Chapter 3: Gamma, Beta, Zeta

Special Functions of Mathematical Physics

after K. A. Novak
Compiled by Jake W. Liu

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Outline

Where we are going

Many special functions begin as discrete objects—factorials, partial sums, p-series—and gain power once they are extended to continuous (or complex) arguments. This chapter develops the two functions that form the basic *language* of that extension:

- **Harmonic numbers** $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and the continuous interpolation that produces the Euler–Mascheroni constant γ.
- The **gamma function** $\Gamma(z)$: the canonical meromorphic interpolation of the shifted factorial. Integral definition, recursion, half-integers, poles, reflection, Euler's limit form, and Stirling.
- The **beta function** $B(p, q)$: the integral that encodes binomial structure, linked to gamma by $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p + q)$.
- **Riemann zeta function** $\zeta(s)$ for $\operatorname{Re} s > 1$: series, integral representation, Euler product, and the Basel value $\zeta(2) = \pi^2/6$.

The theme is discrete $\to$ continuous interpolation, not an essay on algebraic number systems. Write $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Block stacking and the harmonic numbers I

Step 1: find the increment. How far can n identical blocks (length two units, unit mass) overhang a table edge? With one block we balance it half off, so its center of mass sits at the edge $x = 0$; the overhang is $H_1 = 1$.

Suppose $n-1$ stacked blocks already have their center of mass at the edge, with total overhang H_{n-1}. Slip one more block underneath with its center at -1, so its end lines up with the table edge. Each block has unit mass, so the new center of mass x of all n blocks is the mass-weighted average (total mass n times x equals the sum of mass times position): the top $n-1$ blocks contribute mass $n-1$ at position 0, and the new block mass 1 at position -1, so

$$n\,x = (n-1)\cdot 0 + 1\cdot(-1) \implies x = -\tfrac{1}{n}.$$

Shifting the whole stack forward by $1/n$ puts the new center of mass back at the edge. Before this shift, the far end of the topmost block reached $x = H_{n-1}$ (its overhang past the new block's right edge at $x = 0$). The rigid shift moves every point forward by $1/n$, so that far end now reaches $x = H_{n-1} + \frac{1}{n}$; measured from the table edge at $x = 0$, the overhang grows to $H_n = H_{n-1} + \frac{1}{n}$. With the empty-sum base case $H_0 = 0$, induction gives

$$H_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}. \tag{3.1.1}$$

Block stacking and the harmonic numbers II

Step 2: name the sequence. The series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is the *harmonic series*; its partial sums (3.1.1) are the *harmonic numbers* $\{1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \dots\}$. They diverge, but extraordinarily slowly: the estimate derived below places the threshold $H_n > 100$ at roughly 1.5×10^{43} terms.

This raises two questions we now answer.

- What natural analytic function interpolates the discrete sequence H_n?
- How slowly does the sequence diverge?

The harmonic series is the special case $p = 1$ of the p-series $\sum_{k=1}^{\infty} k^{-p}$, which reappears (optionally) as the zeta function.

Interpolating the partial sums I

Step 1: turn the finite sum into an integral. For a positive integer n, the finite geometric identity follows by direct multiplication:

$$\begin{aligned}(1-x)(1+x+x^2+\cdots+x^{n-1}) &= (1+x+\cdots+x^{n-1})-(x+x^2+\cdots+x^n)\\ &= 1-x^n.\end{aligned}$$

Hence, for $x \neq 1$,

$$\frac{1-x^n}{1-x} = 1+x+x^2+\cdots+x^{n-1}.$$

Both sides have the same continuous limit n at $x=1$. Integrating the polynomial term by term over $[0,1]$,

$$\begin{aligned}\int_0^1 \frac{1-x^n}{1-x}\,dx &= \sum_{k=0}^{n-1}\int_0^1 x^k\,dx = \sum_{k=0}^{n-1}\left[\frac{x^{k+1}}{k+1}\right]_0^1\\ &= \sum_{k=0}^{n-1}\frac{1}{k+1} = \sum_{j=1}^{n}\frac{1}{j} = H_n.\end{aligned}$$

Step 2: replace n by a continuous (or complex) variable. This motivates one natural analytic interpolation,

$$f(s) = \int_0^1 \frac{1-x^s}{1-x}\,dx, \qquad \operatorname{Re} s > -1, \tag{3.1.2}$$

Interpolating the partial sums II

where $\operatorname{Re} s$ is the real part of s, $x^s = e^{s \log x}$, and $\log x$ is the real logarithm for $0 < x < 1$. The domain condition is obtained from the endpoints:

- as $x \to 0^+$, the only potentially singular term is x^s, whose magnitude is $x^{\operatorname{Re} s}$, integrable precisely when $\operatorname{Re} s > -1$;
- as $x \to 1$, l'Hôpital's rule gives
$$\lim_{x\to 1} \frac{1-x^s}{1-x} = \lim_{x\to 1} \frac{-sx^{s-1}}{-1} = s,$$
so the apparent singularity at $x = 1$ is removable.

For $s = n \in \mathbb{N}$, the preceding finite-sum computation gives $f(n) = H_n$. The interpolation also satisfies the same first-order difference equation as the harmonic numbers:

$$\begin{aligned} f(s+1) - f(s) &= \int_0^1 \frac{(1-x^{s+1}) - (1-x^s)}{1-x}\,dx \\ &= \int_0^1 \frac{x^s - x^{s+1}}{1-x}\,dx = \int_0^1 x^s\,dx = \left[\frac{x^{s+1}}{s+1}\right]_0^1 = \frac{1}{s+1}, \end{aligned}$$

where the algebraic identity

$$\frac{x^s - x^{s+1}}{1-x} = x^s$$

was used (for $x \neq 1$; the values at the endpoints do not affect the integral). Together with $f(0) = 0$, this reproduces $f(n) = \sum_{k=1}^{n} 1/k$.

Step 3: prove that $H_n - \log n$ converges. Define

$$a_n = H_n - \log n.$$

Its successive difference is

$$\begin{aligned} a_{n+1} - a_n &= \left(H_n + \frac{1}{n+1} - \log(n+1)\right) - (H_n - \log n) \\ &= \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right). \end{aligned}$$

For $x > 0$, let

$$g(x) = \log(1+x) - \frac{x}{1+x}.$$

Then $g(0) = 0$ and

$$g'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0,$$

so $g(x) > 0$. Taking $x = 1/n$ yields

$$\log\left(1 + \frac{1}{n}\right) > \frac{1/n}{1+1/n} = \frac{1}{n+1},$$

and therefore $a_{n+1} < a_n$.

Interpolating the partial sums IV

On the other hand, because $1/x$ decreases,

$$\frac{1}{k} > \int_k^{k+1} \frac{dx}{x} \qquad (k = 1, \ldots, n),$$

so

$$H_n = \sum_{k=1}^{n} \frac{1}{k} > \int_1^{n+1} \frac{dx}{x} = \log(n+1) > \log n.$$

Thus $a_n > 0$. The sequence (a_n) is decreasing and bounded below, hence convergent. Its limit is the Euler–Mascheroni constant:

$$\gamma = \lim_{n\to\infty} (H_n - \log n) = 0.577215664901532\ldots. \tag{3.1.3}$$

Equivalently,

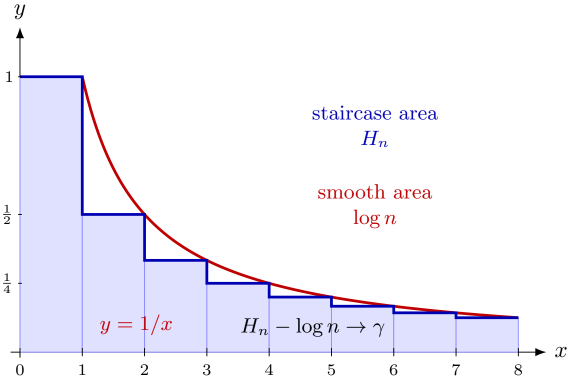
$$H_n = \log n + \gamma + o(1),$$

where $o(1)$ denotes a term that tends to zero as $n \to \infty$. Consequently the rough threshold $H_n \approx 100$ occurs near

$$n \approx e^{100-\gamma} = 1.509\ldots \times 10^{43},$$

which explains the extremely slow divergence.

Euler–Mascheroni as a staircase error



The harmonic number H_n is the area of a unit-width staircase. The logarithm $\log n = \int_1^n dx/x$ is the area under the smooth curve. The limiting excess area is $\gamma = \lim_{n\to\infty}\big(H_n - \log n\big)$.

Definition and the factorial recursion I

Step 1: define the canonical interpolation. Infinitely many functions can agree with $n!$ at the nonnegative integers, so interpolation alone does not imply uniqueness. The standard canonical choice is the *gamma function*, defined initially by Euler's integral of the second kind,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \operatorname{Re} z > 0. \tag{3.2.1}$$

Writing $\sigma = \operatorname{Re} z$, the endpoint conditions are explicit: near $t = 0$,

$$\left| e^{-t} t^{z-1} \right| = e^{-t} t^{\sigma-1} \sim t^{\sigma-1},$$

where $a(t) \sim b(t)$ means $a(t)/b(t) \to 1$ in the stated limit. Also, $\int_0^1 t^{\sigma-1} dt = 1/\sigma < \infty$ exactly when $\sigma > 0$. At infinity, the exponential factor dominates every power, so $\int_1^\infty e^{-t} t^{\sigma-1} dt < \infty$. Thus the integral converges absolutely for $\sigma > 0$, so $\Gamma(z)$ is at least well defined there. Convergence of the t-integral does not by itself make Γ a holomorphic function of the parameter z, so we record the short argument that it is. For $0 < \varepsilon < 1 < R < \infty$ truncate the integral,

$$\Gamma_{\varepsilon,R}(z) = \int_\varepsilon^R e^{-t} t^{z-1}\, dt.$$

For each fixed $t > 0$ the map $z \mapsto t^{z-1} = e^{(z-1)\log t}$ is entire, so for any closed triangle T Cauchy's theorem gives $\oint_{\partial T} t^{z-1}\, dz = 0$. On the compact set $\partial T \times [\varepsilon, R]$

the integrand $e^{-t}t^{z-1}$ is continuous, hence bounded, so the two integrations may be exchanged:

$$\oint_{\partial T} \Gamma_{\varepsilon,R}(z)\,dz = \int_{\varepsilon}^{R} e^{-t}\left(\oint_{\partial T} t^{z-1}\,dz\right) dt = 0.$$

Morera's theorem—the converse of Cauchy's theorem, quoted here rather than proved: a continuous function whose integral around every triangle vanishes is holomorphic—therefore makes each $\Gamma_{\varepsilon,R}$ entire. It remains to remove the truncation. Fix a strip $0 < \delta \le \sigma \le M$. Since $t^{\sigma-1} \le t^{\delta-1}$ for $0 < t \le 1$ and $t^{\sigma-1} \le t^{M-1}$ for $t \ge 1$,

$$\left|\int_0^{\varepsilon} e^{-t}t^{z-1}\,dt\right| \le \int_0^{\varepsilon} t^{\delta-1}\,dt = \frac{\varepsilon^{\delta}}{\delta}, \qquad \left|\int_R^{\infty} e^{-t}t^{z-1}\,dt\right| \le \int_R^{\infty} e^{-t}t^{M-1}\,dt.$$

Neither bound depends on z in the strip, and both tend to 0 as $\varepsilon \to 0$ and $R \to \infty$ (the second because $\int_1^{\infty} e^{-t}t^{M-1}dt < \infty$). Since $\Gamma - \Gamma_{\varepsilon,R}$ is exactly the sum of these two tails, $\Gamma_{\varepsilon,R} \to \Gamma$ uniformly on each such strip, hence uniformly on every compact subset of $\{\operatorname{Re} z > 0\}$, because any such compact set lies in a strip $0 < \delta \le \sigma \le M$. A limit of holomorphic functions that is uniform on compact sets is again holomorphic, so the integral defines a holomorphic function on $\{\operatorname{Re} z > 0\}$.

Definition and the factorial recursion III

Step 2: derive the recursion. Apply integration by parts to $\Gamma(z+1)$, taking

$$u = t^z, \qquad dv = e^{-t}dt, \qquad du = z\, t^{z-1}dt, \qquad v = -e^{-t}.$$

Then

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t} t^z \, dt \\ &= \left[-t^z e^{-t}\right]_0^\infty + z \int_0^\infty e^{-t} t^{z-1} \, dt.\end{aligned}$$

For $\sigma > 0$,

$$|t^z e^{-t}| = t^\sigma e^{-t} \longrightarrow 0 \quad (t \to \infty), \qquad |t^z e^{-t}| = t^\sigma e^{-t} \longrightarrow 0 \quad (t \to 0^+),$$

so the boundary term is zero. Therefore

$$\Gamma(z+1) = z\,\Gamma(z). \tag{3.2.2}$$

Step 3: recover factorials. At $z = 1$,

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = \left[-e^{-t}\right]_0^\infty = 1.$$

Definition and the factorial recursion IV

For $n \in \mathbb{N}$, repeated use of the recursion gives

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n) = n(n-1)\Gamma(n-1) \\ &= \cdots = n(n-1)\cdots 2\cdot 1\,\Gamma(1) = n!.\end{aligned}$$

Together with $\Gamma(1) = 1 = 0!$, this proves

$$\Gamma(n+1) = n!, \qquad n = 0, 1, 2, \ldots. \tag{3.2.3}$$

The shift is essential: $\Gamma(n+1) = n!$, whereas the related pi function $\Pi(z) = \Gamma(z+1)$ satisfies $\Pi(n) = n!$.

A second integral form follows from $t = u^2$. Since $dt = 2u\,du$ and $t^{z-1} = (u^2)^{z-1} = u^{2z-2}$ for $u > 0$,

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-u^2} u^{2z-2}(2u)\,du \\ &= 2\int_0^\infty e^{-u^2} u^{2z-1}\,du, \qquad \operatorname{Re} z > 0.\end{aligned}$$

Hence

$$\Gamma(z) = 2\int_0^\infty e^{-u^2} u^{2z-1}\,du. \tag{3.2.4}$$

Worked examples: half-integer values I

Step 1: evaluate the Gaussian integral. First evaluate the Gaussian integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

The integrand is positive, so $I > 0$. Squaring and writing the product as a double integral (justified by positivity / Tonelli),

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \\ &= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy. \end{aligned}$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the Jacobian is r, so

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta \\ &= \int_0^{2\pi}\left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=\infty} d\theta \\ &= \int_0^{2\pi}\frac{1}{2}\, d\theta = \pi. \end{aligned}$$

Because $I > 0$, $I = \sqrt{\pi}$.

Worked examples: half-integer values II

Step 2: insert $z = \frac{1}{2}$. Set $z = \frac{1}{2}$ in the Gaussian form (3.2.4). Since $2z - 1 = 0$,

$$\begin{aligned} \Gamma\left(\tfrac{1}{2}\right) &= 2\int_0^\infty e^{-u^2}\,du \\ &= \int_{-\infty}^\infty e^{-u^2}\,du = \sqrt{\pi}. \end{aligned}$$

Thus

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}. \tag{3.2.5}$$

One application of the recursion gives

$$\begin{aligned} \Gamma\left(\tfrac{3}{2}\right) &= \left(\tfrac{3}{2} - 1\right)\Gamma\left(\tfrac{3}{2} - 1\right) \\ &= \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}. \end{aligned}$$

More generally, for $m \in \mathbb{N}$,

$$\Gamma\left(m + \tfrac{1}{2}\right) = \left(m - \tfrac{1}{2}\right)\left(m - \tfrac{3}{2}\right)\cdots\frac{1}{2}\sqrt{\pi} = \frac{(2m)!}{4^m m!}\sqrt{\pi}.$$

To obtain the last equality, first write

$$\left(m - \frac{1}{2}\right)\left(m - \frac{3}{2}\right)\cdots\frac{1}{2} = \frac{(2m-1)(2m-3)\cdots 1}{2^m}.$$

Worked examples: half-integer values III

The odd product satisfies

$$1 \cdot 3 \cdot 5 \cdots (2m-1) = \frac{(2m)!}{2 \cdot 4 \cdots 2m} = \frac{(2m)!}{2^m m!}.$$

Therefore

$$\frac{(2m-1)(2m-3)\cdots 1}{2^m} = \frac{(2m)!}{2^m m!}\frac{1}{2^m} = \frac{(2m)!}{4^m m!}.$$

Analytic continuation to the left half-plane I

The defining integral (3.2.1) converges only for $\operatorname{Re} z > 0$, but the recursion can be solved backward:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{1}{z}\int_0^\infty e^{-t} t^z \, dt, \qquad \operatorname{Re} z > -1, \quad z \neq 0.$$

Repeating the recursion m times gives

$$\begin{aligned}\Gamma(z+m) &= (z+m-1)\Gamma(z+m-1) \\ &= (z+m-1)(z+m-2)\Gamma(z+m-2) \\ &= \cdots = (z+m-1)(z+m-2)\cdots(z+1)z\,\Gamma(z).\end{aligned}$$

If m is chosen so that $\operatorname{Re}(z+m) > 0$, then $\Gamma(z+m)$ is defined by the original integral and

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)(z+2)\cdots(z+m-1)}. \tag{3.2.6}$$

The value is independent of the admissible choice of m: replacing m by $m+1$ multiplies the numerator by $z+m$ through $\Gamma(z+m+1) = (z+m)\Gamma(z+m)$, and the denominator acquires the same factor. This gives a meromorphic continuation to the entire complex plane.

Analytic continuation to the left half-plane II

To see the nature of the singularity at $z = -n$, where $n \in \mathbb{N}_0$, set $z = -n + \varepsilon$ and choose $m = n + 1$:

$$\Gamma(-n+\varepsilon) = \frac{\Gamma(1+\varepsilon)}{(-n+\varepsilon)(1-n+\varepsilon)\cdots(-1+\varepsilon)\varepsilon}.$$

As $\varepsilon \to 0$,

$$\Gamma(1+\varepsilon) \to \Gamma(1) = 1$$

and, with the empty product interpreted as 1 when $n = 0$,

$$(-n)(1-n)\cdots(-1) = (-1)^n n!.$$

Therefore

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!\,\varepsilon} + O(1),$$

where $F = O(G)$ means that $|F/G|$ remains bounded in the stated limit. Thus every non-positive integer is a simple pole and

$$\operatorname*{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}. \tag{3.2.7}$$

Here $\operatorname{Res}_{z=z_0}$ denotes the complex residue at z_0.

A more efficient relation between positive and negative arguments is Euler's reflection formula.

Analytic continuation to the left half-plane III

Euler's reflection formula

For $z \notin \mathbb{Z}$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Write a for the variable z throughout the proof. It is enough first to prove $\Gamma(a)\Gamma(1-a) = \pi/\sin(\pi a)$ in the strip $0 < \operatorname{Re} a < 1$; the identity theorem then restores the statement for every $a \notin \mathbb{Z}$, which is the block's formula with a in place of z. Both gamma integrals then converge absolutely:

$$\begin{aligned}\Gamma(a)\Gamma(1-a) &= \left(\int_0^\infty t^{a-1}e^{-t}\,dt\right)\left(\int_0^\infty s^{-a}e^{-s}\,ds\right)\\ &= \int_0^\infty\int_0^\infty t^{a-1}s^{-a}e^{-(t+s)}\,dt\,ds.\end{aligned}$$

Absolute convergence justifies Fubini's theorem. For each fixed $s > 0$, set $t = su$. Then $dt = s\,du$, and

$$t^{a-1}s^{-a}\,dt = (su)^{a-1}s^{-a}(s\,du) = u^{a-1}\,du.$$

Analytic continuation to the left half-plane IV

Consequently

$$\begin{aligned}\Gamma(a)\Gamma(1-a) &= \int_0^\infty \int_0^\infty u^{a-1} e^{-s(1+u)}\, du\, ds \\ &= \int_0^\infty u^{a-1} \left(\int_0^\infty e^{-s(1+u)}\, ds \right) du \\ &= \int_0^\infty u^{a-1} \left[\frac{-e^{-s(1+u)}}{1+u} \right]_{s=0}^{s=\infty} du \\ &= \int_0^\infty \frac{u^{a-1}}{1+u}\, du.\end{aligned}$$

It remains to evaluate

$$I(a) = \int_0^\infty \frac{x^{a-1}}{1+x}\, dx, \qquad 0 < \operatorname{Re} a < 1.$$

Because z^{a-1} is multivalued for noninteger a, it needs a branch cut; place the cut along the positive real axis, where the integrand of $I(a)$ lives. Use a counterclockwise keyhole contour that respects this cut: outward from a small circle $|z| = \varepsilon$ to a large circle $|z| = R$ just above the cut, around $|z| = R$, back inward just below the cut, and around $|z| = \varepsilon$ to the start—a positively oriented closed loop enclosing the single pole $z = -1$ without ever crossing the cut. Fix the branch

$$z^{a-1} = \exp\big((a-1)(\log|z| + i\arg z)\big), \qquad 0 < \arg z < 2\pi.$$

Here $\arg z$ is the polar angle on the chosen branch. The same real point $x > 0$ carries $\arg z = 0$ on the upper side of the cut but $\arg z = 2\pi$ on the lower side, so the branch of z^{a-1} differs by $e^{2\pi i(a-1)} = e^{2\pi ia}$ between the two sides; this is what keeps the two straight parts from cancelling and leaves a nonzero result. For $f(z) = z^{a-1}/(1+z)$, the only pole inside the contour is $z = -1$. Since $1 + z$ has a simple zero there with derivative 1, the residue is just the numerator evaluated at the pole. Because $\arg(-1) = \pi$,

$$\operatorname*{Res}_{z=-1} f(z) = z^{a-1}\Big|_{z=-1} = (-1)^{a-1} = e^{i\pi(a-1)} = -e^{i\pi a}.$$

Let $\sigma = \operatorname{Re} a$. On the circle $|z| = R$,

$$|f(z)| = O(R^{\sigma-2}),$$

Analytic continuation to the left half-plane VI

so the integral over that circle is $O(R^{\sigma-1}) \to 0$ because $\sigma < 1$. On the circle $|z| = \varepsilon$,

$$|f(z)| = O(\varepsilon^{\sigma-1}),$$

so the integral is $O(\varepsilon^{\sigma}) \to 0$ because $\sigma > 0$.

On the upper side of the cut, $\arg z = 0$, and the contribution tends to $I(a)$. On the lower side, $\arg z = 2\pi$, the contour runs from R back to ε, and

$$z^{a-1} = x^{a-1} e^{2\pi i(a-1)} = x^{a-1} e^{2\pi i a}.$$

Hence the lower contribution tends to $-e^{2\pi i a} I(a)$. Adding the four pieces, with the two circular arcs vanishing, the closed keyhole integral equals $I(a) + \left(-e^{2\pi i a} I(a)\right) = (1 - e^{2\pi i a}) I(a)$. The residue theorem $\oint f = 2\pi i \operatorname{Res}_{z=-1} f$ then gives

$$\left(1 - e^{2\pi i a}\right) I(a) = 2\pi i \left(-e^{i\pi a}\right).$$

Now

$$1 - e^{2\pi i a} = e^{i\pi a}\left(e^{-i\pi a} - e^{i\pi a}\right) = -2i\, e^{i\pi a} \sin(\pi a),$$

and therefore

$$I(a) = \frac{-2\pi i\, e^{i\pi a}}{-2i\, e^{i\pi a} \sin(\pi a)} = \frac{\pi}{\sin(\pi a)}.$$

Thus

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

Analytic continuation to the left half-plane VII

in the strip $0 < \operatorname{Re} a < 1$. Both sides are meromorphic functions of a, so the identity theorem extends the formula to every $a \notin \mathbb{Z}$.

To obtain a formula convenient for negative arguments, put $a = -z$:

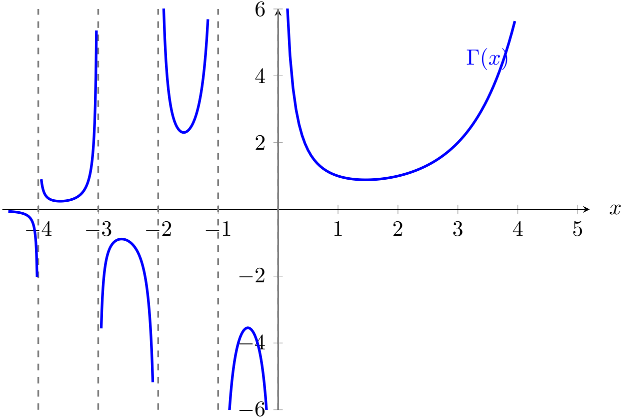
$$\Gamma(-z)\Gamma(1+z) = \frac{\pi}{\sin(-\pi z)} = -\frac{\pi}{\sin(\pi z)}.$$

Since $\Gamma(1+z) = \Gamma(z+1)$,

$$\Gamma(-z) = -\frac{\pi}{\Gamma(z+1)\sin(\pi z)}. \tag{3.2.8}$$

As a function of z, $\Gamma(-z)$ has poles at $z = 0, 1, 2, \ldots$; equivalently, $\Gamma(w)$ has poles at $w = 0, -1, -2, \ldots$.

Real-axis picture of Γ



On the positive real axis, $\Gamma(n+1) = n!$. Because $\Gamma(z) = \Gamma(z+n+1)/\big(z(z+1)\cdots(z+n)\big)$, running the recursion backward makes the factor $z+n$ produce a simple pole at each $z=-n$, with residue $(-1)^n/n!$; the alternating sign of that residue is why Γ flips sign from one pole-interval to the next.

Euler's limit representation I

For $n \in \mathbb{N}$ and $\operatorname{Re} z > 0$, define

$$F_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt. \tag{3.2.9}$$

Step 1: pointwise limit of the integrand. For each fixed $t \geq 0$, once $n > t$,

$$\begin{aligned} n \log\left(1 - \frac{t}{n}\right) &= n\left(-\frac{t}{n} - \frac{t^2}{2n^2} - \frac{t^3}{3n^3} - \cdots\right) \\ &= -t - \frac{t^2}{2n} - \frac{t^3}{3n^2} - \cdots \longrightarrow -t. \end{aligned}$$

Therefore

$$\left(1 - \frac{t}{n}\right)^n \longrightarrow e^{-t}.$$

Step 2: pass the limit through the integral (elementary majorant). Extend the integrand by zero for $t > n$. For $0 \leq t \leq n$, the inequality $\log(1 - y) \leq -y$ for $0 \leq y < 1$ gives

$$0 \leq \left(1 - \frac{t}{n}\right)^n = \exp\left[n \log\left(1 - \frac{t}{n}\right)\right] \leq e^{-t}.$$

Writing $\sigma = \operatorname{Re} z > 0$,

$$0 \leq \mathbf{1}_{[0,n]}(t)\left(1 - \frac{t}{n}\right)^n t^{\sigma-1} \leq e^{-t} t^{\sigma-1},$$

Euler's limit representation II

where $\mathbf{1}_A$ is the indicator of a set A. Fix $T > 0$. On the compact interval $[0, T]$, the integrand converges pointwise to $e^{-t}t^{z-1}$ and is dominated by the integrable function $e^{-t}t^{\sigma-1}$, so

$$\lim_{n\to\infty} \int_0^T \mathbf{1}_{[0,n]}(t)\left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = \int_0^T e^{-t}t^{z-1}\, dt.$$

The tail is uniformly small for all n:

$$\left| \int_T^\infty \mathbf{1}_{[0,n]}(t)\left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt \right| \le \int_T^\infty e^{-t}t^{\sigma-1}\, dt,$$

which tends to 0 as $T \to \infty$. Therefore

$$\lim_{n\to\infty} F_n(z) = \int_0^\infty e^{-t}t^{z-1}\, dt = \Gamma(z).$$

Step 3: evaluate the approximating integral exactly. Now set $t = nu$. Then $dt = n\, du$, $t^{z-1} = n^{z-1}u^{z-1}$, and

$$F_n(z) = n^z \int_0^1 (1-u)^n u^{z-1}\, du.$$

Define

$$I_j = \int_0^1 (1-u)^{n-j} u^{z+j-1}\, du, \qquad j = 0, 1, \dots, n.$$

Euler's limit representation III

For $j < n$, integrate by parts with

$$A = (1-u)^{n-j}, \qquad dB = u^{z+j-1}du, \qquad B = \frac{u^{z+j}}{z+j}.$$

Because $\mathrm{Re}(z+j) > 0$, the boundary term vanishes:

$$\left[(1-u)^{n-j}\frac{u^{z+j}}{z+j}\right]_{u=0}^{u=1} = 0.$$

Also,

$$dA = -(n-j)(1-u)^{n-j-1}du.$$

Hence

$$\begin{aligned} I_j &= -\int_0^1 B\,dA \\ &= \frac{n-j}{z+j}\int_0^1 (1-u)^{n-j-1}u^{z+j}\,du \\ &= \frac{n-j}{z+j}I_{j+1}. \end{aligned}$$

Iterating this identity from $j = 0$ through $j = n-1$,

Euler's limit representation IV

$$
\begin{aligned}
I_0 &= \frac{n}{z} I_1 \\
&= \frac{n}{z} \frac{n-1}{z+1} I_2 \\
&= \cdots = \frac{n!}{z(z+1)\cdots(z+n-1)} I_n.
\end{aligned}
$$

The remaining integral is

$$
I_n = \int_0^1 u^{z+n-1}\, du = \left[\frac{u^{z+n}}{z+n} \right]_0^1 = \frac{1}{z+n}.
$$

Thus

$$
F_n(z) = n^z I_0 = \frac{n^z n!}{z(z+1)\cdots(z+n)}.
$$

Combining this exact identity with the limit above gives

$$
\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\cdots(z+n)}, \qquad \operatorname{Re} z > 0. \tag{3.2.10}
$$

Euler's limit representation V

The same limit continues meromorphically away from the non-positive integers. For $z \notin \{0, -1, -2, \dots\}$ choose $m \in \mathbb{N}$ with $\mathrm{Re}(z+m) > 0$, apply (3.2.10) to $\Gamma(z+m)$, and divide by $z(z+1)\cdots(z+m-1)$ as in the continuation (3.2.6):

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)} = \lim_{N\to\infty} A_N, \qquad A_N = \frac{N^{z+m} N!}{z(z+1)\cdots(z+N+m)},$$

since the product $(z+m)(z+m+1)\cdots(z+m+N)$ supplied by (3.2.10) is completed by the divisor $z(z+1)\cdots(z+m-1)$ into $z(z+1)\cdots(z+N+m)$.

This is not yet the shape of (3.2.10): the numerator carries N^{z+m} rather than N^z, and the denominator stops at $z+N+m$ rather than $z+N$. Both discrepancies wash out in the limit. Write

$$B_M = \frac{M^z M!}{z(z+1)\cdots(z+M)}$$

for the approximant we want, and compare A_N with B_M at $M = N+m$, which is the choice that makes the two denominators literally the same product. Only the numerators then survive in the quotient:

$$\frac{A_N}{B_{N+m}} = \frac{N^{z+m} N!}{(N+m)^z (N+m)!} = \left(\frac{N}{N+m}\right)^z \frac{N^m}{(N+1)(N+2)\cdots(N+m)},$$

Euler's limit representation VI

where the last equality splits $N^{z+m} = N^z N^m$ and uses $N!/(N+m)! = 1/\big((N+1)(N+2)\cdots(N+m)\big)$. Each of the two factors tends to 1. The base $N/(N+m)$ is positive and tends to 1, so

$$\left(\frac{N}{N+m}\right)^z = \exp\left(z \log \frac{N}{N+m}\right) \longrightarrow e^0 = 1,$$

and, dividing each factor of the product by N,

$$\frac{N^m}{(N+1)(N+2)\cdots(N+m)} = \prod_{k=1}^{m}\left(1+\frac{k}{N}\right)^{-1} \longrightarrow 1,$$

because m is fixed, so this is a fixed finite number of factors each tending to 1. Therefore $B_{N+m} = A_N\big(A_N/B_{N+m}\big)^{-1} \to \Gamma(z)\cdot 1 = \Gamma(z)$, and as N runs through the integers $M = N+m$ runs through all sufficiently large integers, so

$$\Gamma(z) = \lim_{N\to\infty} \frac{N^z N!}{z(z+1)\cdots(z+N)}$$

everywhere the right-hand side is defined.

Summary of gamma formulas

$$
\begin{aligned}
\Gamma(z) &= \int_0^\infty e^{-t} t^{z-1}\,dt, && \operatorname{Re} z > 0,\\
\Gamma(z+1) &= z\Gamma(z),\\
\Gamma\left(\tfrac{1}{2}\right) &= \sqrt{\pi},\\
\Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)}, && z \notin \mathbb{Z},\\
\Gamma(z) &= \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\cdots(z+n)}, && z \notin \{0,-1,-2,\dots\}.
\end{aligned}
$$

For $n \in \mathbb{N}_0$, $\Gamma(n+1) = n!$. The meromorphic continuation has simple poles at $0, -1, -2, \ldots$, with residue $(-1)^n/n!$ at $z = -n$.

Worked examples: gamma integrals I

Example 1. Evaluate

$$\int_0^\infty e^{-x^p}\,dx, \qquad p > 0.$$

Set $t = x^p$. Then

$$x = t^{1/p}, \qquad dx = \frac{1}{p} t^{1/p-1}\,dt,$$

and the endpoints $x = 0, \infty$ map to $t = 0, \infty$. Therefore

$$\begin{aligned}\int_0^\infty e^{-x^p}\,dx &= \frac{1}{p}\int_0^\infty e^{-t} t^{1/p-1}\,dt \\ &= \frac{1}{p}\Gamma\left(\frac{1}{p}\right).\end{aligned}$$

Using

$$\Gamma\left(1+\frac{1}{p}\right) = \frac{1}{p}\Gamma\left(\frac{1}{p}\right),$$

we obtain

$$\int_0^\infty e^{-x^p}\,dx = \Gamma\left(1+\frac{1}{p}\right). \tag{3.2.11}$$

Worked examples: gamma integrals II

Example 2. For $p > 1$, evaluate $\int_0^\infty \cos(x^p)\,dx$. (The restriction $p > 1$ is needed for endpoint convergence at infinity: $\cos(x^p) = O(1)$ does not guarantee integrability of the oscillatory integral at ∞ when $p \le 1$, whereas $p > 1$ makes the phase stationary only at the origin and yields absolute convergence of the contour estimates below.) Write

$$\int_0^\infty \cos(x^p)\,dx = \operatorname{Re} \int_0^\infty e^{ix^p}\,dx.$$

Consider the sector contour $0 \le \arg z \le \pi/(2p)$ with outer radius R and inner radius ε. For noninteger p the symbol z^p is multivalued, so before integrating we fix the branch

$$z^p = \exp\bigl(p \operatorname{Log} z\bigr), \qquad \operatorname{Log} z = \log|z| + i \arg z, \quad |\arg z| < \pi,$$

the principal branch: it is analytic on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ and agrees with the real power x^p on the positive real axis, where $\arg z = 0$. Because $p > 1$ we have $\pi/(2p) < \pi/2$, so the whole sector $0 \le \arg z \le \pi/(2p)$ lies in the open right half plane and never comes near the cut; on it $\arg z$ is a continuous single-valued function and $z \mapsto e^{iz^p}$ is analytic for $z \ne 0$. The origin is a branch point when $p \notin \mathbb{N}$, which is precisely why the contour is closed off by a small arc of radius ε instead of being run through 0. The closed contour (segment $[\varepsilon, R]$, outer arc, ray inward, inner arc) therefore has an integrand analytic on and inside it and encloses no singularity, and that is what licenses the appeal to Cauchy's theorem below. On the slanted ray $z = re^{i\pi/(2p)}$,

$$z^p = r^p e^{i\pi/2} = ir^p, \qquad iz^p = i \cdot ir^p = -r^p, \qquad dz = e^{i\pi/(2p)}\, dr,$$

so

$$\int_{\mathrm{ray}} e^{iz^p}\, dz = e^{i\pi/(2p)} \int_0^\infty e^{-r^p}\, dr.$$

The substitution $t = r^p$, $r = t^{1/p}$, $dr = (1/p)t^{1/p-1}\, dt$ converts the real integral into a gamma value:

$$\int_0^\infty e^{-r^p}\, dr = \frac{1}{p}\int_0^\infty e^{-t} t^{1/p-1}\, dt = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) = \Gamma\left(1 + \frac{1}{p}\right).$$

On the outer arc $z = Re^{i\theta}$ with $0 \le \theta \le \pi/(2p)$,

$$iz^p = i\, R^p e^{ip\theta} = R^p e^{i(\pi/2 + p\theta)},$$

so

$$\mathrm{Re}(iz^p) = -R^p \sin(p\theta) \le 0$$

throughout the sector (since $p\theta \in [0, \pi/2]$ and $\sin \ge 0$ there). Thus

$$\left| e^{iz^p} \right| = e^{\mathrm{Re}(iz^p)} = e^{-R^p \sin(p\theta)} \le 1.$$

Worked examples: gamma integrals IV

The bound 1 by itself is not enough on the outer arc, whose length $R\pi/(2p)$ grows with R; we must keep the decaying exponential. With $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\, d\theta$, so that $|dz| = R\, d\theta$, the substitution $\varphi = p\theta$ (hence $d\theta = d\varphi/p$) gives

$$\left|\int_{\text{outer arc}} e^{iz^p}\, dz\right| \le R \int_0^{\pi/(2p)} e^{-R^p \sin(p\theta)}\, d\theta = \frac{R}{p}\int_0^{\pi/2} e^{-R^p \sin\varphi}\, d\varphi.$$

On $[0, \pi/2]$ the sine is concave, so its graph lies above the chord joining $(0, 0)$ to $(\pi/2, 1)$, that is $\sin\varphi \ge 2\varphi/\pi$; therefore $e^{-R^p \sin\varphi} \le e^{-2R^p\varphi/\pi}$ and

$$\frac{R}{p}\int_0^{\pi/2} e^{-2R^p\varphi/\pi}\, d\varphi = \frac{R}{p}\cdot\frac{\pi}{2R^p}\left(1 - e^{-R^p}\right) \le \frac{\pi}{2p}\, R^{1-p} \xrightarrow[R\to\infty]{} 0,$$

the limit holding because $p > 1$ makes the exponent $1 - p$ negative. This is where the hypothesis $p > 1$ actually does its work: at $p = 1$ the same bound is the constant $\pi/2$, the arc contribution does not drop out, and indeed $\int_0^\infty \cos x\, dx$ does not converge. The inner arc vanishes as $\varepsilon \to 0$ by length $\to 0$. By Cauchy's theorem the real-axis integral equals the ray integral, so

$$\int_0^\infty e^{ix^p}\, dx = e^{i\pi/(2p)}\, \frac{1}{p}\, \Gamma\left(\frac{1}{p}\right) = e^{i\pi/(2p)}\, \Gamma\left(1 + \frac{1}{p}\right).$$

Worked examples: gamma integrals V

Taking the real part yields

$$\int_0^\infty \cos(x^p)\,dx = \frac{1}{p}\,\Gamma\left(\frac{1}{p}\right)\cos\left(\frac{\pi}{2p}\right) = \Gamma\left(1+\frac{1}{p}\right)\cos\left(\frac{\pi}{2p}\right), \qquad p>1. \tag{3.2.12}$$

Stirling's approximation I

This is Laplace's method (Chapter 2): a large-parameter integral is dominated by a neighborhood of the maximum of the real phase, and that neighborhood contributes a Gaussian. Start from the gamma integral

$$n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t}\, dt.$$

Set $t = nx$, so $dt = n\, dx$. Then

$$\begin{aligned} n! &= \int_0^\infty (nx)^n e^{-nx}\, n\, dx \\ &= n^{n+1} \int_0^\infty e^{n \log x - nx}\, dx \\ &= n^{n+1} e^{-n} \int_0^\infty e^{n(\log x - x + 1)}\, dx. \end{aligned}$$

Define

$$h(x) = \log x - x + 1.$$

Its derivatives are

$$h'(x) = \frac{1}{x} - 1, \qquad h''(x) = -\frac{1}{x^2} < 0.$$

Stirling's approximation II

Thus h is strictly concave, has its unique maximum at $x = 1$, and

$$h(1) = 0, \qquad h''(1) = -1.$$

The mass of the integral is therefore concentrated near $x = 1$.

Localize near the maximum. Taylor's theorem with $h(1) = 0$, $h'(1) = 0$, $h''(1) = -1$ gives $n\,h(x) \approx -\frac{n}{2}(x-1)^2$ near the maximum, so $e^{nh(x)}$ stays of order one only while $(x-1)^2 \lesssim 1/n$, that is $|x-1| \sim n^{-1/2}$. This fixes the natural width of the peak; rescaling it to order one motivates writing

$$x = 1 + \frac{u}{\sqrt{n}}, \qquad dx = \frac{du}{\sqrt{n}}.$$

As x runs from 0 to ∞, u runs from $-\sqrt{n}$ to ∞. Hence

$$\int_0^\infty e^{nh(x)}\,dx = \frac{1}{\sqrt{n}} \int_{-\sqrt{n}}^\infty \exp\left[n\left(\log\left(1 + \frac{u}{\sqrt{n}}\right) - \frac{u}{\sqrt{n}}\right)\right] du.$$

Use the Taylor expansion

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots.$$

Stirling's approximation III

With $y = u/\sqrt{n}$,

$$n\left[\log\left(1+\frac{u}{\sqrt{n}}\right)-\frac{u}{\sqrt{n}}\right] = n\left[-\frac{u^2}{2n}+\frac{u^3}{3n^{3/2}}-\frac{u^4}{4n^2}+\cdots\right]$$
$$= -\frac{u^2}{2}+\frac{u^3}{3\sqrt{n}}-\frac{u^4}{4n}+\cdots.$$

For each fixed u, the exponent tends to $-u^2/2$.

Tails are negligible; reduce to a Gaussian. Away from $x = 1$, h is strictly negative. Quantitatively: fix $\delta \in (0,1)$. On the compact set $\{|x-1| \geq \delta\} \cap [0, M]$ one has $h \leq -c_\delta < 0$, so

$$\int_{\{|x-1|\geq\delta\}\cap[0,M]} e^{nh(x)}\,dx \leq Me^{-nc_\delta} = O(e^{-nc_\delta}).$$

For large x, $\log x - x + 1 \leq -x/2$ once $x \geq M$ is large enough, hence

$$\int_M^\infty e^{nh(x)}\,dx \leq \int_M^\infty e^{-nx/2}\,dx = \frac{2}{n}\,e^{-nM/2}.$$

Stirling's approximation IV

Both contributions are exponentially small in n. Near $x = 1$, after $u = \sqrt{n}(x-1)$, the integrand tends pointwise to $e^{-u^2/2}$, and Taylor with $h''(1) = -1$ supplies a local Gaussian majorant $e^{-c'u^2}$. Extending the limits $-\sqrt{n} \to -\infty$ therefore yields

$$\sqrt{n} \int_0^\infty e^{nh(x)}\, dx \longrightarrow \int_{-\infty}^\infty e^{-u^2/2}\, du = \sqrt{2\pi},$$

where the last equality uses $v = u/\sqrt{2}$ and $\int e^{-v^2}\, dv = \sqrt{\pi}$. Thus

$$\int_0^\infty e^{nh(x)}\, dx \sim \sqrt{\frac{2\pi}{n}}.$$

Substitute this into the exact scaled gamma integral:

$$\begin{aligned} n! &\sim n^{n+1} e^{-n} \sqrt{\frac{2\pi}{n}} \\ &= \sqrt{2\pi}\, n^{n+1-1/2} e^{-n} \\ &= \sqrt{2\pi}\, n^{n+1/2} e^{-n} \\ &= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \end{aligned}$$

Stirling's approximation V

Therefore

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \sqrt{2\pi}\, n^{n+1/2} e^{-n}. \tag{3.2.13}$$

Here $a_n \sim b_n$ means $a_n/b_n \to 1$. Taking logarithms gives the corresponding leading expansion

$$\log(n!) = n \log n - n + \frac{1}{2}\log(2\pi n) + o(1).$$

Definition and trigonometric form I

The *Euler beta function* is

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad \operatorname{Re} p > 0, \quad \operatorname{Re} q > 0. \tag{3.3.1}$$

The endpoint restrictions follow from a convergence test. For real $t > 0$, $|t^{p-1}| = |e^{(p-1)\log t}| = t^{\operatorname{Re} p - 1}$, while $|(1-t)^{q-1}| \to 1$ as $t \to 0$; since $\int_0^{1/2} t^{\,a}\,dt$ converges precisely when $a > -1$, integrability near $t = 0$ needs $\operatorname{Re} p - 1 > -1$, i.e. $\operatorname{Re} p > 0$. Symmetrically, near $t = 1$ the integrand behaves like $(1-t)^{\operatorname{Re} q - 1}$, so integrability there needs $\operatorname{Re} q > 0$.

The function is symmetric. Starting from $B(q,p)$, let $u = 1 - t$, so $dt = -du$:

$$\begin{aligned} B(q,p) &= \int_0^1 t^{q-1}(1-t)^{p-1}\,dt \\ &= \int_1^0 (1-u)^{q-1}u^{p-1}(-du) \\ &= \int_0^1 u^{p-1}(1-u)^{q-1}\,du \\ &= B(p,q). \end{aligned}$$

Definition and trigonometric form II

For nonnegative integers m, n, set $I_{m,n} = \int_0^1 t^m(1-t)^n\,dt = B(m+1, n+1)$. Then $I_{m,0} = \int_0^1 t^m\,dt = 1/(m+1)$. For $n \geq 1$, integrate by parts with

$$u = (1-t)^n, \qquad dv = t^m\,dt, \qquad du = -n(1-t)^{n-1}dt, \qquad v = \frac{t^{m+1}}{m+1}.$$

Then

$$I_{m,n} = \left[\frac{t^{m+1}}{m+1}(1-t)^n\right]_0^1 - \int_0^1 \frac{t^{m+1}}{m+1}\left(-n(1-t)^{n-1}\right)dt.$$

The bracket vanishes at both endpoints: at $t = 1$ because $(1-t)^n = 0$ there for $n \geq 1$, and at $t = 0$ because $t^{m+1} = 0$ there for $m \geq 0$. In the remaining integral the minus sign of the integration-by-parts formula cancels the minus sign carried by du, leaving the recursion

$$I_{m,n} = \frac{n}{m+1}\int_0^1 t^{m+1}(1-t)^{n-1}\,dt = \frac{n}{m+1}I_{m+1,n-1}.$$

Apply the recursion twice more:

$$\begin{aligned} I_{m,n} &= \frac{n}{m+1}I_{m+1,n-1} = \frac{n}{m+1}\cdot\frac{n-1}{m+2}I_{m+2,n-2} \\ &= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdot\frac{n-2}{m+3}I_{m+3,n-3}. \end{aligned}$$

After n steps the second index reaches 0:

$$I_{m,n} = \frac{n(n-1)\cdots 1}{(m+1)(m+2)\cdots(m+n)}\, I_{m+n,0} = \frac{n!}{(m+1)\cdots(m+n)} \cdot \frac{1}{m+n+1}.$$

The rising product is $(m+1)\cdots(m+n) = (m+n)!/m!$, so

$$B(m+1,n+1) = I_{m,n} = \frac{m!\,n!}{(m+n+1)!} = \frac{1}{(m+n+1)\binom{m+n}{m}}. \tag{3.3.2}$$

For the trigonometric form, set

$$t = \sin^2\theta, \qquad dt = 2\sin\theta\cos\theta\,d\theta.$$

As t runs from 0 to 1, θ runs from 0 to $\pi/2$, and

$$1-t = 1-\sin^2\theta = \cos^2\theta.$$

Therefore

$$t^{p-1} = (\sin^2\theta)^{p-1} = \sin^{2p-2}\theta,$$
$$(1-t)^{q-1} = (\cos^2\theta)^{q-1} = \cos^{2q-2}\theta.$$

Definition and trigonometric form IV

Substitution into (3.3.1) yields

$$\begin{aligned} B(p,q) &= \int_0^{\pi/2} \sin^{2p-2}\theta\ \cos^{2q-2}\theta\, 2\sin\theta\cos\theta\, d\theta \\ &= 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta\, d\theta. \qquad (3.3.3) \end{aligned}$$

Beta in terms of gamma I

Beta–gamma identity

For $\operatorname{Re} p > 0$ and $\operatorname{Re} q > 0$,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Use the Gaussian gamma representation (3.2.4):

$$\Gamma(p) = 2\int_0^\infty e^{-x^2} x^{2p-1}\,dx, \qquad \Gamma(q) = 2\int_0^\infty e^{-y^2} y^{2q-1}\,dy.$$

Because the product is absolutely integrable for the stated parameter range, Fubini's theorem gives

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2p-1} y^{2q-1}\,dx\,dy.$$

In the first quadrant, set

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r \ge 0, \quad 0 \le \theta \le \frac{\pi}{2}.$$

The Jacobian determinant is

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Beta in terms of gamma II

Also,

$$x^{2p-1} = r^{2p-1} \cos^{2p-1} \theta,$$
$$y^{2q-1} = r^{2q-1} \sin^{2q-1} \theta.$$

Hence

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4 \int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2p-1} r^{2q-1} r \\ &\qquad \times \cos^{2p-1} \theta \sin^{2q-1} \theta \, d\theta \, dr \\ &= 4 \left(\int_0^\infty e^{-r^2} r^{2(p+q)-1} \, dr \right) \left(\int_0^{\pi/2} \cos^{2p-1} \theta \sin^{2q-1} \theta \, d\theta \right).\end{aligned}$$

Split $4 = 2 \cdot 2$. The radial factor is

$$2 \int_0^\infty e^{-r^2} r^{2(p+q)-1} \, dr = \Gamma(p+q).$$

The angular factor is, by (3.3.3),

$$2 \int_0^{\pi/2} \sin^{2q-1} \theta \cos^{2p-1} \theta \, d\theta = B(q,p).$$

Beta in terms of gamma III

Since $B(q, p) = B(p, q)$,

$$\Gamma(p)\Gamma(q) = \Gamma(p + q)B(p, q).$$

Here $\mathrm{Re}(p + q) > 0$, so $\Gamma(p + q)$ is finite. It is also nonzero: at positive integers this follows from the factorial values, and at nonintegers it follows from the reflection formula, whose right-hand side is nonzero. Division by $\Gamma(p + q)$ therefore gives

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}. \tag{3.3.4}$$

Summary of beta formulas

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)},$$

$$B(m+1,n+1) = \frac{m!\,n!}{(m+n+1)!} = \frac{1}{(m+n+1)\binom{m+n}{m}},$$

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \quad \operatorname{Re} p, \operatorname{Re} q > 0,$$

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta\,\cos^{2q-1}\theta\,d\theta.$$

The first line lets us evaluate many integrals in closed form once we recognize their $t^{p-1}(1-t)^{q-1}$ shape.

Example 1. Evaluate

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

Set $u = x^4$. Then

$$x = u^{1/4}, \qquad dx = \frac{1}{4}u^{-3/4}\,du,$$

and the limits remain 0 and 1. Hence

$$\begin{aligned} I &= \frac{1}{4}\int_0^1 u^{-3/4}(1-u)^{-1/2}\,du \\ &= \frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right) \\ &= \frac{1}{4}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} \\ &= \frac{\sqrt{\pi}\,\Gamma(1/4)}{4\Gamma(3/4)}. \end{aligned}$$

The reflection formula at $z = \frac{1}{4}$ gives

$$\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = \frac{\pi}{\sin(\pi/4)} = \pi\sqrt{2}.$$

Thus $1/\Gamma(3/4) = \Gamma(1/4)/(\pi\sqrt{2})$, so

$$I = \frac{\sqrt{\pi}\,\Gamma(1/4)}{4} \cdot \frac{1}{\Gamma(3/4)} = \frac{\sqrt{\pi}\,\Gamma(1/4)}{4} \cdot \frac{\Gamma(1/4)}{\pi\sqrt{2}} = \frac{\sqrt{\pi}}{4\pi\sqrt{2}}\,\Gamma\left(\tfrac{1}{4}\right)^2.$$

Writing $\pi = (\sqrt{\pi})^2$ in the denominator cancels one factor of $\sqrt{\pi}$:

$$\frac{\sqrt{\pi}}{\pi\sqrt{2}} = \frac{\sqrt{\pi}}{(\sqrt{\pi})^2\sqrt{2}} = \frac{1}{\sqrt{\pi}\,\sqrt{2}} = \frac{1}{\sqrt{2\pi}},$$

and therefore

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}}. \tag{3.3.5}$$

Example 2. Evaluate

$$I_n = \int_0^\pi \sin^n x\,dx, \qquad n > -1.$$

The condition $n > -1$ ensures integrability at 0 and π. Since $\sin(\pi - x) = \sin x$,

$$I_n = 2\int_0^{\pi/2} \sin^n x\,dx.$$

Worked examples: beta integrals III

Compare this with

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x \, dx.$$

To obtain $\sin^n x$ and no cosine factor, solve

$$2p - 1 = n, \qquad 2q - 1 = 0.$$

Therefore

$$p = \frac{n+1}{2}, \qquad q = \frac{1}{2}.$$

It follows that

$$\begin{aligned} I_n &= B\left(\frac{n+1}{2}, \frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \\ &= \sqrt{\pi}\,\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \end{aligned}$$

Worked examples: beta integrals IV

Hence

$$\int_0^{\pi} \sin^n x \, dx = \sqrt{\pi} \, \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}. \tag{3.3.6}$$

From the p-series to $\zeta(s)$ I

Step 1: define ζ and check convergence. For $s \in \mathbb{C}$ with $\sigma = \operatorname{Re} s > 1$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad n^{-s} = e^{-s \log n}. \tag{3.4.1}$$

The series converges absolutely. Since $x^{-\sigma}$ decreases for $\sigma > 1$, the integral test gives

$$\sum_{n=1}^{\infty} |n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma} \le 1 + \int_1^{\infty} x^{-\sigma}\, dx = 1 + \frac{1}{\sigma - 1} < \infty.$$

At $\sigma = 1$ the series is the divergent harmonic series; for $\sigma \le 1$ the general term does not tend to zero fast enough for absolute convergence.

Step 2: convert the series into an integral. Weighting by t^{s-1} and integrating over $t \in (0, \infty)$ produces

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt, \qquad \operatorname{Re} s > 1. \tag{3.4.2}$$

The same condition is visible from the integral. As $t \to 0$,

$$e^t - 1 = t + O(t^2),$$

so

$$\frac{t^{s-1}}{e^t - 1} = t^{s-2}\big(1 + O(t)\big),$$

whose magnitude is locally integrable precisely when $\sigma > 1$. At infinity,

$$\left| \frac{t^{s-1}}{e^t - 1} \right| = O(t^{\sigma-1}e^{-t}),$$

which is integrable for every fixed σ.

Step 3: expand and integrate term by term. To prove (3.4.2), use the geometric series. For $t > 0$,

$$\begin{aligned} \frac{1}{e^t - 1} &= \frac{e^{-t}}{1 - e^{-t}} \\ &= e^{-t} \sum_{k=0}^{\infty} e^{-kt} \\ &= \sum_{n=1}^{\infty} e^{-nt}. \end{aligned}$$

The interchange of summation and integration is justified by absolute convergence:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left| t^{s-1} e^{-nt} \right| \, dt = \sum_{n=1}^{\infty} \int_0^{\infty} t^{\sigma-1} e^{-nt} \, dt.$$

From the p-series to $\zeta(s)$ III

In the n-th integral set $u = nt$, so $t = u/n$ and $dt = du/n$:

$$\begin{aligned}\int_0^\infty t^{\sigma-1}e^{-nt}\,dt &= \int_0^\infty \left(\frac{u}{n}\right)^{\sigma-1} e^{-u}\,\frac{du}{n} \\ &= n^{-\sigma}\int_0^\infty u^{\sigma-1}e^{-u}\,du \\ &= n^{-\sigma}\Gamma(\sigma).\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty}\int_0^\infty \left|t^{s-1}e^{-nt}\right|\,dt = \Gamma(\sigma)\zeta(\sigma) < \infty.$$

Fubini's theorem now gives

$$\int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt = \sum_{n=1}^{\infty}\int_0^\infty t^{s-1}e^{-nt}\,dt.$$

From the p-series to $\zeta(s)$ IV

Again put $u = nt$, now retaining the complex exponent:

$$\begin{aligned}\int_0^\infty t^{s-1}e^{-nt}\,dt &= \int_0^\infty \left(\frac{u}{n}\right)^{s-1} e^{-u}\frac{du}{n}\\ &= n^{-s}\int_0^\infty u^{s-1}e^{-u}\,du\\ &= n^{-s}\Gamma(s).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^\infty \frac{t^{s-1}}{e^t-1}\,dt &= \Gamma(s)\sum_{n=1}^\infty n^{-s}\\ &= \Gamma(s)\zeta(s).\end{aligned}$$

Division by $\Gamma(s)$ proves (3.4.2).

The Euler product over primes I

Let $s \in \mathbb{C}$ with $\sigma = \operatorname{Re} s > 1$. Multiplying the zeta series by 2^{-s} gives

$$2^{-s}\zeta(s) = 2^{-s} + 4^{-s} + 6^{-s} + 8^{-s} + \cdots,$$

because $2^{-s}n^{-s} = (2n)^{-s}$. Subtracting this from

$$\zeta(s) = 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$$

removes exactly the terms with even indices:

$$(1 - 2^{-s})\zeta(s) = \sum_{\substack{m \geq 1 \\ 2 \nmid m}} m^{-s}.$$

Here $a \mid m$ means that a divides m, while $a \nmid m$ means that it does not. Continuing through finitely many primes removes all integers divisible by any prime in that finite set.

A direct finite-product expansion makes the limiting argument precise. Each factor is about to be replaced by a geometric series, so we first check that its ratio is small. Since $\log p$ is real, the definition $p^{-s} = e^{-s \log p}$ gives

$$\left|p^{-s}\right| = \left|e^{-s \log p}\right| = e^{\operatorname{Re}(-s \log p)} = e^{-\sigma \log p} = p^{-\sigma}.$$

Every prime obeys $p \geq 2$, and $x \mapsto x^{-\sigma}$ decreases for $\sigma > 0$, so

$$\left|p^{-s}\right| = p^{-\sigma} \leq 2^{-\sigma} < \tfrac{1}{2} < 1, \qquad \sigma > 1.$$

The Euler product over primes II

The geometric series $1/(1-x) = \sum_{k=0}^{\infty} x^k$, valid for $|x| < 1$, therefore applies with $x = p^{-s}$, and it converges absolutely because $\sum_{k \geq 0} |p^{-ks}| = 1/(1-p^{-\sigma}) < \infty$. For a bound P,

$$\prod_{p \leq P} \frac{1}{1-p^{-s}} = \prod_{p \leq P} \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots\right).$$

Only finitely many absolutely convergent series are being multiplied here, so the product may be expanded term by term and the resulting terms collected in any order. Each choice of exponents $a_p \in \mathbb{N}_0$ contributes

$$\prod_{p \leq P} p^{-a_p s} = \left(\prod_{p \leq P} p^{a_p}\right)^{-s}.$$

By unique factorization, different exponent choices give different integers, and every integer whose prime factors are at most P occurs exactly once. Hence

$$\prod_{p \leq P} \frac{1}{1-p^{-s}} = \sum_{\substack{m \geq 1 \\ \text{all prime divisors of } m \leq P}} m^{-s}.$$

The Euler product over primes III

The omitted integers have a prime divisor larger than P, so in particular $m > P$. Therefore the absolute error is bounded by

$$\sum_{m>P} |m^{-s}| = \sum_{m>P} m^{-\sigma} \longrightarrow 0.$$

It follows that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \qquad \operatorname{Re} s > 1. \tag{3.4.3}$$

Absolute convergence in this half-plane justifies all rearrangements.

Basel value (and a cultural note)

A classical evaluation (Fourier series of x^2, or residue calculus with $\pi\cot(\pi z)/z^2$) gives Basel's identity

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We quote the value here rather than re-deriving it. We do *not* develop meromorphic continuation of ζ in this chapter. The main point is that gamma supplies the weight in (3.4.2), so ζ lives in the same integral language as Γ and B.

Riemann Hypothesis

Every nontrivial zero of the continued $\zeta(s)$ is conjectured to lie on $\operatorname{Re} s = \frac{1}{2}$. Unsolved.

Where we have been

- **Gamma:** $\Gamma(n+1) = n!$, $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ for $z \notin \mathbb{Z}$; poles at $0, -1, -2, \ldots$ with residue $(-1)^n/n!$; Euler limit and Stirling $n! \sim \sqrt{2\pi n}\,(n/e)^n$.
- **Beta:**
$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad \operatorname{Re} p, \operatorname{Re} q > 0.$$
- **Zeta:**
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\,dt, \qquad \operatorname{Re} s > 1,$$
with the classical value $\zeta(2) = \pi^2/6$ quoted (not re-derived here).

Gamma and beta are the core language; later special functions are largely built from them.