'' - xy = 0. \tag{4.1.11}$$

Equivalently, $y'' = xy$. For $x < 0$ this can be written $y'' + |x|y = 0$, which has an oscillatory sign; for $x > 0$ it is $y'' - xy = 0$, which has an exponential sign. Because the coefficient $|x|$ or x varies with x, these are qualitative statements, not constant-coefficient sine or exponential solutions.

Use

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}, \qquad a_0 \neq 0.$$

Then

$$y'(x) = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1},$$

$$y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2},$$

$$xy(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}.$$

Example: Airy's equation by series II

Substitution into (4.1.11) gives

$$\sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2} - \sum_{k=0}^{\infty} a_k x^{k+r+1} = 0.$$

The second sum carries powers x^{k+r+1}, which we rewrite in the form x^{j+r-2} to match the first sum. Equating exponents, $k+r+1 = j+r-2$, hence

$$j = k+3, \qquad k = j-3.$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty} a_k x^{k+r+1} &= \sum_{j=3}^{\infty} a_{j-3} x^{j+r-2} \\ &= \sum_{k=3}^{\infty} a_{k-3} x^{k+r-2}.\end{aligned}$$

Reindexing aligns every sum on the same power of x. With

$$a_{-3} = a_{-2} = a_{-1} = 0,$$

Example: Airy's equation by series III

this becomes a sum starting at $k = 0$. Hence

$$\sum_{k=0}^{\infty} \left[(k+r)(k+r-1)a_k - a_{k-3}\right] x^{k+r-2} = 0,$$

and every coefficient must satisfy

$$(k+r)(k+r-1)a_k - a_{k-3} = 0, \qquad k = 0, 1, 2, \ldots. \tag{4.1.12}$$

At $k = 0$,

$$r(r-1)a_0 = 0.$$

Since $a_0 \neq 0$,

$$r(r-1) = 0 \quad \Longrightarrow \quad r = 0 \quad \text{or} \quad r = 1.$$

Because $x = 0$ is an ordinary point, the ordinary Taylor choice $r = 0$ is sufficient to obtain both independent solutions. The root $r = 1$ only reindexes the solution whose leading term is proportional to x; it does not produce a third solution.

Set $r = 0$. The first three coefficient equations are

$$\begin{aligned}
k = 0: &\quad 0 \cdot (-1)a_0 - a_{-3} = 0, &\quad 0 = 0, &\quad a_0 \text{ is free},\\
k = 1: &\quad 1 \cdot 0a_1 - a_{-2} = 0, &\quad 0 = 0, &\quad a_1 \text{ is free},\\
k = 2: &\quad 2 \cdot 1a_2 - a_{-1} = 0, &\quad 2a_2 = 0, &\quad a_2 = 0.
\end{aligned}$$

Example: Airy's equation by series IV

For $k \geq 3$, (4.1.12) gives

$$a_k = \frac{a_{k-3}}{k(k-1)}.$$

Replacing k by $k+3$ produces the forward recurrence

$$a_{k+3} = \frac{a_k}{(k+3)(k+2)}, \qquad k = 0, 1, 2, \ldots . \tag{4.1.13}$$

The recurrence advances by three. The a_{3j+2} chain vanishes because $a_2 = 0$. For the a_0 chain,

$$\begin{aligned} a_3 &= \frac{a_0}{3 \cdot 2} = \frac{a_0}{6}, \\ a_6 &= \frac{a_3}{6 \cdot 5} = \frac{a_0}{(3 \cdot 2)(6 \cdot 5)} = \frac{a_0}{180}, \\ a_9 &= \frac{a_6}{9 \cdot 8} = \frac{a_0}{(3 \cdot 2)(6 \cdot 5)(9 \cdot 8)} = \frac{a_0}{12960}. \end{aligned}$$

In general,

$$a_{3j} = \frac{a_0}{\displaystyle\prod_{\ell=1}^{j} (3\ell)(3\ell - 1)}.$$

Example: Airy's equation by series V

The factorial $(3j)!$ factors into the three residue classes modulo 3:

$$(3j)! = \left[\prod_{\ell=1}^{j}(3\ell)\right]\left[\prod_{\ell=1}^{j}(3\ell-1)\right]\left[\prod_{\ell=1}^{j}(3\ell-2)\right].$$

The last product is the triple factorial $(3j-2)!!! = (3j-2)(3j-5)\cdots 1$ (with $(-2)!!! = 1$ when $j=0$), so

$$\prod_{\ell=1}^{j}(3\ell)(3\ell-1) = \frac{(3j)!}{\prod_{\ell=1}^{j}(3\ell-2)} = \frac{(3j)!}{(3j-2)!!!}.$$

Therefore

$$a_{3j} = \frac{(3j-2)!!!}{(3j)!}a_0,$$

where $n!!! = n(n-3)(n-6)\cdots$ and $(-2)!!! = (-1)!!! = 1$ for the $j=0$ terms.

For the a_1 chain,

$$\begin{aligned} a_4 &= \frac{a_1}{4\cdot 3} = \frac{a_1}{12}, \\ a_7 &= \frac{a_4}{7\cdot 6} = \frac{a_1}{(4\cdot 3)(7\cdot 6)} = \frac{a_1}{504}, \\ a_{10} &= \frac{a_7}{10\cdot 9} = \frac{a_1}{45360}. \end{aligned}$$

Example: Airy's equation by series VI

Thus

$$a_{3j+1} = \frac{a_1}{\displaystyle\prod_{\ell=1}^{j}(3\ell+1)(3\ell)}.$$

Here the factorial $(3j+1)!$ factors into the same three residue classes modulo 3:

$$(3j+1)! = \left[\prod_{\ell=1}^{j}(3\ell)\right]\left[\prod_{\ell=0}^{j}(3\ell+1)\right]\left[\prod_{\ell=1}^{j}(3\ell-1)\right],$$

with $\prod_{\ell=1}^{j}(3\ell-1) = 2\cdot 5\cdots(3j-1) = (3j-1)!!!$. The $\ell = 0$ factor of the middle product is 1, so the denominator above is $\prod_{\ell=1}^{j}(3\ell+1)(3\ell) = (3j+1)!/(3j-1)!!!$. Therefore

$$a_{3j+1} = \frac{(3j-1)!!!}{(3j+1)!}a_1.$$

Example: Airy's equation by series VII

The complete Taylor solution is

$$\begin{aligned} y(x) = a_0 \sum_{j=0}^{\infty} \frac{(3j-2)!!!}{(3j)!} x^{3j} \\ + a_1 \sum_{j=0}^{\infty} \frac{(3j-1)!!!}{(3j+1)!} x^{3j+1}. \end{aligned} \tag{4.1.14}$$

Equivalently, define

$$f_0(x) = 1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \cdots,$$
$$f_1(x) = x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \cdots.$$

Then

$$y(x) = a_0 f_0(x) + a_1 f_1(x),$$

and the constants are exactly the initial data:

$$a_0 = y(0), \qquad a_1 = y'(0).$$

Example: Airy's equation by series VIII

Thus $\{f_0, f_1\}$ is a basis of the two-dimensional solution space. For comparison, choosing $r = 1$ gives a free coefficient multiplying x and forces the next two coefficients to zero; its recurrence reproduces f_1 only. The next frame names the classical combinations Ai and Bi of this same basis.

Standard Airy solutions Ai and Bi I

The two independent series solutions of Airy's equation are the Taylor basis

$$f_0(x) = 1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \cdots, \qquad f_1(x) = x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \cdots$$

from the previous frame, so every solution is

$$y(x) = a_0 f_0(x) + a_1 f_1(x)$$

with free data $a_0 = y(0)$ and $a_1 = y'(0)$.

The classical Airy functions of the first and second kinds are the particular linear combinations fixed by the standard initial values

$$\begin{aligned} Ai(0) &= \frac{1}{3^{2/3}\Gamma(2/3)}, & Ai'(0) &= -\frac{1}{3^{1/3}\Gamma(1/3)}, \\ Bi(0) &= \frac{1}{3^{1/6}\Gamma(2/3)} = \sqrt{3}\,Ai(0), & Bi'(0) &= \frac{3^{1/6}}{\Gamma(1/3)} = -\sqrt{3}\,Ai'(0). \end{aligned} \tag{4.1.15}$$

(We quote these constants; they come from classical integral representations of Ai and Bi, not from the power-series calculation itself.) Explicitly,

$$\begin{aligned} Ai(x) &= Ai(0)\,f_0(x) + Ai'(0)\,f_1(x), \\ Bi(x) &= Bi(0)\,f_0(x) + Bi'(0)\,f_1(x). \end{aligned} \tag{4.1.16}$$

Standard Airy solutions Ai and Bi II

Equation (4.1.16) is a change of basis: it writes the pair Ai, Bi in terms of f_0, f_1 through the coefficient matrix

$$\begin{pmatrix} Ai \\ Bi \end{pmatrix} = \begin{pmatrix} Ai(0) & Ai'(0) \\ Bi(0) & Bi'(0) \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix},$$

so Ai and Bi span the same two-dimensional space as f_0, f_1 exactly when this matrix is invertible, that is, when

$$\begin{vmatrix} Ai(0) & Ai'(0) \\ Bi(0) & Bi'(0) \end{vmatrix} = Ai(0)Bi'(0) - Ai'(0)Bi(0) \neq 0.$$

That the entries really are the values and derivatives at the origin follows from $f_0(0) = 1,\ f_0'(0) = 0,\ f_1(0) = 0,\ f_1'(0) = 1$, read off the two series above. This determinant is the *Wronskian* $W[u, v] = uv' - u'v$ of Ai and Bi evaluated at $x = 0$

— the same combination used later in this chapter for reduction of order. We evaluate it from the initial values (4.1.15):

$$\begin{aligned} W[Ai, Bi](0) &= Ai(0)Bi'(0) - Ai'(0)Bi(0) \\ &= \left(\frac{1}{3^{2/3}\Gamma(2/3)}\right)\left(\frac{3^{1/6}}{\Gamma(1/3)}\right) - \left(-\frac{1}{3^{1/3}\Gamma(1/3)}\right)\left(\frac{1}{3^{1/6}\Gamma(2/3)}\right) \\ &= \frac{3^{-2/3+1/6}}{\Gamma(1/3)\Gamma(2/3)} + \frac{3^{-1/3-1/6}}{\Gamma(1/3)\Gamma(2/3)}. \end{aligned}$$

The two exponents simplify independently:

$$-2/3 + 1/6 = -4/6 + 1/6 = -1/2, \qquad -1/3 - 1/6 = -2/6 - 1/6 = -1/2,$$

so each power of 3 is $3^{-1/2} = 1/\sqrt{3}$, and

$$W[Ai, Bi](0) = \frac{1/\sqrt{3} + 1/\sqrt{3}}{\Gamma(1/3)\Gamma(2/3)} = \frac{2/\sqrt{3}}{\Gamma(1/3)\Gamma(2/3)}.$$

The reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ at $z = 1/3$ gives

$$\Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin(\pi/3)} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}.$$

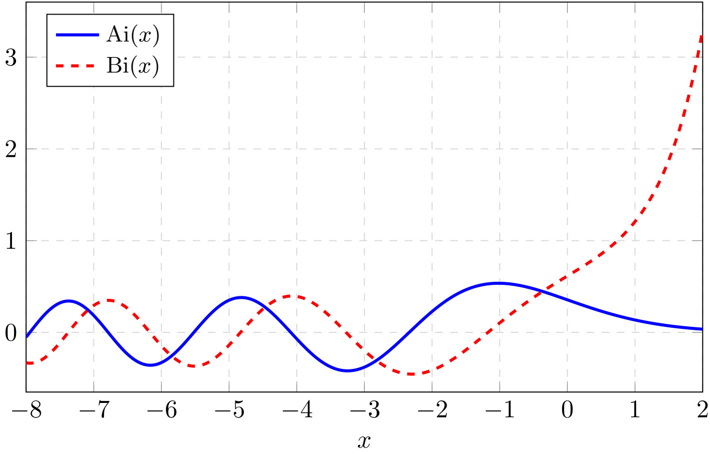
Standard Airy solutions Ai and Bi IV

Therefore

$$W[Ai, Bi](0) = \frac{2/\sqrt{3}}{2\pi/\sqrt{3}} = \frac{1}{\pi} \neq 0.$$

The determinant is nonzero, so the coefficient matrix above is invertible and f_0, f_1 can be recovered from Ai, Bi; thus Ai and Bi form a basis of the two-dimensional solution space. For large positive x, $Ai(x)$ decays while $Bi(x)$ grows; for $x < 0$ both oscillate. Those asymptotic roles explain why the classical names are preferred to the bare series basis f_0, f_1 in applications.

The Airy functions *Ai* and *Bi*



The plot shows the classical combinations (4.1.16) of the two power series f_0 and f_1. For $x < 0$ both oscillate; for $x > 0$, $Ai(x)$ decays while $Bi(x)$ grows, so boundary conditions usually decide which combination is physically allowed.

Table of contents

Classifying the point x_0 I

Consider

$$y'' + P(x)y' + Q(x)y = 0$$

and set

$$z = x - x_0.$$

The point x_0 is classified as follows:

- It is an **ordinary point** if P and Q are analytic at x_0.
- It is a **regular singular point** if x_0 is singular but $zP(x)$ and $z^2Q(x)$ are analytic at $z = 0$.
- It is an **irregular singular point** otherwise.

Classifying the point x_0 II

At a regular singular point, define analytic functions

$$p(z) = zP(x) = \sum_{j=0}^{\infty} p_j z^j, \qquad q(z) = z^2 Q(x) = \sum_{j=0}^{\infty} q_j z^j.$$

Here p_j and q_j are *numbers*—the Taylor coefficients of the rescaled functions $p(z) = zP(x)$ and $q(z) = z^2 Q(x)$. They are not the coefficient functions $p_0(x), p_1(x), p_2(x)$ of the undivided equation on the recipe slide: that division has already been carried out, and the equation under study is the normalized one, $y'' + P(x)y' + Q(x)y = 0$. In particular $p_0 = \lim_{z\to 0} zP(x)$ and $q_0 = \lim_{z\to 0} z^2 Q(x)$. Multiplying the differential equation by z^2 gives

$$z^2 y'' + zp(z)y' + q(z)y = 0.$$

Substitute

$$y = \sum_{n=0}^{\infty} a_n z^{n+r}, \qquad a_0 \neq 0.$$

Classifying the point x_0 III

The three terms become

$$z^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n z^{n+r},$$

$$zp(z)y' = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} p_j (n+r) a_n z^{j+n+r},$$

$$q(z)y = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} q_j a_n z^{j+n+r}.$$

Collect like powers. A term contributes to the coefficient of z^{N+r} exactly when its exponent equals $N+r$: the $z^2 y''$ sum contributes only its $n=N$ term, while in each double sum $j+n=N$ forces $n=N-j$ with j running from 0 to N. Separating the $j=0$ part of the two double sums, namely $[p_0(N+r)+q_0]\, a_N$, and merging it with the $z^2 y''$ term $(N+r)(N+r-1)a_N$ leaves the $j \geq 1$ terms as the coupling to lower coefficients:

$$\begin{aligned} 0 = {} & [(N+r)(N+r-1) + p_0(N+r) + q_0]\, a_N \\ & + \sum_{j=1}^{N} [p_j(N-j+r) + q_j]\, a_{N-j}. \end{aligned}$$

Classifying the point x_0 IV

At $N = 0$, the sum is empty, so the lowest-power equation is

$$[r(r-1) + p_0 r + q_0]\, a_0 = 0.$$

Because $a_0 \neq 0$, the general indicial equation is

$$r(r-1) + p_0 r + q_0 = 0. \tag{4.2.1}$$

The local existence results below are classical and are quoted here without proof; the two statements for a regular singular point, items 2 and 3, are together known as **Fuchs' theorem**:

1. At an ordinary point, there are two linearly independent Taylor-series solutions. Their series converge at least throughout any disk centered at x_0 in which both P and Q remain analytic.

2. At a regular singular point, Fuchs' theorem guarantees at least one Frobenius solution. Let $R > 0$ be the distance from x_0 to the nearest other singularity of P and Q, so that $p(z)$ and $q(z)$ are analytic on $|z| < R$. Label the roots so that $\operatorname{Re} r_1 \geq \operatorname{Re} r_2$. Then the coefficients a_n delivered by the recurrence give a series $\sum_{n\geq 0} a_n z^n$ with radius of convergence at least R, so

$$y_1 = z^{r_1} \sum_{n=0}^{\infty} a_n z^n$$

solves the equation for $0 < z < R$. If the indicial roots satisfy $r_1 - r_2 \notin \mathbb{Z}$, there are two independent Frobenius series, both convergent on that same interval.

3. If $r_1 - r_2 = N$ is a nonnegative integer, the r_1 branch is a Frobenius solution. The second solution may be another Frobenius series, or it may have the form

$$y_2 = C\, y_1 \log z + z^{r_2} \sum_{n=0}^{\infty} b_n z^n.$$

Where this form comes from, and why the logarithm can drop out when $N \geq 1$, is derived on the next frame. A repeated root is the case $N = 0$.

4. At an irregular singular point, there is no general Frobenius guarantee. Some equations have no solution of the assumed Frobenius form.

Classifying the point x_0 VI

Throughout, z^r and $\log z$ are read on the ray $z > 0$, where $z^r := e^{r \ln z}$ with the real logarithm; for noninteger r both are multivalued near $z = 0$, so some such choice is necessary. For $z < 0$ replace z by $|z|$, and for complex z fix one branch by cutting the punctured disk $0 < |z| < R$ along a ray from x_0, using that same branch in z^r and in the logarithm of item 3.

Item 3 above displays the second solution as

$$y_2 = C\, y_1 \log z + z^{r_2} \sum_{n=0}^{\infty} b_n z^n.$$

That form is not a guess: it is what reduction of order returns, and the derivation also says when $C = 0$. Anticipating (4.3.3) of the next section, reduction of order fed the larger-root solution $y_1 = z^{r_1} \sum_{n\geq 0} a_n z^n$, $a_0 \neq 0$, gives

$$y_2 = y_1 \int^z \frac{\exp\left(-\int^s P\right)}{y_1^2(s)}\, ds,$$

where $\int^z$ denotes any fixed antiderivative; changing it shifts y_2 by a multiple of y_1, which is harmless.

Since $P = p(z)/z = p_0/z + p_1 + p_2 z + \cdots$, integrating term by term gives

$$\int^z P = p_0 \log z + A(z), \qquad A \text{ analytic near } z = 0,$$

and therefore

$$\exp\left(-\int^z P\right) = z^{-p_0} h(z), \qquad h = e^{-A}, \quad h(0) \neq 0.$$

Where the logarithm in the second solution comes from II

With $y_1^2 = z^{2r_1}\left(a_0 + a_1 z + \cdots\right)^2$, the integrand is

$$\frac{\exp\left(-\int^z P\right)}{y_1^2} = z^{-p_0-2r_1} g(z), \qquad g = \frac{h}{\left(a_0 + a_1 z + \cdots\right)^2}, \quad g(0) = \frac{h(0)}{a_0^2} \neq 0.$$

The exponent is not arbitrary. Writing (4.2.1) as $r^2 + (p_0 - 1)r + q_0 = 0$, the sum of its roots is $r_1 + r_2 = 1 - p_0$, that is $p_0 = 1 - r_1 - r_2$, so

$$-p_0 - 2r_1 = -(1 - r_1 - r_2) - 2r_1 = -(r_1 - r_2) - 1 = -N - 1.$$

Expand $g(z) = \sum_{k \geq 0} g_k z^k$ and integrate term by term. Every power $g_k z^{k-N-1}$ integrates to another power except the single term with $k = N$, which is g_N / z and integrates to a logarithm:

$$\int^z s^{-N-1} g(s)\, ds = g_N \log z + \sum_{k \neq N} \frac{g_k}{k - N}\, z^{k-N}.$$

Multiply by y_1. The logarithmic part is $g_N\, y_1 \log z$, and since $y_1 z^{k-N} = z^{r_1 - N} z^k \sum_n a_n z^n = z^{r_2} z^k \sum_n a_n z^n$ with $k \geq 0$, the power part collects into a single Frobenius series with exponent r_2. This is exactly the displayed form, with

$$C = g_N.$$

Two consequences. For a repeated root, $N = 0$ and $C = g_0 \neq 0$: the logarithm is unavoidable. For $N \geq 1$ only the single number g_N has to vanish for the logarithm to

Where the logarithm in the second solution comes from III

disappear, and that is what happens in the Euler example two frames on: there $P = 0$ and $y_1 = x^3$, so $h \equiv 1$, $a_0 = 1$, $g \equiv 1$, hence $g_5 = 0$ with $N = 3 - (-2) = 5$, and

$$y_2 = x^3 \int^x s^{-6}\, ds = -\tfrac{1}{5} x^{-2},$$

a pure Frobenius series with no logarithm.

Examples: ordinary and irregular singular points I

Airy equation. For

$$y'' - xy = 0,$$

we have

$$P(x) = 0, \qquad Q(x) = -x.$$

Both are analytic at $x = 0$, so the origin is an ordinary point. The two Taylor solutions f_0 and f_1 found earlier are therefore expected.

The equation $x^4y'' + 2x^3y' + y = 0$. For $x \neq 0$, divide by x^4:

$$y'' + \frac{2}{x}y' + \frac{1}{x^4}y = 0.$$

Thus

$$P(x) = \frac{2}{x}, \qquad Q(x) = \frac{1}{x^4}.$$

Although

$$xP(x) = 2$$

is analytic, the regular-singular test fails because

$$x^2Q(x) = \frac{1}{x^2}$$

is not analytic at the origin. Hence $x = 0$ is an irregular singular point.

Examples: ordinary and irregular singular points II

A direct Frobenius substitution confirms the failure. With $y = \sum_{k=0}^{\infty} a_k x^{k+r}$,

$$x^4 y'' + 2x^3 y' + y = \sum_{k=0}^{\infty} \left[(k+r)(k+r-1) + 2(k+r)\right] a_k x^{k+r+2} + \sum_{k=0}^{\infty} a_k x^{k+r}.$$

The lowest power is x^r, and it occurs only in the second sum. Its coefficient is a_0, so the equation would require

$$a_0 = 0,$$

contrary to the Frobenius assumption $a_0 \neq 0$. Thus no nonzero solution of the assumed Frobenius form exists.

(For curiosity, the equation can be solved in closed form. Because z already denotes $x - x_0$ here, call the new independent variable t: set

$$t = \frac{1}{x}, \qquad u(t) = y\left(\frac{1}{t}\right), \qquad \text{so that} \qquad y(x) = u\left(\frac{1}{x}\right).$$

Only the independent variable changes. From $\dfrac{dt}{dx} = -\dfrac{1}{x^2} = -t^2$, the chain rule gives

$$\frac{dy}{dx} = \frac{du}{dt}\frac{dt}{dx} = -t^2 u'(t),$$

Examples: ordinary and irregular singular points III

and, differentiating once more and using the chain rule again,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left(-t^2u'\right)\frac{dt}{dx}\\ &= \left(-2tu' - t^2u''\right)\left(-t^2\right)\\ &= t^4u'' + 2t^3u'.\end{aligned}$$

Since $x = 1/t$ we have $x^4 = t^{-4}$ and $x^3 = t^{-3}$, so

$$\begin{aligned}x^4y'' + 2x^3y' + y &= t^{-4}\left(t^4u'' + 2t^3u'\right) + 2t^{-3}\left(-t^2u'\right) + u\\ &= u'' + \frac{2}{t}u' - \frac{2}{t}u' + u\\ &= u'' + u,\end{aligned}$$

the two $\frac{2}{t}u'$ terms cancelling: it is exactly the $2x^3y'$ term of the original equation that removes the first-derivative term produced by the change of variable. Hence $u'' + u = 0$, so $u(t) = c_1\cos t + c_2\sin t$ and

$$y(x) = u\left(\frac{1}{x}\right) = c_1\cos\bigl(1/x\bigr) + c_2\sin\bigl(1/x\bigr).$$

These functions have essential singularities at $x = 0$, with infinitely many negative Laurent powers—far outside the Frobenius class $x^r\sum_{k\geq 0}a_kx^k$.)

Example: an Euler equation with an integer root gap I

Consider

$$x^2y'' - 6y = 0.$$

For $x \neq 0$,

$$y'' - \frac{6}{x^2}y = 0,$$

so $P(x) = 0$ and $x^2Q(x) = -6$ is analytic. The origin is a regular singular point. Substitute

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}.$$

Then

$$\begin{aligned} x^2y'' - 6y &= \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r} - 6\sum_{k=0}^{\infty} a_k x^{k+r} \\ &= \sum_{k=0}^{\infty}\left[(k+r)(k+r-1) - 6\right] a_k x^{k+r}. \end{aligned}$$

Thus

$$\left[(k+r)(k+r-1) - 6\right] a_k = 0.$$

Example: an Euler equation with an integer root gap II

At $k = 0$,

$$
\begin{aligned}
r(r-1) - 6 &= 0, \\
r^2 - r - 6 &= 0, \\
(r-3)(r+2) &= 0,
\end{aligned}
$$

so

$$r = 3 \qquad \text{or} \qquad r = -2.$$

For $r = 3$,

$$
\begin{aligned}
(k+3)(k+2) - 6 &= k^2 + 5k + 6 - 6 \\
&= k(k+5).
\end{aligned}
$$

The $k = 0$ equation is $0 = 0$, so a_0 is free. For every $k \geq 1$, $k(k+5) \neq 0$, and hence $a_k = 0$. This branch is

$$y_1 = x^3.$$

For $r = -2$,

$$
\begin{aligned}
(k-2)(k-3) - 6 &= k^2 - 5k + 6 - 6 \\
&= k(k-5).
\end{aligned}
$$

The $k = 0$ equation leaves a_0 free. For $k = 1, 2, 3, 4$, the factor $k(k-5)$ is nonzero, so

$$a_1 = a_2 = a_3 = a_4 = 0.$$

Example: an Euler equation with an integer root gap III

At $k = 5$, the equation is again $0 = 0$, so a_5 is also free. For $k \geq 6$, all remaining coefficients vanish. The lower-root ansatz therefore gives

$$y = a_0 x^{-2} + a_5 x^3.$$

The free a_5 term is a consequence of the integer root gap $3 - (-2) = 5$; it duplicates the larger-root solution. A convenient independent basis is

$$y_1 = x^3, \qquad y_2 = x^{-2},$$

and the general solution is

$$y(x) = c_1 x^3 + c_2 x^{-2}.$$

Example: Bessel's regular singular point I

Bessel equation. Divide

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

by x^2:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{m^2}{x^2}\right) y = 0.$$

Then

$$xP(x) = 1, \qquad x^2Q(x) = x^2 - m^2,$$

and both expressions are analytic at the origin. Hence $x = 0$ is a regular singular point. In the notation of (4.2.1),

$$p_0 = 1, \qquad q_0 = -m^2.$$

Therefore

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= r(r-1) + r - m^2 \\ &= r^2 - m^2, \end{aligned}$$

which reproduces the Bessel indicial equation $r = \pm m$. The earlier irregular example $(x^4y'' + \cdots)$ already shows what happens when the regular-singular test fails: the lowest power can force $a_0 = 0$ and kill the Frobenius ansatz outright.

Summary: when Frobenius works

For $y'' + P(x)y' + Q(x)y = 0$ near x_0:

- If P and Q are analytic, x_0 is an ordinary point and there are two independent Taylor-series solutions.
- If P or Q fails to be analytic at x_0, but $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic, then x_0 is a regular singular point and at least one Frobenius solution exists.
- If the indicial roots differ by a noninteger, both solutions are Frobenius series. If they coincide or differ by an integer, the second solution may contain a logarithm or may arise through a compatibility degeneracy.
- At an irregular singular point, Frobenius has no general existence guarantee. The lowest-power coefficient may directly contradict $a_0 \neq 0$.

A zero denominator in a recurrence must be returned to the undivided coefficient equation and checked for consistency.

Table of contents

Reduction of order: building the second solution I

Step 1: eliminate the unknown Q term. Let P and Q be continuous on an interval I, and suppose a known solution v of

$$u'' + P(x)u' + Q(x)u = 0$$

is nonzero on the subinterval under consideration. Seek another solution y. The two equations are

$$y'' + Py' + Qy = 0, \qquad v'' + Pv' + Qv = 0.$$

Multiply the first by v, the second by y, and subtract:

$$\begin{aligned} 0 &= v(y'' + Py' + Qy) - y(v'' + Pv' + Qv) \\ &= vy'' - yv'' + P(vy' - yv') + Q(vy - yv). \end{aligned}$$

The Q term vanishes because $vy - yv = 0$. Hence

$$vy'' - yv'' + P(vy' - yv') = 0.$$

Define the Wronskian in the order

$$W[v, y] = vy' - v'y = vy' - yv'.$$

Its derivative is

$$\begin{aligned} W'[v, y] &= (vy' - v'y)' \\ &= v'y' + vy'' - v''y - v'y' \\ &= vy'' - yv''. \end{aligned}$$

Reduction of order: building the second solution II

Therefore

$$W' + PW = 0. \tag{4.3.1}$$

Step 2: solve the Wronskian equation. Choose a reference point $x_* \in I$ and define the integrating factor

$$\mu(x) = \exp\left(\int_{x_*}^{x} P(t)\,dt\right).$$

By the fundamental theorem of calculus and the chain rule,

$$\mu'(x) = P(x)\mu(x).$$

Multiplying (4.3.1) by μ gives

$$\begin{aligned} 0 &= \mu W' + \mu P W \\ &= \mu W' + \mu' W \\ &= (\mu W)'. \end{aligned}$$

Thus

$$\mu(x)W(x) = C_1,$$

or

$$W[v, y](x) = C_1 \exp\left(-\int_{x_*}^{x} P(t)\,dt\right). \tag{4.3.2}$$

Reduction of order: building the second solution III

This is Abel's identity for the Wronskian.

Step 3: integrate once more to recover y. Next observe that

$$\left(\frac{y}{v}\right)' = \frac{vy' - yv'}{v^2} = \frac{W[v, y]}{v^2}.$$

Using (4.3.2),

$$\left(\frac{y}{v}\right)' = C_1 \frac{\exp\left(-\int_{x_*}^{x} P(t)\,dt\right)}{v^2(x)}.$$

Integrate from x_* to x:

$$\frac{y(x)}{v(x)} = C_0 + C_1 \int_{x_*}^{x} \frac{\exp\left(-\int_{x_*}^{s} P(t)\,dt\right)}{v^2(s)}\,ds.$$

Reduction of order: building the second solution IV

Therefore the general expression produced by reduction of order is

$$y(x) = C_0 v(x) + C_1 v(x) \int_{x_*}^{x} \frac{\exp\left(-\int_{x_*}^{s} P(t)\,dt\right)}{v^2(s)}\,ds.$$

The first term is only a multiple of the known solution. A convenient second solution is therefore

$$y_2(x) = v(x) \int_{x_*}^{x} \frac{\exp\left(-\int_{x_*}^{s} P(t)\,dt\right)}{v^2(s)}\,ds. \tag{4.3.3}$$

To verify independence, let the integral in (4.3.3) be $I(x)$. Then

$$y_2 = vI, \qquad y_2' = v'I + vI'.$$

Consequently,

$$\begin{aligned} W[v, y_2] &= v(v'I + vI') - v'(vI) \\ &= v^2 I' \\ &= \exp\left(-\int_{x_*}^{x} P(t)\,dt\right) \neq 0. \end{aligned}$$

Thus v and y_2 are linearly independent on the chosen interval.

Reduction-of-order example: harmonic oscillator I

For

$$y'' + y = 0,$$

we have

$$P(x) = 0, \qquad v(x) = \cos x.$$

Work on an interval containing 0 but no zero of $\cos x$, for example $(-\pi/2, \pi/2)$. With $x_* = 0$,

$$\exp\left(-\int_0^s P(t)\,dt\right) = \exp(0) = 1.$$

Formula (4.3.3) gives

$$\begin{aligned} y_2(x) &= \cos x \int_0^x \frac{1}{\cos^2 s}\,ds \\ &= \cos x \int_0^x \sec^2 s\,ds \\ &= \cos x\,[\tan s]_0^x \\ &= \cos x \tan x \\ &= \sin x. \end{aligned}$$

Reduction-of-order example: harmonic oscillator II

The Wronskian check is

$$\begin{aligned} W[\cos x, \sin x] &= \cos x(\cos x) - (-\sin x)(\sin x) \\ &= \cos^2 x + \sin^2 x \\ &= 1, \end{aligned}$$

so the two solutions are independent.

Reduction-of-order example: Bessel's second solution I

For Bessel's equation in normalized form,

$$y'' + \frac{1}{x}y' + \left(1 - \frac{m^2}{x^2}\right) y = 0,$$

we have

$$P(x) = \frac{1}{x}, \qquad v(x) = J_m(x).$$

Choose an interval contained in $(0, \infty)$ on which J_m has no zero. For a reference point $x_* > 0$,

$$\begin{aligned} \exp\left(-\int_{x_*}^{s} \frac{dt}{t}\right) &= \exp\left[-\log\left(\frac{s}{x_*}\right)\right] \\ &= \frac{x_*}{s}. \end{aligned}$$

The constant factor x_* can be absorbed into the arbitrary normalization. Thus a second solution is

$$\widetilde{Y}_m(x) = J_m(x) \int_{x_*}^{x} \frac{ds}{sJ_m^2(s)}. \tag{4.3.4}$$

Reduction-of-order example: Bessel's second solution II

Its Wronskian follows directly from the definition. Writing $\widetilde{Y}_m = J_m\, I$ with $I(x) = \int_{x_*}^{x} \frac{ds}{sJ_m^2(s)}$, so that $I'(x) = \frac{1}{xJ_m^2(x)}$,

$$W[J_m, \widetilde{Y}_m] = J_m\big(J_m' I + J_m I'\big) - J_m'\, J_m I = J_m^2\, I' = \frac{1}{x}.$$

The general reduction-of-order formula would read $\exp\left(-\int_{x_*}^{x} dt/t\right) = x_*/x$; the two differ only by the constant x_* absorbed into the normalization above. Only the scale of $\widetilde{Y}_m$ is still undetermined, and the standard choice is the one carried by the Bessel function of the second kind Y_m. That function is constructed in Chapter 6: for noninteger order one sets

$$Y_\nu = \frac{J_\nu \cos(\pi\nu) - J_{-\nu}}{\sin(\pi\nu)}, \qquad Y_m = \lim_{\nu \to m} Y_\nu.$$

Chapter 6 first obtains $W[J_\nu, J_{-\nu}] = -\dfrac{2\sin(\pi\nu)}{\pi x}$ from the leading small-x terms of the two series; then linearity of the Wronskian in its second argument, together with $W[J_\nu, J_\nu] = 0$, gives

$$W[J_\nu, Y_\nu] = \frac{\cos(\pi\nu) W[J_\nu, J_\nu] - W[J_\nu, J_{-\nu}]}{\sin(\pi\nu)} = -\frac{W[J_\nu, J_{-\nu}]}{\sin(\pi\nu)} = \frac{2}{\pi x},$$

Reduction-of-order example: Bessel's second solution III

a value free of ν, so letting $\nu \to m$ leaves

$$W[J_m, Y_m](x) = \frac{2}{\pi x}.$$

We quote that normalization here and use it only to fix the scale. Matching this Wronskian fixes the scale of $\widetilde{Y}_m$ but not an additive multiple of J_m. Therefore, on the interval under consideration,

$$Y_m(x) = \frac{2}{\pi}\,\widetilde{Y}_m(x) + CJ_m(x), \tag{4.3.5}$$

where changing the lower integration limit only changes the harmless constant C.

The singular behavior at the origin follows directly from the integral. For $m \geq 1$,

$$J_m(x) \sim \frac{x^m}{2^m m!} \qquad (x \to 0^+).$$

Hence

$$\begin{aligned} \frac{1}{sJ_m^2(s)} &\sim \frac{1}{s}\left(\frac{2^m m!}{s^m}\right)^2 \\ &= 2^{2m}(m!)^2 s^{-2m-1}. \end{aligned}$$

Integrating toward the origin,

$$\int_{x_*}^{x} \frac{ds}{sJ_m^2(s)} \sim 2^{2m}(m!)^2 \int_{x_*}^{x} s^{-2m-1}\,ds = 2^{2m}(m!)^2 \left[-\frac{s^{-2m}}{2m}\right]_{x_*}^{x}.$$

Reduction-of-order example: Bessel's second solution IV

As $x \to 0^+$ with x_* fixed, $x^{-2m} \to \infty$ dwarfs the bounded constant x_*^{-2m}, so only the x-endpoint survives:

$$\int_{x_*}^{x} \frac{ds}{sJ_m^2(s)} \sim -\frac{2^{2m}(m!)^2}{2m} x^{-2m}.$$

Reduction-of-order example: Bessel's second solution V

Multiplying by $J_m(x)$:
$$\begin{aligned}\widetilde{Y}_m(x) &\sim \frac{x^m}{2^m m!}\left[-\frac{2^{2m}(m!)^2}{2m}x^{-2m}\right]\\ &= -\frac{2^{2m}(m!)^2}{2^m\, m!\,(2m)}\,x^{-m}\\ &= -\frac{2^m m!}{2m}\,x^{-m}\\ &= -2^{m-1}(m-1)!\,x^{-m},\end{aligned}$$

using $\dfrac{2^{2m}}{2^m} = 2^m$, $\dfrac{(m!)^2}{m!} = m!$, and $\dfrac{m!}{2m} = \dfrac{(m-1)!}{2}$. After the factor $2/\pi$ in (4.3.5),
$$Y_m(x) \sim -\frac{(m-1)!}{\pi}\left(\frac{2}{x}\right)^m, \qquad m \geq 1.$$

A brief remark on logarithms: after reduction of order the integrand satisfies $1/(sJ_m^2(s)) \sim c\, s^{-2m-1}$ near $s = 0$, so the leading antiderivative is a pure power s^{-2m} (for $m \geq 1$). A $\log s$ term would appear only if the Laurent expansion of the integrand contained an s^{-1} coefficient. Any such logarithmic contribution multiplies $J_m(x) \sim x^m$ and is therefore subdominant to the pure power singularity x^{-m} as

Reduction-of-order example: Bessel's second solution VI

$x \to 0^+$. The leading small-x law for Y_m is thus the pure power above. For $m = 0$, reduction of order starts from

$$\widetilde{Y}_0(x) = J_0(x) \int_{x_*}^{x} \frac{ds}{sJ_0^2(s)}.$$

Near the origin the Taylor series is

$$J_0(x) = 1 - \frac{x^2}{4} + O(x^4),$$

so

$$\begin{aligned} J_0^2(x) &= 1 - \frac{x^2}{2} + O(x^4), \\ \frac{1}{J_0^2(x)} &= 1 + \frac{x^2}{2} + O(x^4), \\ \frac{1}{xJ_0^2(x)} &= \frac{1}{x} + \frac{x}{2} + O(x^3). \end{aligned}$$

Integrating,

$$\int_{x_*}^{x} \frac{ds}{sJ_0^2(s)} = \log x + C_* + O(x^2),$$

Reduction-of-order example: Bessel's second solution VII

where C_* absorbs the lower-limit constant and the regular part of the integrand. Since $J_0(x) \to 1$,

$$\widetilde{Y}_0(x) = \log x + C_* + O(x^2|\log x|).$$

The conventional normalization $Y_0 = (2/\pi)\widetilde{Y}_0 + CJ_0$ is fixed by the Wronskian requirement $W[J_0, Y_0] = 2/(\pi x)$. Indeed, if

$$Y_0(x) = A \log x + B + o(1) \qquad (x \to 0^+),$$

then $Y_0' = A/x + o(x^{-1})$, while $J_0 \to 1$ and $J_0' \to 0$, so

$$W[J_0, Y_0] = J_0\, Y_0' - J_0'\, Y_0 \sim \frac{A}{x}.$$

Matching $A/x = 2/(\pi x)$ forces the logarithmic coefficient

$$A = \frac{2}{\pi}.$$

(The factor $2/\pi$ in front of $\widetilde{Y}_0$ is the same scale that converts $W[J_0, \widetilde{Y}_0] = 1/x$ into $W[J_0, Y_0] = 2/(\pi x)$.) The constant term is not fixed by the Wronskian at all: replacing Y_0 by $Y_0 + cJ_0$ changes $W[J_0, Y_0]$ by $c\, W[J_0, J_0] = 0$. It is fixed instead by the limit $Y_0 = \lim_{\nu \to 0} Y_\nu$, carried out in Chapter 6, where differentiating the series in

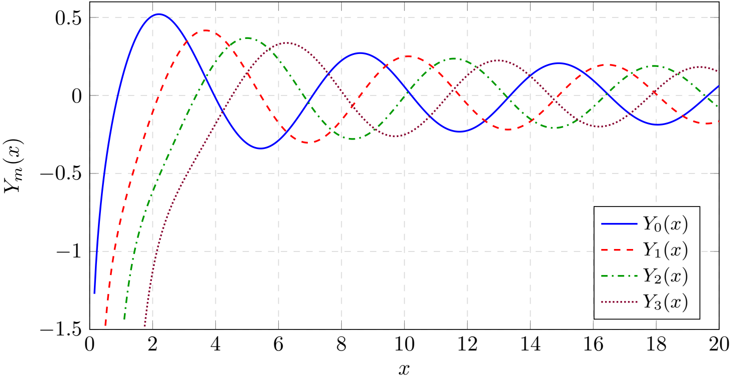
Reduction-of-order example: Bessel's second solution VIII

ν brings in the digamma value $\psi(1) = \Gamma'(1)/\Gamma(1) = -\gamma$ and hence Euler's constant γ. The sharper small-x formula is therefore

$$Y_0(x) = \frac{2}{\pi}\left[\log\left(\frac{x}{2}\right) + \gamma\right] + O\left(x^2|\log x|\right) \qquad (x \to 0^+).$$

Thus every Y_m is singular at the origin.

The Bessel functions of the second kind $Y_0, \dots, Y_3$



Unlike J_m, each Y_m plunges to $-\infty$ as $x \to 0^+$: the second Bessel solution is singular at the center, so it is excluded for a drum (finite there) but needed on annular domains.

Chapter summary

- A Frobenius expansion has the form $y = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r}$ with $a_0 \neq 0$.
- The lowest-power coefficient gives the indicial equation. Higher coefficients give a recurrence, but every zero recurrence denominator must be checked in the original, undivided coefficient equation.
- The harmonic oscillator produces the even and odd Taylor chains $\cos x$ and $\sin x$.
- Bessel's equation has indicial roots $\pm m$. The regular root gives J_m; for positive integer m, the negative-root branch fails a compatibility equation, and the missing solution is Y_m.
- Airy's equation (ordinary point) has two Taylor solutions f_0, f_1 generated by $y(0)$ and $y'(0)$. The classical Ai and Bi are the linear combinations with the standard initial values (4.1.15).
- Ordinary points have two analytic solutions. Regular singular points have at least one Frobenius solution; an integer root gap may introduce a logarithm or a compatibility degeneracy. Irregular singular points have no general Frobenius guarantee.
- Reduction of order gives
$$y_2 = v \int \frac{e^{-\int P}}{v^2}\, dx,$$
with consistent limits understood; its nonzero Wronskian proves independence from v.