Chapter 2: Complex Analysis

Special Functions of Mathematical Physics

after K. A. Novak
Compiled by Jake W. Liu

July 27, 2026

Outline

Where we are going

To analyze the special functions of physics we first build the tools of *complex analysis*.

- **Taylor and Laurent series** represent a function near a point, even near a singularity.
- **Analytic functions** (complex differentiable) obey the *Cauchy–Riemann equations* and admit power-series expansions.
- **Cauchy's integral theorem** and the **residue theorem** turn contour integrals into a count of poles—the machine used later for Gamma/Beta identities and real integrals.
- A short catalog of **contour templates** (semicircle, keyhole, Jordan) shows how orientation and vanishing-arc estimates convert complex residues into real answers.
- The **method of steepest descent** extracts large-parameter asymptotics of contour integrals. Its real special cases (Laplace method, stationary phase) reappear for Stirling (Ch. 3) and for large-argument Bessel expansions (Ch. 6).

Matching a polynomial to a function I

Let

$$p_N(x) = \sum_{j=0}^{N} a_j x^j.$$

We choose its coefficients so that it matches a sufficiently differentiable function f at $x = 0$. For $0 \leq k \leq N$, differentiate term by term. Each monomial $a_j x^j$ with $j < k$ has degree below k, so differentiating it k times gives 0; only the terms with $j \geq k$ survive, which is why the sum now begins at $j = k$:

$$p_N^{(k)}(x) = \sum_{j=k}^{N} a_j\, j(j-1)\cdots(j-k+1)x^{j-k} = \sum_{j=k}^{N} a_j \frac{j!}{(j-k)!} x^{j-k}.$$

At $x = 0$, every term with $j > k$ contains the positive power 0^{j-k} and vanishes, whereas the $j = k$ term equals $a_k k!$. Therefore

$$p_N^{(k)}(0) = k! a_k.$$

Imposing $p_N^{(k)}(0) = f^{(k)}(0)$ yields, one coefficient at a time,

$$a_k = \frac{f^{(k)}(0)}{k!}, \qquad k = 0, 1, \ldots, N. \tag{2.1.1}$$

Matching a polynomial to a function II

For example,

$$a_0 = f(0), \qquad a_1 = f'(0), \qquad a_2 = \frac{f''(0)}{2!}, \qquad a_3 = \frac{f^{(3)}(0)}{3!}.$$

The resulting degree-N Taylor polynomial about 0 is

$$T_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k.$$

About an arbitrary center x_0, apply the preceding calculation to the shifted variable $h = x - x_0$ and the function $F(h) = f(x_0 + h)$. Since $F^{(k)}(0) = f^{(k)}(x_0)$,

$$T_N(x; x_0) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k. \tag{2.1.2}$$

If f has $N+1$ continuous derivatives on the closed interval with endpoints x_0 and x, Taylor's theorem from single-variable calculus tells us exactly how much (2.1.2) misses f. Give that error a name,

$$R_N(x) := f(x) - T_N(x; x_0), \qquad \text{equivalently} \qquad f(x) = T_N(x; x_0) + R_N(x),$$

Matching a polynomial to a function III

and the theorem states that there is a point ξ strictly between x_0 and x — its location depending on both N and x, and not known explicitly — such that

$$R_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}\,(x - x_0)^{N+1}. \tag{2.1.3}$$

This is the *Lagrange remainder*: all the unknown information about f has been pushed into the single point ξ, so any bound on $|f^{(N+1)}|$ between x_0 and x bounds the error. Now let N grow and consider the formal series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{2.1.4}$$

Its partial sum through $k = N$ is precisely $T_N(x; x_0)$, so the identity $f = T_N + R_N$ says that the distance from $f(x)$ to that partial sum equals $|R_N(x)|$. Therefore (2.1.4) converges to $f(x)$ if and only if $R_N(x) \to 0$ as $N \to \infty$ with x held fixed. (C^∞ alone is not enough: the flat function e^{-1/x^2} for $x > 0$, extended by 0 at the origin, is a standard counterexample.)

Exponential. The function $f(x) = e^x$ satisfies $f^{(k)}(x) = e^x$ for every $k \geq 0$, so $f^{(k)}(0) = 1$. Substitution in (2.1.4) gives

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots . \tag{2.1.5}$$

For the absolute values of consecutive terms,

$$\frac{|x|^{k+1}/(k+1)!}{|x|^k/k!} = \frac{|x|}{k+1} \longrightarrow 0.$$

The ratio test therefore proves convergence for every real or complex x.

Convergence of the series is not yet representation, so we check the criterion itself. For real x every derivative of e^x is again e^x, so with $x_0 = 0$ the point ξ supplied by (2.1.3) lies between 0 and x and contributes $f^{(N+1)}(\xi) = e^{\xi}$. Since e^t is increasing and $\xi \leq |x|$, we get $e^{\xi} \leq e^{|x|}$, hence

$$|R_N(x)| = \frac{e^{\xi}\,|x|^{N+1}}{(N+1)!} \leq e^{|x|}\,\frac{|x|^{N+1}}{(N+1)!} \xrightarrow[N\to\infty]{} 0,$$

because $|x|^{N+1}/(N+1)!$ is the general term of the series just shown to converge, the terms of a convergent series tend to 0, and $e^{|x|}$ is a constant for fixed x. So $R_N(x) \to 0$ for every real x and (2.1.5) is a genuine identity. For complex x no remainder question arises: the right-hand side of (2.1.5), which the ratio test has just shown to converge everywhere, is what we shall *take* as the definition of e^x in Sec. 2.2, where it produces Euler's formula.

Cosine. Successive derivatives cycle as

$$\cos x,\ -\sin x,\ -\cos x,\ \sin x,\ \cos x, \ldots.$$

At $x = 0$ the *derivative values* therefore repeat with period 4,

$$f^{(k)}(0) = 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1, \ldots \qquad \text{that is} \qquad f^{(k)}(0) = \begin{cases} (-1)^{k/2}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

The Taylor coefficients are these divided by $k!$, as (2.1.1) requires, so the odd-order terms drop out of (2.1.4) and only the even indices survive. Writing the surviving even index as $2k$, so that k now counts the surviving terms, and using $(-1)^{(2k)/2} = (-1)^k$,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \qquad (2.1.6)$$

Sine. Successive derivatives cycle as

$$\sin x,\ \cos x,\ -\sin x,\ -\cos x,\ \sin x, \ldots.$$

At $x = 0$ the derivative values are $f^{(k)}(0) = 0, 1, 0, -1, 0, 1, \ldots$, that is $f^{(k)}(0) = 0$ for even k and $f^{(k)}(0) = (-1)^{(k-1)/2}$ for odd k. Dividing by $k!$ and writing the surviving odd index as $2k+1$ turns (2.1.4) into

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots . \tag{2.1.7}$$

Both series converge for all x, by the ratio test or by comparison with the absolutely convergent exponential series.

Again convergence is not representation, and again the remainder settles it. Every derivative of sin or cos is one of $\pm\sin, \pm\cos$, so $|f^{(N+1)}(t)| \leq 1$ for every real t, and in particular at the point ξ that Taylor's theorem supplies, wherever between 0 and x it happens to lie. With $x_0 = 0$, (2.1.3) therefore gives, for each fixed real x,

$$|R_N(x)| = \frac{|f^{(N+1)}(\xi)|}{(N+1)!} |x|^{N+1} \leq \frac{|x|^{N+1}}{(N+1)!} \xrightarrow[N\to\infty]{} 0,$$

since $|x|^{N+1}/(N+1)!$ is the general term of the convergent series $\sum_{k\geq 0} |x|^k/k!$ and the terms of a convergent series tend to 0. Hence $R_N(x) \to 0$ at every real x, and (2.1.6) and (2.1.7) are genuine identities, not merely convergent series — which is what licenses their use in Sec. 2.2 to identify the real and imaginary parts of $e^{i\theta}$.

Imaginary unit, Cartesian and polar form I

Define the imaginary unit i by $i^2 = -1$; the other square root of -1 is $-i$. A complex number has Cartesian form

$$z = x + iy, \qquad x = \operatorname{Re} z, \quad y = \operatorname{Im} z,$$

where $x, y \in \mathbb{R}$; $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of z. The powers of i repeat with period four:

$$i^{4q} = 1, \qquad i^{4q+1} = i, \qquad i^{4q+2} = -1, \qquad i^{4q+3} = -i.$$

Substitute $i\theta$ into the absolutely convergent exponential series and separate even and odd powers:

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{i^{2k}\theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1}\theta^{2k+1}}{(2k+1)!} \end{aligned}$$

and since $i^{2k} = (i^2)^k = (-1)^k$ and $i^{2k+1} = i\, i^{2k} = i(-1)^k$,

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k+1}}{(2k+1)!}.$$

Imaginary unit, Cartesian and polar form II

Comparison with (2.1.6)–(2.1.7) gives Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{2.2.1}$$

The substitution just made deserves one word of justification, because it is the hinge for everything that follows. The ratio test above showed that $\sum_k x^k/k!$ converges absolutely for every complex x, so we may simply *define* the exponential of a complex number by that series, $e^w := \sum_{n\geq 0} w^n/n!$; for real w this reproduces (2.1.5). With that definition the familiar addition law survives. Both series converge absolutely, so they may be multiplied out and the products regrouped by total degree $n = j + m$ (a Cauchy product):

$$\begin{aligned} e^{w_1}e^{w_2} &= \Big(\sum_{j=0}^{\infty}\frac{w_1^j}{j!}\Big)\Big(\sum_{m=0}^{\infty}\frac{w_2^m}{m!}\Big) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{w_1^j}{j!}\,\frac{w_2^{\,n-j}}{(n-j)!} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}w_1^j w_2^{\,n-j} = \sum_{n=0}^{\infty}\frac{(w_1+w_2)^n}{n!} = e^{w_1+w_2}, \end{aligned}$$

where the third equality only rewrites $1/\big(j!\,(n-j)!\big) = \binom{n}{j}/n!$ and the fourth is the binomial theorem. Two special cases are used constantly below. Taking $w_1 = u$ and $w_2 = iv$ with $u, v \in \mathbb{R}$ and then applying (2.2.1) to the second factor,

$$e^{u+iv} = e^u e^{iv} = e^u(\cos v + i\sin v), \tag{2.2.2}$$

Imaginary unit, Cartesian and polar form III

which splits the complex exponential into a real modulus e^u and a unit-modulus phase. Taking instead $w_1 = i\theta_1$ and $w_2 = i\theta_2$ with $\theta_1, \theta_2 \in \mathbb{R}$ gives $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, the statement that phases add.

For $z \neq 0$, set

$$r = |z| = \sqrt{x^2 + y^2} > 0.$$

Choose any angle θ satisfying $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{2.2.3}$$

The angle is not unique: if θ is one argument, then every argument is $\theta + 2\pi k$, $k \in \mathbb{Z}$. The principal argument is commonly chosen as $\operatorname{Arg} z \in (-\pi, \pi]$ (other endpoint conventions are also used). The number $z = 0$ has modulus 0 but no defined argument.

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then the phases add by the addition law above, so

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)},$$

so moduli multiply and arguments add modulo 2π.

The complex conjugate is

$$\bar{z} = x - iy = re^{-i\theta}.$$

Imaginary unit, Cartesian and polar form IV

Consequently

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = r^2 = |z|^2,$$

and adding or subtracting z and $\bar{z}$ isolates the Cartesian components:

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im} z = \frac{z - \bar{z}}{2i}. \tag{2.2.4}$$

Replacing θ by $-\theta$ in Euler's formula gives $e^{-i\theta} = \cos\theta - i\sin\theta$. Hence

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= 2\cos\theta, \\ e^{i\theta} - e^{-i\theta} &= 2i\sin\theta, \end{aligned}$$

and therefore

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{2.2.5}$$

Multivalued functions and branch cuts I

An inverse relation need not define a single-valued function. For example, $y = x^2$ gives $x = \pm\sqrt{y}$ for $y > 0$. A *branch* selects one value continuously on a chosen domain; the usual real principal square root selects $+\sqrt{y}$ for $y \geq 0$.

Likewise, if $w = \sin x$ with $|w| < 1$, then on each period $\sin x = w$ is attained twice: once where sin is increasing, $x = \arcsin w + 2n\pi$, and once where it is decreasing, $x = \pi - \arcsin w + 2n\pi$. Setting $k = 2n$ in the first family and $k = 2n+1$ in the second merges both into a single formula,

$$x = (-1)^k \arcsin w + k\pi, \qquad k \in \mathbb{Z},$$

since $(-1)^{2n+1} \arcsin w + (2n+1)\pi = \pi - \arcsin w + 2n\pi$. Here the principal real branch has $\arcsin w \in [-\pi/2, \pi/2]$ for $w \in [-1, 1]$. For $w = 0$, this formula gives $x = k\pi$.

The two endpoints $w = \pm 1$ are the exception to the count, and they are worth a moment because they are exactly where the two branches meet. There $\arcsin(\pm 1) = \pm\pi/2$, so $\pi - \arcsin w + 2n\pi = \arcsin w + 2n\pi$ up to a multiple of 2π: the increasing and decreasing families collapse onto the single set of turning points $x = \pm\pi/2 + 2n\pi$, where $\cos x = 0$ and sin is stationary. Each period then contributes one solution rather than two. The merged formula $x = (-1)^k \arcsin w + k\pi$ still holds; it simply lists that one solution twice.

Multivalued functions and branch cuts II

The complex logarithm. Let $z \neq 0$ and write $z = re^{i\theta}$ with $r > 0$. To solve $e^w = z$, write $w = u + iv$ with $u, v \in \mathbb{R}$. The splitting (2.2.2) gives

$$e^w = e^{u+iv} = e^u(\cos v + i \sin v).$$

Equality with $r(\cos\theta + i\sin\theta)$ requires

$$e^u = r \quad \Longrightarrow \quad u = \log r,$$

and

$$\cos v = \cos\theta, \quad \sin v = \sin\theta \quad \Longrightarrow \quad v = \theta + 2\pi k, \qquad k \in \mathbb{Z}.$$

Thus the complete multivalued logarithm is

$$\log z = \log|z| + i(\theta + 2\pi k), \qquad k \in \mathbb{Z}. \tag{2.2.6}$$

A principal branch is obtained by choosing one interval of length 2π, commonly

$$-\pi < \operatorname{Arg} z < \pi, \qquad \operatorname{Log} z = \log|z| + i \operatorname{Arg} z.$$

Because this principal branch deletes the entire nonpositive real axis (the branch cut introduced below), $\operatorname{Arg} z$ ranges over the *open* interval $(-\pi, \pi)$ on its domain; the endpoint values $\pm\pi$ occur only as boundary limits on the two sides of the cut, as in the $\operatorname{Log}(-r \pm i0)$ formulas below. (The half-open convention $\operatorname{Arg} z \in (-\pi, \pi]$ mentioned earlier instead keeps the negative real axis, assigning it $\theta = \pi$.)

Multivalued functions and branch cuts III

A continuous argument cannot be defined on every closed loop around 0: after one counterclockwise turn, the continuous angle increases by 2π. A *branch point* is a point around which no small loop returns a chosen branch to its starting value; for log these are $z = 0$, where the argument gains 2π on each encircling loop, and $z = \infty$. To obtain a single-valued logarithm, remove a *branch cut*, a curve joining the branch points 0 and ∞, so that no loop can enclose just one of them. For the principal logarithm that curve is the nonpositive real axis. The boundary values satisfy

$$\mathrm{Log}(-r + i0) = \log r + i\pi, \qquad \mathrm{Log}(-r - i0) = \log r - i\pi,$$

so crossing the cut changes the logarithm by $2\pi i$.

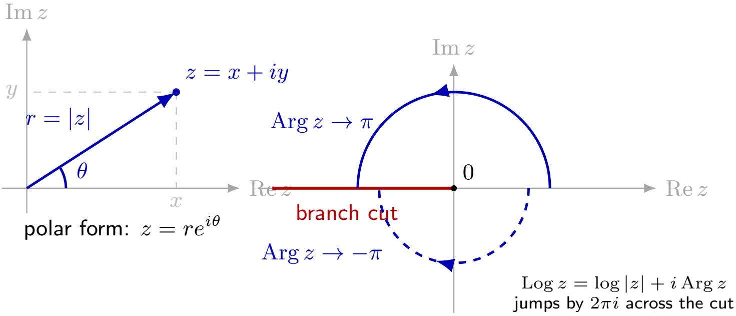
The same mechanism applies to roots. For example,

$$z^{1/2} = \sqrt{r}\, e^{i(\theta+2\pi k)/2} = (-1)^k \sqrt{r}\, e^{i\theta/2},$$

so there are two values. A branch cut permits one of them to be selected continuously.

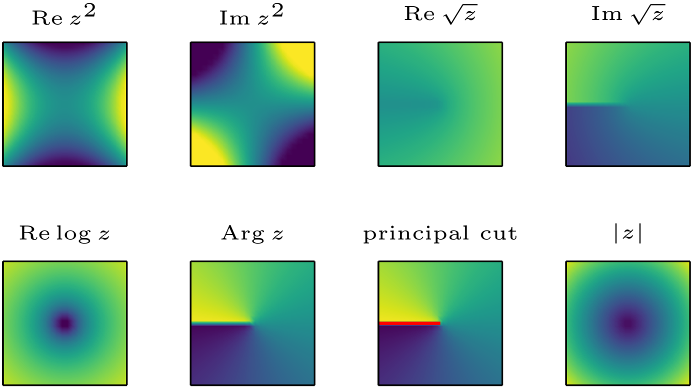
Pictures: branch cuts and component plots I

A complex map $z = x + iy \mapsto f = u + iv$ has two real inputs and two real outputs, so we use cut planes and component plots rather than a single graph $y = f(x)$.



Across a chosen ray the continuous argument jumps by 2π; a single-valued branch lives on one side of that ray only.

Pictures: branch cuts and component plots II



Gallery: Re / Im of z^2, principal square root, and principal log; the jump in Arg z is the branch cut.

The complex derivative and Cauchy–Riemann I

The complex derivative at z_0 is

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}, \tag{2.3.1}$$

the limit being taken over *all* complex increments $h \to 0$, not merely along one preferred direction. If $f'(z_0)$ exists then every direction of approach must of course return the same value, and that one consequence is already strong enough to pin down the structure of f, as we now see. (The converse is not automatic: matching limits along all straight-line approaches does not by itself create the limit in (2.3.1), exactly as in real two-variable calculus, where a function can have both partial derivatives at a point without being differentiable there.) A function is *holomorphic* (analytic) on an open set if this derivative exists at every point of that set; it is *entire* if the set is all of $\mathbb{C}$.

Write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. Along a real increment $h = \Delta x$, the difference quotient is

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i\,\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$
$$\xrightarrow[\Delta x \to 0]{} u_x + iv_x,$$

The complex derivative and Cauchy–Riemann II

so $f'(z) = u_x + iv_x$. Along an imaginary increment $h = i\Delta y$ the bookkeeping is the same, with the extra factor $1/i$ pulled out of the denominator:

$$\frac{f(z+i\Delta y)-f(z)}{i\,\Delta y} = \frac{1}{i}\left[\frac{u(x,y+\Delta y)-u(x,y)}{\Delta y} + i\,\frac{v(x,y+\Delta y)-v(x,y)}{\Delta y}\right].$$

The left-hand side converges as $\Delta y \to 0$ because $f'(z)$ exists; multiplying by i, the bracket converges too, and a complex quantity converges exactly when its real and imaginary parts do, so the two real difference quotients converge separately — that is, u_y and v_y exist. Passing to the limit and using $1/i = -i$,

$$f'(z) = \frac{u_y + iv_y}{i} = -i(u_y + iv_y) = v_y - iu_y.$$

Because f is complex differentiable, every direction of approach must return the *same* value $f'(z)$; equating $u_x + iv_x = v_y - iu_y$ and matching real and imaginary parts gives the Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x. \tag{2.3.2}$$

They are necessary for complex differentiability.

The equations are also sufficient under the standard regularity hypothesis that u, v have continuous first partial derivatives in a neighborhood. Let $h = \Delta x + i\Delta y$, so $|h| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Continuity of the partials is exactly what upgrades "the partials exist" to a first-order expansion with a controlled remainder, and the upgrade is worth seeing once. Split the increment of u into one change in x followed by one change in y,

$$u(x+\Delta x, y+\Delta y) - u(x,y) = \Big[u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\Big] + \Big[u(x, y+\Delta y) - u(x,y)\Big].$$

Each bracket moves only one variable, so the one-variable mean value theorem applies to each (legitimate because u_x and u_y exist throughout the neighborhood): there are x^* between x and $x+\Delta x$, and y^* between y and $y+\Delta y$, with

$$u(x+\Delta x, y+\Delta y) - u(x,y) = u_x(x^*, y+\Delta y)\,\Delta x + u_y(x, y^*)\,\Delta y.$$

Both sample points approach (x,y) as $h \to 0$, so continuity of u_x, u_y at (x,y) lets us write

$$u_x(x^*, y+\Delta y) = u_x + \varepsilon_1(h), \qquad u_y(x, y^*) = u_y + \varepsilon_2(h), \qquad \varepsilon_1, \varepsilon_2 \xrightarrow[h\to 0]{} 0.$$

Since $|\Delta x| \le |h|$ and $|\Delta y| \le |h|$, the error they contribute obeys

$$\Big|\varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y\Big| \le \Big(|\varepsilon_1| + |\varepsilon_2|\Big)\,|h| = o(|h|),$$

which is precisely the first-order expansion

$$u(x+\Delta x, y+\Delta y) = u + u_x\Delta x + u_y\Delta y + o(|h|),$$

The complex derivative and Cauchy–Riemann IV

and the identical argument applies to v. Packing them into $f = u + iv$ yields

$$f(z+h) - f(z) = (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + o(|h|).$$

Here $o(|h|)$ means a remainder $\varepsilon(h)$ with $\varepsilon(h)/|h| \to 0$ as $h \to 0$ (the same remainder for every direction of approach). Using $u_y = -v_x$ and $v_y = u_x$,

$$u_y + iv_y = -v_x + iu_x = i(u_x + iv_x).$$

Therefore

$$\begin{aligned} f(z+h) - f(z) &= (u_x + iv_x)\Delta x + i(u_x + iv_x)\Delta y + o(|h|) \\ &= (u_x + iv_x)(\Delta x + i\Delta y) + o(|h|) \\ &= (u_x + iv_x)h + o(|h|). \end{aligned}$$

After division by h, the remainder satisfies $o(|h|)/h \to 0$, so

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Equivalently, the Cauchy–Riemann equations imply $f_y = if_x$.

Example: z^2. Since

$$z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy),$$

we have

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x.$$

Thus z^2 is entire and $f'(z) = 2x + i(2y) = 2z$.

Example: a branch of $\log z$**.** On the cut plane $\mathbb{C} \setminus (-\infty, 0]$ write $\operatorname{Log} z = u + iv$ with $u = \frac{1}{2}\log(x^2+y^2) = \log r$ and $v = \operatorname{Arg} z = \theta$. The chain rule gives the partials of u, and the polar-angle partials computed in Chapter 1 from $\tan\theta = y/x$ give those of v:

$$u_x = \frac{1}{2}\cdot\frac{2x}{x^2+y^2} = \frac{x}{r^2}, \quad u_y = \frac{y}{r^2}, \qquad v_x = -\frac{y}{r^2}, \quad v_y = \frac{x}{r^2}.$$

Hence $u_x = v_y$ and $u_y = -v_x$, so (2.3.2) holds; the four partials are continuous wherever $r > 0$, so the sufficiency criterion above makes Log holomorphic off the cut, with

$$(\operatorname{Log} z)' = u_x + iv_x = \frac{x - iy}{r^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z},$$

the last step by $z\overline{z} = |z|^2$. For $1/z$ no Cauchy–Riemann computation is needed, since (2.3.1) can be evaluated directly: for $z \neq 0$ and $0 < |h| < |z|$ (so that $z + h \neq 0$),

$$\frac{1}{h}\left(\frac{1}{z+h} - \frac{1}{z}\right) = \frac{1}{h}\cdot\frac{z-(z+h)}{z(z+h)} = \frac{-1}{z(z+h)} \xrightarrow[h\to 0]{} -\frac{1}{z^2},$$

so $1/z$ is holomorphic on $\mathbb{C}\setminus\{0\}$ with derivative $-1/z^2$. More generally, the sum, product, quotient and chain rules hold verbatim for the complex derivative: their real-variable proofs use nothing but the algebra of limits of difference quotients, and that algebra is identical for complex h.

Counterexample: $\bar{z} = x - iy$**.** Then $u_x = 1 \neq -1 = v_y$, so CR fails everywhere. Equivalently $\overline{h}/h$ is 1 for real h and -1 for imaginary h: no direction-independent derivative.

Contour integrals and a key circle integral I

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable path. A partition $a = t_0 < t_1 < \cdots < t_N = b$ and sample points $\tau_j \in [t_j, t_{j+1}]$ give the Riemann sum

$$\sum_{j=0}^{N-1} f(\gamma(\tau_j))\big(\gamma(t_{j+1}) - \gamma(t_j)\big).$$

Its limit, when it exists, is the contour integral

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{2.4.1}$$

The second equality follows because $\gamma(t_{j+1}) - \gamma(t_j) \approx \gamma'(\tau_j)(t_{j+1} - t_j)$. Reversing the path reverses the sign:

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

A closed contour may be traversed in either direction. "Positively oriented" means counterclockwise for the boundary of a bounded planar region.

Integer powers on a circle. Let $n \in \mathbb{Z}$ and parameterize the positively oriented circle $|z| = R$ by

$$z(\theta) = Re^{i\theta}, \qquad dz = iRe^{i\theta}\,d\theta, \qquad 0 \le \theta \le 2\pi.$$

Contour integrals and a key circle integral II

Then

$$\oint_{|z|=R} z^n\,dz = \int_0^{2\pi} R^n e^{in\theta}\left(iRe^{i\theta}\right)d\theta$$
$$= iR^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta.$$

If $n=-1$, the exponential is 1, so

$$\oint_{|z|=R}\frac{dz}{z} = i\int_0^{2\pi} d\theta = 2\pi i.$$

If $n\neq -1$, then

$$iR^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = iR^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}$$
$$= \frac{R^{n+1}}{n+1}\left(e^{2\pi i(n+1)}-1\right) = 0,$$

because $n+1$ is an integer. Let $\delta_{n,-1}$ denote the Kronecker delta: it is 1 when $n=-1$ and 0 otherwise. Translating the circle gives

$$\oint_{|z-z_0|=R}(z-z_0)^n\,dz = 2\pi i\,\delta_{n,-1}. \tag{2.4.2}$$

Contour integrals and a key circle integral III

For a general closed path γ avoiding z_0, write $\gamma(t) - z_0 = r(t)e^{i\theta(t)}$ with a continuous argument lift $\theta(t)$. Then

$$\frac{\gamma'(t)}{\gamma(t) - z_0} = \frac{r'(t)}{r(t)} + i\theta'(t),$$

so, using the definition (2.4.1) and then integrating term by term,

$$\begin{aligned}\oint_\gamma \frac{dz}{z - z_0} &= \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\,dt = \int_a^b \Big(\frac{r'(t)}{r(t)} + i\theta'(t)\Big)dt \\ &= \Big[\log r + i\theta\Big]_a^b = \log\frac{r(b)}{r(a)} + i\big(\theta(b) - \theta(a)\big).\end{aligned}$$

Closedness now does all the remaining work. Since $\gamma(b) = \gamma(a)$, taking moduli gives $r(b) = |\gamma(b) - z_0| = |\gamma(a) - z_0| = r(a)$, and this common value is *positive* because the path avoids z_0; hence $\log\big(r(b)/r(a)\big) = \log 1 = 0$, so the real part of the bracket cancels for every closed path. Dividing the identity $r(b)e^{i\theta(b)} = r(a)e^{i\theta(a)}$ by $r(a) = r(b) > 0$ leaves $e^{i(\theta(b)-\theta(a))} = 1$, and for real ϕ the equation $e^{i\phi} = \cos\phi + i\sin\phi = 1$ holds exactly when ϕ is an integer multiple of 2π. Therefore $\theta(b) - \theta(a) = 2\pi q$ for some $q \in \mathbb{Z}$, and

$$\oint_\gamma \frac{dz}{z - z_0} = 2\pi i\, q.$$

Contour integrals and a key circle integral IV

The integer q records the net number of turns the vector $\gamma(t) - z_0$ makes as t runs from a to b. Solving the last display for it gives the *winding number* (index) of γ about z_0,

$$q = \mathrm{Ind}(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0}.$$

One positive circuit gives $q = 1$; clockwise gives $q = -1$; no net encirclement gives $q = 0$.

Antiderivatives. If $F' = f$ with F single-valued and holomorphic along the path, then $\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$; closed-path integrals vanish. For $n \neq -1$, z^n has the single-valued primitive $z^{n+1}/(n+1)$ on $\mathbb{C} \setminus \{0\}$ (or on all of $\mathbb{C}$ when $n \geq 0$), explaining why those circle integrals are zero. For $n = -1$ no single-valued logarithm exists on a full punctured neighborhood of 0, matching $\oint dz/z = 2\pi i$.

Cauchy's integral theorem and formula I

A *domain* is an open, connected set. A domain is *simply connected* if every closed curve in it can be continuously contracted to a point without leaving the domain; for the planar regions used here, this is the precise version of "the region has no holes."

Cauchy's integral theorem

Let D be a simply connected domain, let f be holomorphic on D, and let γ be a closed piecewise smooth contour in D. Then

$$\oint_\gamma f(z)\,dz = 0.$$

Sketch (Green): suppose $f = u + iv$ has continuous first partials and $\gamma = \partial\Omega$ is the positively oriented boundary of the bounded region Ω it encloses. Recall Green's theorem from vector calculus: for real P, Q with continuous first partials on Ω and its boundary,

$$\oint_{\partial\Omega} \big(P\,dx + Q\,dy\big) = \iint_\Omega \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dA.$$

Multiplying out $f\,dz = (u + iv)(dx + i\,dy)$ and collecting real and imaginary parts,

$$f\,dz = (u\,dx - v\,dy) + i\,(v\,dx + u\,dy).$$

Cauchy's integral theorem and formula II

Applying Green's theorem to the first bracket with $P = u$, $Q = -v$, and to the second with $P = v$, $Q = u$,

$$\oint_{\partial\Omega}(u\,dx - v\,dy) = \iint_{\Omega}\left(-v_x - u_y\right)dA, \qquad \oint_{\partial\Omega}(v\,dx + u\,dy) = \iint_{\Omega}\left(u_x - v_y\right)dA.$$

The Cauchy–Riemann equations (2.3.2) kill both integrands: $u_x = v_y$ gives $u_x - v_y = 0$, and $u_y = -v_x$ gives $-v_x - u_y = u_y - u_y = 0$. Hence both area integrals vanish and $\oint_\gamma f\,dz = 0$. (Cauchy–Goursat removes the continuous-partials hypothesis; we use the continuous case.)

Cauchy's integral formula

Let γ be a positively oriented simple closed contour, let f be holomorphic on an open set containing γ and its interior, and let z_0 lie inside γ. Then

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz. \tag{2.4.3}$$

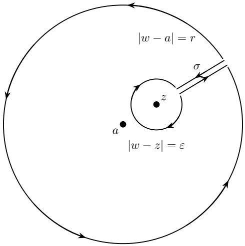
Choose a small circle $C_\varepsilon : |z - z_0| = \varepsilon$ inside γ. The function $f(z)/(z - z_0)$ is holomorphic in the region between the two curves. Join the two boundary curves by a simple slit lying in that region. The cut region is simply connected, so the stated

Cauchy's integral theorem and formula III

form of Cauchy's theorem applies; the integrals along the two oppositely oriented copies of the slit cancel. The remaining boundary consists of γ counterclockwise and C_ε clockwise. Writing CCW and CW for counterclockwise and clockwise orientation, respectively, we obtain

$$0 = \oint_\gamma \frac{f(z)}{z - z_0}\,dz - \oint_{C_\varepsilon,\mathrm{CCW}} \frac{f(z)}{z - z_0}\,dz,$$

so the two counterclockwise integrals are equal.



Schematic of the slit construction: outer contour, small circle about the puncture, and connecting slit (labels in the figure may use w, z where the text uses z, z_0).

Cauchy's integral theorem and formula IV

On the small circle $z = z_0 + \varepsilon e^{i\theta}$,

$$\oint_{C_\varepsilon} \frac{f(z)}{z - z_0}\, dz = i \int_0^{2\pi} f(z_0 + \varepsilon e^{i\theta})\, d\theta.$$

Subtracting $2\pi i f(z_0) = i \int_0^{2\pi} f(z_0)\, d\theta$ and using continuity of f,

$$\left| \oint_{C_\varepsilon} \frac{f}{z - z_0}\, dz - 2\pi i f(z_0) \right| \leq 2\pi \max_{|z - z_0| = \varepsilon} |f(z) - f(z_0)| \longrightarrow 0.$$

Hence $\oint_\gamma f(z)/(z - z_0)\, dz = 2\pi i f(z_0)$, which is (2.4.3).

Differentiate (2.4.3) in z_0 (contour fixed, away from z_0). Since $\partial_{z_0}^n (z - z_0)^{-1} = n!/(z - z_0)^{n+1}$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz. \tag{2.4.4}$$

In particular every holomorphic function is infinitely differentiable, and

$$\oint_\gamma \frac{f(z)}{(z - z_0)^m}\, dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(z_0).$$

Worked example: partial fractions and Cauchy I

Evaluate

$$I = \oint_{|z|=2} \frac{e^z}{z^2(z-1)}\, dz,$$

with positive orientation. Write

$$\frac{1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1}.$$

Multiplication by $z^2(z-1)$ gives

$$\begin{aligned} 1 &= Az(z-1) + B(z-1) + Cz^2 \\ &= (A+C)z^2 + (-A+B)z - B. \end{aligned}$$

Matching the coefficients of z^2, z, and 1 yields

$$A + C = 0, \qquad -A + B = 0, \qquad -B = 1.$$

Solving in order,

$$B = -1, \qquad A = B = -1, \qquad C = -A = 1.$$

Worked example: partial fractions and Cauchy II

Therefore

$$\begin{aligned}\frac{e^z}{z^2(z-1)} &= e^z\left(-\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z-1}\right)\\ &= \frac{e^z}{z-1}-\frac{(z+1)e^z}{z^2}.\end{aligned}$$

For the simple pole at $z=1$, Cauchy's formula gives

$$\oint_{|z|=2}\frac{e^z}{z-1}\,dz = 2\pi ie.$$

For the double-pole term, define

$$h(z)=(z+1)e^z.$$

Then

$$\begin{aligned}h'(z) &= (z+1)'e^z+(z+1)(e^z)'\\ &= e^z+(z+1)e^z=(z+2)e^z,\end{aligned}$$

so $h'(0)=2$. The $m=2$ form of Cauchy's derivative formula gives

$$\oint_{|z|=2}\frac{h(z)}{z^2}\,dz = 2\pi ih'(0)=4\pi i.$$

Worked example: partial fractions and Cauchy III

Remembering the minus sign in the decomposition,

$$\begin{aligned} I &= 2\pi i e - 4\pi i \\ &= 2\pi i(e-2). \end{aligned}$$

Both singularities, 0 and 1, lie inside $|z| = 2$; no other singularities are present.

Taylor series from contour integration I

Taylor series theorem

If f is holomorphic on the disk $|z-z_0|<R$, then for every $|z-z_0|<R$,

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

Choose a radius ρ satisfying

$$|z-z_0|<\rho<R$$

and integrate on $C_\rho : |\xi-z_0|=\rho$. Cauchy's formula gives

$$f(z)=\frac{1}{2\pi i}\oint_{C_\rho}\frac{f(\xi)}{\xi-z}\,d\xi.$$

Rewrite the denominator as

$$\xi-z=(\xi-z_0)-(z-z_0)=(\xi-z_0)\left(1-\frac{z-z_0}{\xi-z_0}\right).$$

Since

$$\left|\frac{z-z_0}{\xi-z_0}\right|=\frac{|z-z_0|}{\rho}<1,$$

Taylor series from contour integration II

the geometric series converges uniformly on the contour:

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n .$$

Uniform convergence permits term-by-term integration:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n d\xi \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \left[\frac{1}{2\pi i} \oint_{C_\rho} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right] \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \end{aligned}$$

where the last line uses (2.4.4).

The same geometric series, stopped after $N + 1$ terms, makes the rate of convergence explicit. For $w \neq 1$,

$$\sum_{n=0}^{N} w^n = \frac{1 - w^{N+1}}{1 - w} \quad \Longrightarrow \quad \frac{1}{1 - w} = \sum_{n=0}^{N} w^n + \frac{w^{N+1}}{1 - w}.$$

Taylor series from contour integration III

Put $w = \dfrac{z - z_0}{\xi - z_0}$ and divide by $\xi - z_0$; since $(\xi - z_0)(1 - w) = \xi - z$ by the factorization above,

$$\frac{1}{\xi - z} = \sum_{n=0}^{N} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} + \frac{1}{\xi - z}\left(\frac{z - z_0}{\xi - z_0}\right)^{N+1}.$$

Insert this in $f(z) = \frac{1}{2\pi i}\oint_{C_\rho} \frac{f(\xi)}{\xi - z}\, d\xi$. The sum is finite, so no interchange of limits is needed, and (2.4.4) turns its n-th term into $\frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$. Hence

$$f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + R_N(z), \qquad R_N(z) = \frac{1}{2\pi i}\oint_{C_\rho} \frac{f(\xi)}{\xi - z}\left(\frac{z - z_0}{\xi - z_0}\right)^{N+1} d\xi.$$

On C_ρ we have $|\xi - z_0| = \rho$ and $|f(\xi)| \le M_\rho := \max_{C_\rho} |f|$, while the triangle inequality in the form $|\xi - z| \ge |\xi - z_0| - |z - z_0|$ gives

$$|\xi - z| \ge \rho - |z - z_0| > 0.$$

Taylor series from contour integration IV

The modulus of the integrand is therefore at most $\dfrac{M_\rho}{\rho - |z - z_0|} \dfrac{|z - z_0|^{N+1}}{\rho^{N+1}}$, and $\mathrm{length}(C_\rho) = 2\pi\rho$, so the *ML* estimate $\left|\oint_C g\right| \leq (\max|g|)\,\mathrm{length}(C)$ yields

$$|R_N(z)| \leq \frac{1}{2\pi}(2\pi\rho)\frac{M_\rho}{\rho - |z - z_0|}\frac{|z - z_0|^{N+1}}{\rho^{N+1}} = \frac{M_\rho \rho}{\rho - |z - z_0|}\left(\frac{|z - z_0|}{\rho}\right)^{N+1} \longrightarrow 0,$$

because $|z - z_0|/\rho < 1$. Thus the Taylor series converges to f throughout the disk of holomorphy, and the error after $N + 1$ terms decays geometrically at the rate $|z - z_0|/\rho$.

Generalized binomial series. When m is not a nonnegative integer the symbol $(1 + z)^m$ first has to be given a meaning, and the principal logarithm of the branch-cut frame supplies it:

$$(1 + z)^m := e^{m\,\mathrm{Log}(1+z)}.$$

On $|z| < 1$ we have $\mathrm{Re}(1 + z) = 1 + \mathrm{Re}\, z > 0$, so $1 + z$ never meets the cut of Log and this formula is holomorphic on the whole disk; it extends holomorphically to the cut plane $\mathbb{C} \setminus (-\infty, -1]$, whose endpoint $z = -1$ is the branch point named below. Since $\mathrm{Log}\, 1 = 0$ the value at $z = 0$ is $e^0 = 1$, so this is the branch we want.

Differentiating $e^{\operatorname{Log} w} = w$ gives $e^{\operatorname{Log} w} \frac{d}{dw} \operatorname{Log} w = 1$, that is $\frac{d}{dw} \operatorname{Log} w = 1/w$; hence by the chain rule, and using $\frac{1}{1+z} = e^{-\operatorname{Log}(1+z)}$,

$$\frac{d}{dz}(1+z)^m = e^{m \operatorname{Log}(1+z)} \frac{m}{1+z} = m\, e^{(m-1)\operatorname{Log}(1+z)} = m(1+z)^{m-1}.$$

Applying this rule repeatedly — by induction on n: the case $n = 0$ is the definition, and differentiating $m(m-1)\cdots(m-n+1)(1+z)^{m-n}$ once more multiplies it by $m - n$ and lowers the exponent by one, which is the same formula with $n+1$ in place of n — we obtain

$$\frac{d^n}{dz^n}(1+z)^m = m(m-1)\cdots(m-n+1)(1+z)^{m-n}.$$

At $z = 0$,

$$\frac{f^{(n)}(0)}{n!} = \frac{m(m-1)\cdots(m-n+1)}{n!} = \binom{m}{n}.$$

Hence

$$(1+z)^m = \sum_{n=0}^{\infty} \binom{m}{n} z^n = 1 + mz + \frac{m(m-1)}{2!} z^2 + \frac{m(m-1)(m-2)}{3!} z^3 + \cdots . \quad (2.5.1)$$

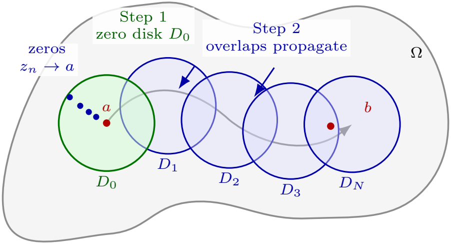
If $m \in \{0, 1, 2, \ldots\}$, the factor $m - n + 1$ becomes zero and the series terminates, so the result is a polynomial valid for every z. Otherwise the nearest obstruction to the

Taylor series from contour integration VI

expansion about 0 is $z = -1$: it is a pole when m is a negative integer and a branch point when m is nonintegral. Therefore the radius of convergence is 1.

Analytic continuation I

A Taylor series is a local formula. Overlapping disks $D_0, D_1, \ldots, D_N$ can carry that formula along a chain: this is *analytic continuation*.



Analytic continuation II

Identity / splicing

Let U and V be domains whose overlap $U \cap V$ is nonempty and connected, let f_1 be holomorphic on U and f_2 holomorphic on V, and suppose $f_1 = f_2$ on some nonempty open subset of $U \cap V$. Then $f_1 = f_2$ on all of $U \cap V$, so

$$f(z) = \begin{cases} f_1(z), & z \in U, \\ f_2(z), & z \in V, \end{cases}$$

is well defined, and it is holomorphic on $U \cup V$ because holomorphy is a local property and f coincides with f_1 near each point of U and with f_2 near each point of V. In particular a continuation of f_1 from U to V, if one exists, is unique.

Along a chain of disks the overlap $D_j \cap D_{j+1}$ is a lens, hence convex and in particular connected, so the block applies at every link and the formula carried along the chain is unambiguous. One analytic fact stands behind the block, and the Taylor theorem

of the previous frame proves it.

Identity theorem

Let g be holomorphic on a domain Ω and vanish on some nonempty open subset of Ω. Then $g \equiv 0$ on Ω.

Consider the set of points at which g vanishes together with all its derivatives,

$$S = \{p \in \Omega : \; g^{(n)}(p) = 0 \text{ for every } n \geq 0\}.$$

S is nonempty: each point of the open set on which g vanishes has a disk about it where g is identically zero, and every derivative of the zero function is zero. S is open: if $p \in S$ and $D \subset \Omega$ is a disk centred at p, the Taylor theorem applies to g on D and gives

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(p)}{n!}(z-p)^n = 0 \qquad (z \in D),$$

so every derivative of g vanishes throughout D and $D \subset S$. S is closed in Ω: by (2.4.4) a holomorphic function is infinitely differentiable and each $g^{(n)}$ is again holomorphic, hence continuous, so $S = \bigcap_{n \geq 0} \left(g^{(n)}\right)^{-1}(\{0\})$ is an intersection of sets closed in Ω. A nonempty subset of a connected set that is both open and closed in it is the whole set, so $S = \Omega$ and in particular $g \equiv 0$.

Analytic continuation IV

Applying this to $g = f_1 - f_2$, holomorphic on the domain $U \cap V$ and vanishing on a nonempty open subset of it, gives $f_1 \equiv f_2$ on $U \cap V$, which is the block above. The same theorem is what later lets an identity proved only on a strip be promoted to the whole connected domain on which both sides are holomorphic. Continuation along different paths around a branch point may land on different branches (*monodromy*); that is not a failure of local uniqueness on a fixed slit plane.

Suppose f is holomorphic on an annulus

$$A = \{z : r < |z - z_0| < R\}.$$

Fix z in the annulus and choose radii $r < \rho_1 < |z - z_0| < \rho_2 < R$. The function

$$g(\xi) = \frac{f(\xi)}{\xi - z}$$

is holomorphic on the closed subannulus $\rho_1 \leq |\xi - z_0| \leq \rho_2$ except at the single point $\xi = z$. Note that f is *not* assumed holomorphic inside $|\xi - z_0| = \rho_1$, so (2.4.3) cannot be applied to the outer circle alone; instead repeat the slit construction used to prove it, now with one puncture more. Choose $\varepsilon > 0$ so small that the circle $C_\varepsilon : |\xi - z| = \varepsilon$ and its interior lie strictly between the two circles, and cut the region between the three circles with two slits, one joining $|\xi - z_0| = \rho_1$ to C_ε and one joining C_ε to $|\xi - z_0| = \rho_2$. The cut region is simply connected and g is holomorphic on it, so Cauchy's integral theorem makes the integral of g around its boundary vanish; each slit is traversed once in each direction and cancels, and the boundary that survives is $|\xi - z_0| = \rho_2$ counterclockwise together with $|\xi - z_0| = \rho_1$ and C_ε clockwise. Writing all three circles counterclockwise, the two clockwise contributions change sign and

$$\oint_{|\xi - z_0| = \rho_2} g\, d\xi - \oint_{|\xi - z_0| = \rho_1} g\, d\xi - \oint_{C_\varepsilon} g\, d\xi = 0.$$

Laurent series II

The last integral was already evaluated in the proof of (2.4.3): on $\xi = z + \varepsilon e^{i\theta}$ one has $d\xi = i\varepsilon e^{i\theta}\, d\theta$, so

$$\oint_{C_\varepsilon} \frac{f(\xi)}{\xi - z}\, d\xi = i \int_0^{2\pi} f(z + \varepsilon e^{i\theta})\, d\theta \longrightarrow 2\pi i\, f(z) \qquad (\varepsilon \to 0)$$

by continuity of f at z. The other two integrals do not depend on ε, so letting $\varepsilon \to 0$ and dividing by $2\pi i$ gives, with both circles written counterclockwise,

$$f(z) = \frac{1}{2\pi i} \oint_{|\xi - z_0| = \rho_2} \frac{f(\xi)}{\xi - z}\, d\xi - \frac{1}{2\pi i} \oint_{|\xi - z_0| = \rho_1} \frac{f(\xi)}{\xi - z}\, d\xi.$$

On the outer circle, $|z - z_0| < |\xi - z_0|$, so

$$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}.$$

On the inner circle, $|\xi - z_0| < |z - z_0|$, so

$$\begin{aligned} \frac{1}{\xi - z} &= -\frac{1}{z - z_0} \frac{1}{1 - \frac{\xi - z_0}{z - z_0}} \\ &= -\sum_{m=0}^{\infty} \frac{(\xi - z_0)^m}{(z - z_0)^{m+1}}. \end{aligned}$$

Laurent series III

Substitute these geometric series into Cauchy's formula and integrate term by term (justified by uniform convergence on the compact circles). The outer integral contributes only nonnegative powers of $(z - z_0)$:

$$\frac{1}{2\pi i} \oint_{|\xi - z_0| = \rho_2} \frac{f(\xi)}{\xi - z} \, d\xi = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{|\xi - z_0| = \rho_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right) (z - z_0)^n.$$

The inner integral, after the minus sign in Cauchy's formula cancels the leading minus in the geometric series, contributes only negative powers:

$$-\frac{1}{2\pi i} \oint_{|\xi - z_0| = \rho_1} \frac{f(\xi)}{\xi - z} \, d\xi = \sum_{m=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{|\xi - z_0| = \rho_1} f(\xi) \, (\xi - z_0)^m \, d\xi \right) (z - z_0)^{-(m+1)}.$$

Collecting coefficients of like powers of $(z - z_0)$ yields the Laurent series below; writing $n = -(m+1)$ for the principal part recovers the single-index form $a_n = \frac{1}{2\pi i} \oint f(\xi)(\xi - z_0)^{-n-1} \, d\xi$ on any circle in the annulus.

Laurent series IV

Thus f has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \tag{2.5.2}$$

convergent throughout the annulus. To extract a coefficient, multiply by $(z - z_0)^{-k-1}$ and integrate on a centered circle $\gamma : |z - z_0| = \rho$, where $r < \rho < R$, oriented counterclockwise:

$$\begin{aligned} \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{k+1}}\, dz &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} a_n \oint_\gamma (z - z_0)^{n-k-1}\, dz \\ &= \frac{1}{2\pi i} a_k (2\pi i) = a_k. \end{aligned}$$

Therefore, for every integer k,

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{k+1}}\, dz. \tag{2.5.3}$$

Term-by-term integration is justified by uniform convergence on compact subannuli.

Laurent series V

Annulus picture. An inner hole of radius r (possible singularity) and outer wall of radius R (next obstruction) license

$$f(z) = \underbrace{\sum_{n\geq 0} a_n (z-z_0)^n}_{\text{regular}} + \underbrace{\sum_{n\geq 1} a_{-n}(z-z_0)^{-n}}_{\text{principal part}}.$$

The Taylor case is the case of an empty principal part, and that happens exactly when f is holomorphic on the *whole* disk $|z-z_0| < R$: then $f(\xi)(\xi-z_0)^{n-1}$ is holomorphic there for every $n \geq 1$, so (2.5.3) with $k=-n$ and Cauchy's integral theorem give

$$a_{-n} = \frac{1}{2\pi i}\oint_\gamma f(\xi)(\xi - z_0)^{n-1}\,d\xi = 0 \qquad (n \geq 1),$$

and (2.5.2) collapses to the Taylor series. Taking merely $r=0$ is not enough: the region is then the punctured disk $0 < |z-z_0| < R$, on which the principal part may still be nonzero, with no terms (removable), finitely many terms (pole) or infinitely many (essential), as the example $1/(z(z-1))$ below shows.

Laurent series VI

Coefficient extraction. On $0 < |z| < 1$,

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n,$$

so $a_{-1} = -1$ and $a_n = -1$ for $n \geq 0$. The coefficient formula (2.5.3) with $k = -1$ on the circle $|z| = \frac{1}{2}$ reproduces this: $a_{-1} = \frac{1}{2\pi i} \oint_{|z|=1/2} \frac{dz}{z(z-1)} = -1$, by (2.4.3) applied to the function $1/(z-1)$, which is holomorphic on $|z| < 1$ and equals -1 at $z = 0$. Because a_{-1} is the only coefficient that survives such a contour integration, it is given a name of its own, the *residue* of f at z_0, written $\operatorname{Res}(f, z_0)$; it is the subject of the next section, and here $\operatorname{Res}(f, 0) = a_{-1} = -1$. (On $1 < |z| < \infty$ the expansion of $1/(z-1)$ changes, and so do the coefficients.)

Removable example: sinc. Dividing the sine series by z gives

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!},$$

entire after setting $\operatorname{sinc} 0 = 1$.

Classifying singularities I

A point z_0 is a *singular point* of f when f fails to be holomorphic there. It is an *isolated singularity* if there exists $R > 0$ such that f is holomorphic on the punctured disk

$$0 < |z - z_0| < R.$$

Only isolated singularities admit an ordinary Laurent classification. Write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Then:

- The singularity is **removable** if $a_n = 0$ for every $n < 0$. Defining $f(z_0) = a_0$ makes f holomorphic at z_0. Example: $\sin z / z$ at 0.

Classifying singularities II

- It is a **pole of order** m, where $m \geq 1$ is an integer, if $a_{-m} \neq 0$ and $a_n = 0$ for every $n < -m$. The restriction $m \geq 1$ is what separates this case from the previous one: it forces at least one negative-power coefficient to be nonzero, whereas a removable singularity has none. It also makes the order unique, since m is then the largest k with $a_{-k} \neq 0$. Equivalently,
$$f(z) = \frac{h(z)}{(z - z_0)^m}, \qquad h(z_0) \neq 0,$$
with h holomorphic near z_0. To see that the two descriptions agree, recall that a power series may be differentiated term by term inside its disk of convergence, so its sum is holomorphic there. If $a_n = 0$ for every $n < -m$, then multiplying (2.5.2) by $(z - z_0)^m$ and relabelling with $k = n + m$ (so that $n \geq -m$ becomes $k \geq 0$),
$$(z - z_0)^m f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^{n+m} = \sum_{k=0}^{\infty} a_{k-m} (z - z_0)^k =: h(z).$$
The right-hand side is a power series converging for $0 < |z - z_0| < R$, hence, being a power series, on the whole disk $|z - z_0| < R$; so h is holomorphic there and $h(z_0) = a_{-m} \neq 0$. Dividing by $(z - z_0)^m$ gives the displayed form. Conversely, if $f(z) = h(z)/(z - z_0)^m$ with h holomorphic near z_0 and

Classifying singularities III

$h(z_0) \neq 0$, the Taylor series theorem gives $h(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j$ with $c_j = h^{(j)}(z_0)/j!$ and $c_0 = h(z_0) \neq 0$, so, putting $n = j - m$,

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^{j-m} = \sum_{n=-m}^{\infty} c_{n+m} (z - z_0)^n,$$

whence $a_n = 0$ for $n < -m$ and $a_{-m} = c_0 \neq 0$, which is the coefficient condition. The same term-by-term fact justifies the removable case above: there f agrees on the punctured disk with the power series $\sum_{n \geq 0} a_n (z - z_0)^n$, so declaring $f(z_0) = a_0$ makes f equal to that holomorphic sum on the whole disk.

- It is **essential** if infinitely many negative-power coefficients are nonzero. For example,

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3! z^3} + \frac{1}{5! z^5} - \cdots$$

has an essential singularity at 0.

Classifying singularities IV

A function holomorphic except for isolated poles is *meromorphic*. For example, $1/\sin z$ is meromorphic, with simple poles at $z = n\pi$, $n \in \mathbb{Z}$.

Why those are all the poles, and why they are simple. The computation leading to (2.2.1) used only the absolute convergence of the exponential series, which holds for every complex argument, so reading (2.2.5) with θ replaced by a complex z is legitimate: $\sin z = (e^{iz} - e^{-iz})/(2i)$, where $\sin z$ means (2.1.7) evaluated at z. By the addition law $e^{w_1} e^{w_2} = e^{w_1 + w_2}$ established on the Euler frame, $e^{iz} e^{-iz} = e^0 = 1$, so e^{iz} never vanishes, and multiplying $\sin z = 0$ by $2i\, e^{iz}$ shows that $\sin z = 0$ is equivalent to $e^{2iz} = 1$. Write $z = x + iy$ with x, y real; the same law together with (2.2.1) gives

$$e^{2iz} = e^{-2y} e^{2ix} = e^{-2y} \big(\cos 2x + i \sin 2x\big), \qquad \left| e^{2iz} \right| = e^{-2y},$$

since $\cos^2 2x + \sin^2 2x = 1$. Hence $e^{2iz} = 1$ forces first $e^{-2y} = 1$, that is $y = 0$, and then $\cos 2x = 1$ together with $\sin 2x = 0$, that is $2x = 2n\pi$. The zeros of sin in $\mathbb{C}$ are therefore exactly the real points $n\pi$, spaced π apart and so isolated.

Each zero is simple. Put $u = z - n\pi$; since $e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n$ by (2.2.1), the addition law gives

$$\sin(n\pi + u) = \frac{e^{in\pi} e^{iu} - e^{-in\pi} e^{-iu}}{2i} = (-1)^n \, \frac{e^{iu} - e^{-iu}}{2i} = (-1)^n \sin u,$$

Classifying singularities V

so by (2.1.7)

$$\sin z = u\,g(u), \qquad g(u) = (-1)^n\left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \cdots\right), \qquad g(0) = (-1)^n \neq 0.$$

The series for g converges everywhere, so g is holomorphic, and $g(0) \neq 0$ keeps g nonzero on a neighbourhood of $u = 0$; there $1/g$ is holomorphic as a quotient with nonvanishing denominator. Therefore

$$\frac{1}{\sin z} = \frac{1/g(z - n\pi)}{z - n\pi},$$

which is a pole of order 1 by the criterion above, and the Laurent series of $1/\sin z$ about $n\pi$ begins with $(-1)^n/(z - n\pi)$.

A branch point is not an isolated singularity. For example, analytic continuation of $\log z$ once around 0 changes its value by $2\pi i$, so no single-valued branch is holomorphic on a complete punctured disk around 0. The removable/pole/essential trichotomy therefore does not apply to the branch point itself.

The nature of f at infinity is the nature of $F(w) = f(1/w)$ at $w = 0$: a polynomial of degree $m \geq 1$ has a pole of order m at ∞, while e^z has an essential singularity there. (A degree-0 polynomial is a constant, and F is then constant as well, so the singularity at ∞ is removable — consistent with the requirement $m \geq 1$ above.)

The residue theorem I

For an isolated singularity z_0, the residue is the Laurent coefficient

$$\operatorname{Res}(f, z_0) = a_{-1}.$$

Let z_0 be an isolated singularity, so that f is holomorphic on a punctured disk $0 < |z - z_0| < R$, and let C be a positively oriented simple contour lying in that punctured disk and winding once about z_0 — equivalently, C encloses z_0 and no other singularity of f. That restriction is exactly what ties a_{-1} to the residue: the Laurent coefficients depend on the annulus, and for $f(z) = 1/(z(z-1))$ the coefficient a_{-1} on $1 < |z| < \infty$ is 0, whereas $\operatorname{Res}(f, 0) = -1$ (and indeed $\oint_{|z|=2} f(z)\,dz = 0$, since that circle also encloses the pole at 1). On the punctured disk the Laurent series converges uniformly on C, so it may be integrated term by term, and each power is already settled: for an integer $n \neq -1$ the function $(z - z_0)^n$ has the single-valued primitive $(z - z_0)^{n+1}/(n+1)$ on $\mathbb{C} \setminus \{z_0\}$, so $\oint_C (z - z_0)^n\,dz = 0$ for *any* closed contour avoiding z_0, while for $n = -1$ the index formula $\oint_C dz/(z - z_0) = 2\pi i \operatorname{Ind}(C, z_0)$ gives $2\pi i$ because C winds once positively.

The residue theorem II

(Equation (2.4.2) is the special case in which C is a circle centered at z_0.) Only the $n = -1$ term survives:

$$\begin{aligned}\oint_C f(z)\,dz &= \sum_{n=-\infty}^{\infty} a_n \oint_C (z - z_0)^n\,dz \\ &= a_{-1}(2\pi i) \\ &= 2\pi i \,\operatorname{Res}(f, z_0).\end{aligned}$$

Thus

$$\oint_C f(z)\,dz = 2\pi i \,\operatorname{Res}(f, z_0). \tag{2.7.1}$$

For winding number q, the right-hand side is multiplied by q.

Residue theorem

Let f be holomorphic inside and on a positively oriented simple closed contour γ, except for finitely many isolated singularities $z_1, \ldots, z_N$ in the interior. Then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f, z_k). \tag{2.7.2}$$

The residue theorem III

To prove it, remove disjoint disks D_k centered at the singularities, then join each small circle to the outer boundary by nonintersecting slits. On the resulting simply connected cut region, f is holomorphic, so the stated form of Cauchy's theorem applies. The integrals along the oppositely oriented copies of every slit cancel. The remaining boundary consists of γ counterclockwise and each small circle clockwise, hence

$$0 = \oint_{\gamma} f(z)\,dz - \sum_{k=1}^{N} \oint_{\partial D_k,\mathrm{CCW}} f(z)\,dz.$$

Rearranging and shrinking the circles gives

$$\oint_{\gamma} f(z)\,dz = \sum_{k=1}^{N} 2\pi i\,\mathrm{Res}(f, z_k),$$

which proves the theorem. For a non-simple contour, each residue is multiplied by the corresponding winding number.

Residue at a pole of order m. Write

$$h(z) = (z - z_0)^m f(z),$$

The residue theorem IV

where h is holomorphic and $h(z_0) \neq 0$. Taylor expansion gives

$$h(z) = \sum_{j=0}^{\infty} \frac{h^{(j)}(z_0)}{j!}(z - z_0)^j.$$

Dividing by $(z - z_0)^m$,

$$f(z) = \sum_{j=0}^{\infty} \frac{h^{(j)}(z_0)}{j!}(z - z_0)^{j-m}.$$

The exponent equals -1 exactly when $j = m - 1$. Therefore

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]\Big|_{z=z_0}. \tag{2.7.3}$$

For $m = 1$ this reduces to

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z),$$

and for $m = 2$ it becomes the first derivative of $(z - z_0)^2 f(z)$ at z_0.

A quotient at a simple zero. Suppose p and q are holomorphic near z_0 with $q(z_0) = 0$ and $q'(z_0) \neq 0$. Here and below, $g(z) = O\big((z - z_0)^j\big)$ is shorthand for "$|g(z)| \leq C|z - z_0|^j$ for some constant C and all z near z_0": it records the size of a

The residue theorem V

discarded tail without naming its coefficients. Since $q(z_0) = 0$, the Taylor series of q starts at first order, so every term carries at least one factor $z - z_0$:

$$\begin{aligned} q(z) &= q'(z_0)(z - z_0) + \frac{q''(z_0)}{2}(z - z_0)^2 + \cdots \\ &= (z - z_0)\left[q'(z_0) + \frac{q''(z_0)}{2}(z - z_0) + \cdots\right] \\ &= (z - z_0)\left[q'(z_0) + O(z - z_0)\right]. \end{aligned}$$

With $p(z) = p(z_0) + O(z - z_0)$, dividing gives

$$\frac{p(z)}{q(z)} = \frac{1}{z - z_0}\, g(z), \qquad g(z) = \frac{p(z)}{q(z)/(z - z_0)} = \frac{p(z_0) + O(z - z_0)}{q'(z_0) + O(z - z_0)}.$$

The factor g is holomorphic at z_0: its numerator is p, and its denominator $q(z)/(z - z_0)$ is, by the factorization above, holomorphic there with value $q'(z_0) \neq 0$, hence nonzero on a small disk about z_0. So p/q has at worst a simple pole at z_0, and the $m = 1$ case of (2.7.3) reads off the coefficient of $(z - z_0)^{-1}$ as the value $g(z_0)$:

$$\operatorname{Res}\left(\frac{p}{q}, z_0\right) = \lim_{z \to z_0} (z - z_0)\frac{p(z)}{q(z)} = g(z_0) = \frac{p(z_0)}{q'(z_0)}. \tag{2.7.4}$$

The residue theorem VI

For example,

$$\operatorname{Res}\left(\frac{1}{z^2+1}, i\right) = \frac{1}{2i},$$

while the order-2 formula at the double pole of $1/(z^2+1)^2$ gives

$$\operatorname{Res}\left(\frac{1}{(z^2+1)^2}, i\right) = \left.\frac{d}{dz}\frac{1}{(z+i)^2}\right|_{z=i} = -\frac{2}{(2i)^3} = \frac{1}{4i}.$$

Quick checks on $|z|=1$.

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \cdots \quad \Longrightarrow \quad \operatorname{Res}\left(\frac{\sin z}{z^2}, 0\right) = 1,$$

so $\oint_{|z|=1} (\sin z)/z^2 \, dz = 2\pi i$. By contrast $z^2/\sin z$ extends holomorphically through 0 (value 0) and has no other zeros of $\sin z$ inside the unit disk, so

$$\oint_{|z|=1} \frac{z^2}{\sin z} \, dz = 0.$$

Branch points are not residues I

A branch point is not an isolated singularity, so it does not carry an ordinary residue. Consider

$$f(z) = \frac{z^{-k}}{z+1}, \qquad 0 < k < 1,$$

where the noninteger power is defined through the logarithm,

$$z^{-k} = e^{-k \log z}, \qquad \log z = \log|z| + i \arg z, \qquad 0 < \arg z < 2\pi,$$

so that z^{-k} is single valued and holomorphic off the cut along the positive real axis. Every such power must be evaluated from this definition: there is no general law $\left(e^{w}\right)^{c} = e^{cw}$ for complex bases, and it is precisely that law which fails when one crosses to another branch. The point $z = -1$ lies off the cut, so f has an ordinary simple pole there and the $m = 1$ formula (2.7.3) applies:

$$\operatorname{Res}(f, -1) = \lim_{z \to -1} (z+1)\frac{z^{-k}}{z+1} = \lim_{z \to -1} z^{-k} = (-1)^{-k},$$

the limit existing because z^{-k} is holomorphic at $z = -1$. In this branch $|-1| = 1$ and $\arg(-1) = \pi$, so $\log(-1) = \log 1 + i\pi = i\pi$ and

$$(-1)^{-k} = e^{-k\log(-1)} = e^{-k(i\pi)} = e^{-i\pi k}.$$

Branch points are not residues II

At $z = 0$, however, z^{-k} changes by the nontrivial factor $e^{-2\pi ik}$ after one full circuit. Thus 0 is a branch point, not an isolated singularity, and no ordinary residue is defined there.

The inner circular arc C_ε of a keyhole contour, parametrized by $z = \varepsilon e^{i\theta}$ with $0 < \theta < 2\pi$, nevertheless has a vanishing contribution. Take $\varepsilon < 1$. Since k and θ are real,

$$\left|z^{-k}\right| = \left|e^{-k(\log \varepsilon + i\theta)}\right| = e^{-k\log\varepsilon}\left|e^{-ik\theta}\right| = \varepsilon^{-k},$$

the same value for every θ, because $|e^{-ik\theta}| = 1$. For the denominator, the triangle inequality applied to $1 = (z+1) - z$ gives $1 \leq |z+1| + |z|$, that is,

$$|z+1| \geq 1 - |z| = 1 - \varepsilon > 0.$$

Hence on C_ε

$$|f(z)| = \frac{|z^{-k}|}{|z+1|} \leq \frac{\varepsilon^{-k}}{1-\varepsilon},$$

and the arc has length $2\pi\varepsilon$, so the *ML* estimate gives

$$\left|\int_{C_\varepsilon} f(z)\,dz\right| \leq \frac{2\pi\varepsilon^{1-k}}{1-\varepsilon} \longrightarrow 0$$

because $1 - k > 0$. A vanishing small-loop integral does *not* turn a branch point into an isolated singularity; it only means that arc can be discarded in a contour estimate.

Real integrals by a semicircular contour I

Let Γ_R consist of the real segment from $-R$ to R and the upper semicircle $C_R : z = Re^{i\theta}$, $0 \le \theta \le \pi$. For an integrand with no pole on the real segment,

$$\oint_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz. \tag{2.8.1}$$

If $M_R = \max_{z \in C_R} |f(z)|$, then the arc length is πR and

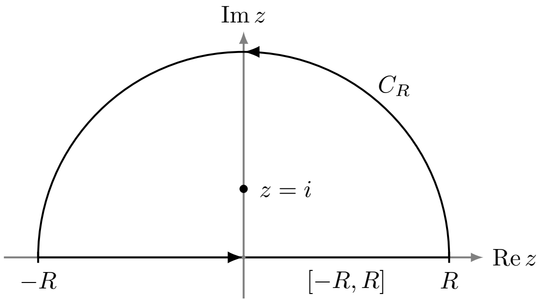
$$\left| \int_{C_R} f(z)\,dz \right| \le \pi R M_R.$$

Thus the arc vanishes whenever $RM_R \to 0$. If the improper real integral converges, the residue theorem then gives

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(f, z_k).$$

When an exponential e^{iaz} is present, its decay in one half-plane often supplies the needed estimate.

Real integrals by a semicircular contour II



Real integrals by a semicircular contour III

Example: $\int_{-\infty}^{\infty} \cos x/(1+x^2)\,dx$. Consider

$$J = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx.$$

For $z = x + iy$ in the upper half-plane,

$$|e^{iz}| = |e^{ix-y}| = e^{-y} \leq 1.$$

On $|z| = R > 1$,

$$|1+z^2| \geq \left||z|^2 - 1\right| = R^2 - 1,$$

so

$$\left|\int_{C_R} \frac{e^{iz}}{1+z^2}\,dz\right| \leq \frac{\pi R}{R^2-1} \longrightarrow 0.$$

Only the pole $z = i$ lies in the upper half-plane. Since $1 + z^2 = (z-i)(z+i)$,

$$\begin{aligned} \operatorname{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) &= \left.\frac{e^{iz}}{z+i}\right|_{z=i} \\ &= \frac{e^{-1}}{2i}. \end{aligned}$$

Real integrals by a semicircular contour IV

Hence

$$J = 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}.$$

Taking real parts gives

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}. \qquad (2.8.2)$$

The imaginary part vanishes consistently because $\sin x/(1+x^2)$ is odd.

Example: $\int_{-\infty}^{\infty} dx/(1+x^4)$. Close in the upper half-plane. The upper arc vanishes by ML: $|\int_{C_R}| \leq \pi R/(R^4-1) \to 0$ as $R \to \infty$. The poles are the fourth roots of -1:

$$z_k = e^{i(\pi+2k\pi)/4}, \qquad k = 0, 1, 2, 3.$$

Those in the upper half-plane are the simple poles

$$z_0 = e^{i\pi/4}, \qquad z_1 = e^{i3\pi/4}.$$

Since $1+z^4 = \prod_{k=0}^{3}(z-z_k)$ and $(1+z^4)' = 4z^3$, each residue is

$$\operatorname{Res}\left(\frac{1}{1+z^4}, z_j\right) = \frac{1}{4z_j^3}.$$

(Equivalently, $z_j^4 = -1$ gives $z_j^3 = -1/z_j$, so the residue is also $-z_j/4$.) Algebraically,

$$\mathrm{Res}(z_0) = \frac{1}{4z_0^3} = -\frac{z_0}{4} = -\frac{e^{i\pi/4}}{4} = -\frac{1+i}{4\sqrt{2}},$$
$$\mathrm{Res}(z_1) = \frac{1}{4z_1^3} = -\frac{z_1}{4} = -\frac{e^{i3\pi/4}}{4} = \frac{1-i}{4\sqrt{2}}.$$

Their sum is

$$\mathrm{Res}(z_0) + \mathrm{Res}(z_1) = \frac{(-1-i)+(1-i)}{4\sqrt{2}} = \frac{-2i}{4\sqrt{2}} = -\frac{i}{2\sqrt{2}}.$$

The residue theorem therefore gives

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i\left(-\frac{i}{2\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}. \tag{2.8.3}$$

Keyhole contour: $\int_0^\infty x^a/(1+x)^2\,dx$ I

Let

$$I(a) = \int_0^\infty \frac{x^a}{(1+x)^2}\,dx, \qquad -1 < a < 1.$$

The endpoint conditions follow directly: near 0 the integrand behaves as x^a, which is integrable when $a > -1$; near infinity it behaves as x^{a-2}, which is integrable when $a < 1$.

For $a \neq 0$, choose the branch of the power whose cut runs along the *positive* real axis,

$$z^a = e^{a\ell(z)}, \qquad \ell(z) = \log|z| + i\arg z, \qquad 0 < \arg z < 2\pi,$$

the same branch already used for z^{-k} on the frame "Branch points are not residues". Note that ℓ is *not* the principal logarithm Log of the earlier slide, whose cut is the *nonpositive* real axis: here the cut has been rotated onto the positive axis so that the keyhole runs along both sides of it and stays on a single branch. Use a counterclockwise keyhole contour with outer radius R and inner radius ε.

On the upper side, $z = x$ and the contribution is

$$I_{\varepsilon,R}(a) = \int_\varepsilon^R \frac{x^a}{(1+x)^2}\,dx.$$

Keyhole contour: $\int_0^\infty x^a/(1+x)^2\,dx$ II

On the lower side, $z = xe^{2\pi i}$ and $z^a = x^a e^{2\pi i a}$. The orientation is from R back to ε, so the contribution is

$$\int_R^\varepsilon \frac{x^a e^{2\pi i a}}{(1+x)^2}\,dx = -e^{2\pi i a} I_{\varepsilon,R}(a).$$

Thus the two straight segments sum to

$$(1-e^{2\pi i a}) I_{\varepsilon,R}(a).$$

On the outer circle, since a is real and $z^a = e^{a(\log R + i \arg z)}$, one has $|z^a| = R^a$. Hence

$$\left|\frac{z^a}{(1+z)^2}\right| \le \frac{R^a}{(R-1)^2} = O(R^{a-2}),$$

so the arc integral is $O(R^{a-1}) \to 0$ because $a < 1$. On the inner circle,

$$\left|\frac{z^a}{(1+z)^2}\right| = O(\varepsilon^a),$$

so the arc integral is $O(\varepsilon^{a+1}) \to 0$ because $a > -1$.

The only enclosed singularity is the double pole at $z = -1$. Since

$$(z+1)^2 \frac{z^a}{(1+z)^2} = z^a,$$

Keyhole contour: $\int_0^\infty x^a/(1+x)^2\,dx$ III

formula (2.7.3) gives

$$\operatorname{Res}\left(\frac{z^a}{(1+z)^2}, -1\right) = \frac{d}{dz} z^a\Big|_{z=-1}.$$

To differentiate the branch, note that $e^{\ell(z)} = z$ gives $\ell'(z)e^{\ell(z)} = 1$, i.e. $\ell'(z) = 1/z$, so

$$\frac{d}{dz} z^a = \frac{d}{dz} e^{a\ell(z)} = \frac{a}{z}\, e^{a\ell(z)} = a\, e^{(a-1)\ell(z)} = a\, z^{a-1}.$$

On this branch $\arg(-1) = \pi$, that is $\ell(-1) = i\pi$, and therefore

$$\frac{d}{dz} z^a\Big|_{z=-1} = a\, e^{(a-1)\ell(-1)} = a\, e^{i\pi(a-1)} = a\, e^{i\pi a} e^{-i\pi} = -a\, e^{i\pi a},$$

which is the value used in the next display. Hence the residue theorem gives, after $R \to \infty$ and $\varepsilon \to 0$,

$$(1 - e^{2\pi ia}) I(a) = -2\pi i a e^{i\pi a}.$$

Factor the denominator:

$$\begin{aligned} 1 - e^{2\pi ia} &= e^{i\pi a}(e^{-i\pi a} - e^{i\pi a}) \\ &= -2ie^{i\pi a}\sin(\pi a). \end{aligned}$$

Therefore

$$I(a) = \frac{-2\pi i a e^{i\pi a}}{-2ie^{i\pi a}\sin(\pi a)} = \frac{\pi a}{\sin(\pi a)}.$$

Thus

$$\int_0^\infty \frac{x^a}{(1+x)^2}\,dx = \frac{\pi a}{\sin(\pi a)}, \qquad -1 < a < 1. \tag{2.8.4}$$

At $a = 0$, the right-hand side has limit 1 because $\sin(\pi a) \sim \pi a$, where $u(a) \sim v(a)$ means $u(a)/v(a) \to 1$ in the stated limit, and directly $\int_0^\infty (1+x)^{-2}dx = 1$.

The same keyhole with the simpler integrand $z^{-a}/(1+z)$ (simple pole at -1) yields the companion evaluation, valid for $0 < a < 1$,

$$\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{\pi}{\sin(\pi a)}. \tag{2.8.5}$$

This is the integral form of the reflection factor that reappears with the Beta and Gamma functions. The bookkeeping runs as for $I(a)$ with one change that has to be made carefully: the exponent is now $-a$, so the phase collected on the lower side of the cut is the *conjugate* of the previous one. Indeed, on the branch $0 < \arg z < 2\pi$

the lower side is $z = xe^{2\pi i}$, whence $z^{-a} = x^{-a}e^{-2\pi ia}$, and the two straight segments sum to

$$(1 - e^{-2\pi ia}) \int_\varepsilon^R \frac{x^{-a}}{1+x}\,dx.$$

The arcs vanish for $0 < a < 1$, one endpoint condition each. On $|z| = R$, $|z^{-a}/(1+z)| \le R^{-a}/(R-1) = O(R^{-a-1})$ and the length is $2\pi R$, so that arc is $O(R^{-a}) \to 0$ because $a > 0$; on $|z| = \varepsilon$, $|z^{-a}/(1+z)| \le \varepsilon^{-a}/(1-\varepsilon) = O(\varepsilon^{-a})$ with length $2\pi\varepsilon$, so that arc is $O(\varepsilon^{1-a}) \to 0$ because $a < 1$. The only enclosed singularity is the simple pole $z = -1$, where the residue is $(-1)^{-a} = e^{-i\pi a}$, so the residue theorem gives

$$(1 - e^{-2\pi ia}) \int_0^\infty \frac{x^{-a}}{1+x}\,dx = 2\pi i\, e^{-i\pi a}.$$

Factoring the prefactor as before,

$$1 - e^{-2\pi ia} = e^{-i\pi a}\left(e^{i\pi a} - e^{-i\pi a}\right) = 2i\, e^{-i\pi a}\sin(\pi a),$$

and dividing,

$$\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{2\pi i\, e^{-i\pi a}}{2i\, e^{-i\pi a}\sin(\pi a)} = \frac{\pi}{\sin(\pi a)},$$

which is (2.8.5).

Jordan's lemma and $\int \sin x/x\,dx$ I

Indentation at a simple pole on the contour. Split f into its singular and regular parts near a simple pole z_0,

$$f(z) = \frac{r_{-1}}{z - z_0} + g(z), \qquad r_{-1} = \mathrm{Res}(f, z_0).$$

Because the pole is simple, the Laurent series of f has only the one negative power, so g has a removable singularity at z_0; it is therefore holomorphic, hence bounded, on some closed disc: $|g(z)| \le M$ for $|z - z_0| \le \varepsilon_0$.

Let C_ε^θ be the circular arc $z = z_0 + \varepsilon e^{i\phi}$ of opening angle θ, traversed in the sense σ, with $\sigma = +1$ for counterclockwise and $\sigma = -1$ for clockwise. Then $dz = i\varepsilon e^{i\phi}\,d\phi$, and the two factors of ε cancel:

$$\frac{r_{-1}}{z - z_0}\,dz = \frac{r_{-1}}{\varepsilon e^{i\phi}}\,i\varepsilon e^{i\phi}\,d\phi = i\,r_{-1}\,d\phi.$$

Integrating a constant over an angular range of length θ swept in the sense σ therefore gives, for *every* $\varepsilon > 0$,

$$\int_{C_\varepsilon^\theta} \frac{r_{-1}}{z - z_0}\,dz = i\sigma\theta\,r_{-1}.$$

Jordan's lemma and $\int \sin x/x\,dx$ II

The regular part is what needs the limit. The arc has length $\varepsilon\theta$, so by the *ML* estimate

$$\left| \int_{C_\varepsilon^\theta} g(z)\,dz \right| \le M\varepsilon\theta \xrightarrow[\varepsilon\to 0]{} 0.$$

Adding the two contributions,

$$\int_{C_\varepsilon^\theta} f(z)\,dz \longrightarrow i\sigma\theta\,\mathrm{Res}(f, z_0), \tag{2.8.6}$$

the first term being exact at each ε and the second vanishing in the limit. A clockwise upper semicircle about a real-axis pole therefore contributes $-i\pi\,\mathrm{Res}(f, z_0)$.

Jordan-type semicircle estimate

Let C_R be the upper semicircle and let $M_R = \max_{z\in C_R} |f(z)|$. If $M_R \to 0$ as $R \to \infty$, then for every $a > 0$,

$$\int_{C_R} f(z)e^{iaz}\,dz \longrightarrow 0.$$

Jordan's lemma and $\int \sin x/x\,dx$ III

Analyticity of f is not needed for this arc estimate itself; it is needed elsewhere when the residue theorem is applied.

Parameterize $z = Re^{i\theta}$. Since $|e^{iaz}| = e^{-aR\sin\theta}$,

$$\left|\int_{C_R} f(z)e^{iaz}\,dz\right| \le M_R R\int_0^{\pi} e^{-aR\sin\theta}\,d\theta$$
$$= 2M_R R\int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta.$$

On $0 \le \theta \le \pi/2$, sin is concave down ($\sin'' = -\sin \le 0$), so its graph lies above the chord joining $(0,0)$ to $(\pi/2, 1)$: $\sin\theta \ge 2\theta/\pi$. Therefore

$$\left|\int_{C_R} f(z)e^{iaz}\,dz\right| \le 2M_R R\int_0^{\pi/2} e^{-2aR\theta/\pi}\,d\theta$$
$$= \frac{\pi M_R}{a}(1 - e^{-aR})$$
$$\longrightarrow 0.$$

Example: the Dirichlet integral. Integrate e^{iz}/z over the upper semicircle, indenting clockwise above $z = 0$. On the large arc, take $f(z) = 1/z$, so

Jordan's lemma and $\int \sin x/x\, dx$ IV

$M_R = 1/R \to 0$ and the Jordan estimate applies. The small semicircle over $z = 0$ is traversed clockwise through angle π, so by the indentation rule (2.8.6) it contributes $i(-1)\pi\,\mathrm{Res}$:

$$\begin{aligned} -i\pi\,\mathrm{Res}\left(\frac{e^{iz}}{z}, 0\right) &= -i\pi \lim_{z\to 0} e^{iz} \\ &= -i\pi. \end{aligned}$$

The indentation excludes the only pole, so e^{iz}/z is holomorphic on and inside the indented contour and Cauchy's theorem makes the total integral zero. Written out, that contour has four pieces — the segment $[-R, -\varepsilon]$, the small semicircle C_ε traversed clockwise from $-\varepsilon$ to $+\varepsilon$, the segment $[\varepsilon, R]$, and the large semicircle C_R:

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int_{C_\varepsilon} \frac{e^{iz}}{z}\, dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\, dx + \int_{C_R} \frac{e^{iz}}{z}\, dz = 0.$$

The two real segments cannot be sent to their limits separately: near the origin $\mathrm{Re}(e^{ix}/x) = \cos x/x \sim 1/x$, which is not integrable on either side, and only the symmetric combination has a limit. That combined limit is what the notation PV records; for a function g with a single interior singularity at the origin we define the *Cauchy principal value*

$$\mathrm{PV}\int_{-\infty}^{\infty} g(x)\,dx := \lim_{\substack{\varepsilon\to 0^+\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} g(x)\,dx + \int_{\varepsilon}^{R} g(x)\,dx\right),$$

whenever this limit exists. The large arc tends to 0 by the Jordan estimate above (with $f(z) = 1/z$, $M_R = 1/R \to 0$ and $a = 1$), and the small arc tends to $-i\pi$ by the indentation rule (2.8.6), as just computed. Letting $R \to \infty$ and $\varepsilon \to 0$ in the four-piece identity therefore gives

$$0 = \mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - i\pi,$$

and hence

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = i\pi. \tag{2.8.7}$$

Taking imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi. \tag{2.8.8}$$

Jordan's lemma and $\int \sin x/x \, dx$ VI

The integrand $\sin x/x$ is continuous at 0 after assigning the value 1, so its integral needs no principal-value interpretation there; the principal value was required only for the auxiliary exponential integral.

The steepest-descent approximation I

Step 1: set up the large-parameter integral. Many asymptotic formulas for special functions come from integrals of the form

$$I(s) = \int_\Gamma g(z)\, e^{sf(z)}\, dz, \qquad s \to +\infty, \tag{2.9.1}$$

where f and g are holomorphic near a contour Γ that may be deformed without crossing singularities of the integrand. Write $f = u + iv$ with real u, v. As s grows, e^{su} makes the integrand largest where u is largest on the deformed contour, so the leading contribution comes from a neighborhood of a *saddle* of u.

Why a saddle rather than a peak? Apply Cauchy's integral formula (2.4.3) on a small circle $z = z_0 + \rho e^{i\psi}$ about any point z_0 where f is holomorphic. There $dz = i\rho e^{i\psi}\, d\psi$ and $z - z_0 = \rho e^{i\psi}$, so $dz/(z - z_0) = i\, d\psi$ and

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + \rho e^{i\psi}\right) d\psi.$$

Taking real parts,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + \rho e^{i\psi}\right) d\psi,$$

so u equals its own average over every small circle about z_0. A strict interior maximum would force that average to be strictly smaller than $u(z_0)$, and a strict interior minimum would force it to be strictly larger; u therefore has neither. From any interior point some direction goes uphill and some goes downhill, so the contour cannot be parked in a basin where u is small in every direction. What we can do is choose the deformation that makes the largest value of u on the contour as small as possible,

$$\min_{\Gamma} \max_{z\in\Gamma} u(z).$$

The reason a critical point appears is now visible, at least as a design principle: if at the highest point z_0 of the optimal contour some direction were still downhill for u, pushing the contour a little that way would strictly lower $\max_\Gamma u$ and give a better deformation — so at the optimum we expect $u_x(z_0) = u_y(z_0) = 0$. The dominant point is therefore a critical point of u; Step 2 shows that for holomorphic f this is exactly $f'(z_0) = 0$, and that every such point is a saddle of u rather than a peak.

Step 2: locate a nondegenerate saddle. For holomorphic f, the Cauchy–Riemann equations give

$$f'(z) = u_x + iv_x = u_x - iu_y.$$

Thus $f'(z_0) = 0$ forces $u_x(z_0) = u_y(z_0) = 0$, and then also $v_x(z_0) = v_y(z_0) = 0$: both u and v are stationary at z_0. The Cauchy–Riemann equations also control the second derivatives. Every holomorphic function is infinitely differentiable by (2.4.4), so u and v have continuous partial derivatives of all orders and mixed partials may

The steepest-descent approximation III

be taken in either order. Differentiating $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y,

$$u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy} = -v_{yx},$$

and adding,

$$u_{xx} + u_{yy} = 0.$$

A function satisfying this Laplace equation is called *harmonic*; in particular $u_{yy} = -u_{xx}$, so u curves downward in one direction exactly as much as it curves upward in the other. Differentiating $f' = u_x + iv_x$ along the real direction and using $v_x = -u_y$ once more,

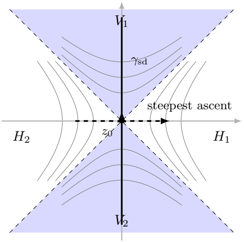
$$f''(z) = u_{xx} + iv_{xx} = u_{xx} - iu_{xy}, \qquad \text{so} \qquad |f''(z_0)|^2 = u_{xx}^2 + u_{xy}^2.$$

Hence the Hessian determinant of u at z_0 is

$$\det D^2 u(z_0) = u_{xx}u_{yy} - u_{xy}^2 = -u_{xx}^2 - u_{xy}^2 = -|f''(z_0)|^2 \leq 0,$$

negative whenever $f''(z_0) \neq 0$: a critical point of f is never a peak or a basin of u, always a saddle. A nondegenerate saddle means $f''(z_0) \neq 0$.

The steepest-descent approximation IV



In the schematic, hill sectors and valley sectors meet at the saddle; a steepest-descent path through the saddle runs into the valleys of $u = \operatorname{Re} f$.

Step 3: choose the descent direction. Taylor expansion at a nondegenerate saddle is

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + O\big((z - z_0)^3\big). \tag{2.9.2}$$

Write

$$f''(z_0) = |f''(z_0)| e^{i\theta}, \qquad z - z_0 = re^{i\phi}.$$

Then

$$\frac{1}{2} f''(z_0)(z - z_0)^2 = \frac{1}{2} |f''(z_0)| r^2 \, e^{i(\theta + 2\phi)}.$$

A *steepest-descent* direction makes this quadratic change negative real, so

$$\theta + 2\phi = \pi \pmod{2\pi}, \qquad \text{that is} \qquad \phi = \frac{\pi - \theta}{2} \pmod{\pi}. \tag{2.9.3}$$

The ambiguity by π is not a defect: it is the two opposite rays leaving z_0, and Step 4 uses both at once by letting r run over all of $\mathbb{R}$. Along that ray,

$$\frac{1}{2} f''(z_0)(z - z_0)^2 = -\frac{1}{2} |f''(z_0)| r^2.$$

(The orthogonal choice $\theta + 2\phi = 0$ is steepest *ascent*.)

Step 4: reduce to a Gaussian integral. Assume the contour can be deformed through z_0 along both descent rays, that this saddle dominates the integral, and that $g(z_0) \neq 0$. Parameterize the full descent line by

$$\alpha = e^{i(\pi-\theta)/2} = i\, e^{-i\theta/2}, \qquad z = z_0 + \alpha r, \qquad r \in \mathbb{R},$$

so $dz = \alpha\, dr$ and $\alpha^2 = -e^{-i\theta}$. Then

$$f''(z_0)\alpha^2 = |f''(z_0)| e^{i\theta}(-e^{-i\theta}) = -|f''(z_0)|.$$

Replacing f by its quadratic Taylor piece and g by the constant $g(z_0)$ yields

$$\begin{aligned} I(s) &\sim g(z_0) e^{sf(z_0)} \alpha \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2} s |f''(z_0)| r^2\Big)\, dr \\ &= g(z_0) e^{sf(z_0)} \alpha \sqrt{\frac{2\pi}{s|f''(z_0)|}}, \end{aligned}$$

where we used $\int_{-\infty}^{\infty} e^{-ar^2}\, dr = \sqrt{\pi/a}$ with $a = \frac{1}{2} s |f''(z_0)|$.
Therefore, for the stated orientation,

$$I(s) \sim g(z_0) e^{sf(z_0)}\, i\, e^{-i\theta/2} \sqrt{\frac{2\pi}{s|f''(z_0)|}}. \tag{2.9.4}$$

The steepest-descent approximation VII

Reversing the contour multiplies by -1. If only one descent ray is available (endpoint saddle), take half the Gaussian. If several saddles share the same dominant real part, sum their contributions.

Real special cases: Laplace method and stationary phase I

The same local-Gaussian idea appears in two real-variable forms used later in the course.

Laplace's method (real, decaying exponential). For a smooth real phase φ with a unique interior maximum at x_0, $\varphi'(x_0) = 0$, $\varphi''(x_0) < 0$, and a smooth amplitude g with $g(x_0) \neq 0$,

$$\int_a^b g(x)\, e^{s\varphi(x)}\, dx \sim g(x_0) e^{s\varphi(x_0)} \sqrt{\frac{2\pi}{s\left|\varphi''(x_0)\right|}}, \qquad s \to +\infty. \tag{2.9.5}$$

This is (2.9.4) restricted to the real line with $\theta = \pi$ (so $f''(x_0) = -|f''(x_0)|$ is already negative real and $\alpha = 1$). Chapter 3 uses this pattern for Stirling's approximation of $n!$ from the gamma integral.

Stationary phase (real, oscillatory). For a real phase φ with $\varphi'(x_0) = 0$, $\varphi''(x_0) \neq 0$,

$$\int g(x)\, e^{is\varphi(x)}\, dx \sim g(x_0) e^{is\varphi(x_0)} e^{i(\pi/4)\ \mathrm{sgn}\, \varphi''(x_0)} \sqrt{\frac{2\pi}{s\left|\varphi''(x_0)\right|}}, \qquad s \to +\infty, \tag{2.9.6}$$

under standard nondegeneracy and endpoint hypotheses. The Fresnel phase factor arises from the local Gaussian limit. Near x_0,

$$\varphi(x) \approx \varphi(x_0) + \tfrac{1}{2}\varphi''(x_0)\,(x - x_0)^2.$$

Real special cases: Laplace method and stationary phase II

Set $u = \sqrt{s}\,(x - x_0)$, so $dx = du/\sqrt{s}$. Then

$$\int g(x)\, e^{is\varphi(x)}\, dx \sim e^{is\varphi(x_0)}\, \frac{g(x_0)}{\sqrt{s}} \int_{-\infty}^{\infty} e^{i\,(\varphi''(x_0)/2)\,u^2}\, du.$$

For $a > 0$, obtain the Fresnel integrals from the real Gaussian by a damped limit. The parameter is now complex, so the real formula does not apply directly; it extends by analytic continuation. For $\operatorname{Re} c > 0$ we have $|e^{-cu^2}| = e^{-(\operatorname{Re} c)u^2}$, so

$$G(c) = \int_{-\infty}^{\infty} e^{-cu^2}\, du$$

converges absolutely, and uniformly for c in any compact subset of the half-plane $\operatorname{Re} c > 0$; the same bound applied to $u^2 e^{-cu^2}$ lets us differentiate under the integral sign, so G is holomorphic there. The principal branch of $\sqrt{\pi/c}$ is holomorphic on that half-plane too, since the half-plane misses the cut. The two agree on the positive real axis, where they reduce to the real Gaussian integral, and agreement on a set with a limit point forces agreement throughout by the identity theorem. Hence

$$\int_{-\infty}^{\infty} e^{-cu^2}\, du = \sqrt{\frac{\pi}{c}}, \qquad \operatorname{Re} c > 0.$$

Real special cases: Laplace method and stationary phase III

With $\varepsilon > 0$ and $c = \varepsilon \mp ia$ this reads

$$\int_{-\infty}^{\infty} e^{-(\varepsilon \mp ia)u^2}\, du = \sqrt{\frac{\pi}{\varepsilon \mp ia}},$$

taking the principal branch of the square root (positive on the positive reals). As $\varepsilon \to 0^+$,

$$\varepsilon \mp ia \to a\, e^{\mp i\pi/2} \qquad \Longrightarrow \qquad \sqrt{\frac{\pi}{\varepsilon \mp ia}} \to \sqrt{\frac{\pi}{a}}\, e^{\pm i\pi/4},$$

The limit may be taken inside the integral. On any bounded interval $e^{-(\varepsilon \mp ia)u^2} \to e^{\pm iau^2}$ uniformly as $\varepsilon \to 0^+$, and the tails are small uniformly in $\varepsilon \geq 0$: writing $e^{-cu^2} = \left(-\frac{1}{2cu}\right)\dfrac{d}{du}e^{-cu^2}$ and integrating by parts,

$$\int_R^{\infty} e^{-cu^2}\, du = \frac{e^{-cR^2}}{2cR} - \int_R^{\infty} \frac{e^{-cu^2}}{2cu^2}\, du,$$

and since $|e^{-cu^2}| \leq 1$ for $\operatorname{Re} c \geq 0$ and $|c| = |\varepsilon \mp ia| \geq a$,

$$\left|\int_R^{\infty} e^{-cu^2}\, du\right| \leq \frac{1}{2|c|R} + \frac{1}{2|c|}\int_R^{\infty} \frac{du}{u^2} = \frac{1}{|c|R} \leq \frac{1}{aR} \xrightarrow[R\to\infty]{} 0.$$

Real special cases: Laplace method and stationary phase IV

The bound is the same at $\varepsilon = 0$, so the Fresnel integral exists as an improper (conditionally convergent) integral and equals the limit of the damped ones; hence

$$\int_{-\infty}^{\infty} e^{\pm i\, a\, u^2}\, du = \sqrt{\frac{\pi}{a}}\, e^{\pm i\pi/4}.$$

If $\varphi''(x_0) > 0$, take $a = \varphi''(x_0)/2$ and the $+$ sign to get $e^{i\pi/4}\sqrt{2\pi/\varphi''(x_0)}$; if $\varphi''(x_0) < 0$, the $-$ sign supplies $e^{-i\pi/4}\sqrt{2\pi/|\varphi''(x_0)|}$. In both cases the phase is $e^{i(\pi/4)\ \mathrm{sgn}\,\varphi''(x_0)}$ and the amplitude is $\sqrt{2\pi/(s|\varphi''(x_0)|)}$, which is (2.9.6). Chapter 6 evaluates the large-argument expansion of $J_n(x)$ by stationary phase on the unit-circle integral representation.

Worked example: large binomial coefficient I

Let $0 < m < n$ with both m and $n - m$ large, and set $t = m/n \in (0,1)$. From $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$, Cauchy's coefficient formula gives

$$\binom{n}{m} = \frac{1}{2\pi i} \oint \frac{(1+z)^n}{z^{m+1}}\, dz = \oint g(z) e^{nf(z)}\, dz,$$

where

$$g(z) = \frac{1}{2\pi i z}, \qquad f(z) = \log(1+z) - t \log z$$

(with branches chosen locally near the positive saddle below).

Saddle. Differentiate:

$$f'(z) = \frac{1}{1+z} - \frac{t}{z}, \qquad f''(z) = -\frac{1}{(1+z)^2} + \frac{t}{z^2}.$$

Set $f'(z_0) = 0$:

$$z_0 - t(1+z_0) = 0 \quad \Longrightarrow \quad z_0 = \frac{t}{1-t} = \frac{m}{n-m} > 0.$$

Worked example: large binomial coefficient II

Then $1 + z_0 = 1/(1-t)$, and

$$f(z_0) = \log \frac{1}{1-t} - t \log \frac{t}{1-t} = -t \log t - (1-t)\log(1-t),$$

$$f''(z_0) = -\frac{1}{(1+z_0)^2} + \frac{t}{z_0^2} = -(1-t)^2 + \frac{(1-t)^2}{t}$$

$$= (1-t)^2\left(\frac{1}{t} - 1\right) = \frac{(1-t)^3}{t} > 0.$$

Thus $\theta = \arg f''(z_0) = 0$, so the descent direction from (2.9.3) is $\phi = \pi/2$: the local descent line is vertical. Take the contour in Cauchy's coefficient formula to be the circle $|z| = z_0$. This is allowed because $(1+z)^n z^{-m-1}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, so by Cauchy's theorem every circle about the origin gives the same integral. To see that the saddle dominates on this circle, write $z = z_0 e^{i\psi}$ and use

$$\left|1 + z_0 e^{i\psi}\right|^2 = (1 + z_0 \cos\psi)^2 + (z_0 \sin\psi)^2 = 1 + 2z_0\cos\psi + z_0^2,$$

while $|z| = z_0$ is constant. Since $\operatorname{Re} f = \log|1+z| - t\log|z|$,

$$u(\psi) = \operatorname{Re} f\left(z_0 e^{i\psi}\right) = \tfrac{1}{2}\log\left(1 + 2z_0\cos\psi + z_0^2\right) - t\log z_0.$$

Only $\cos\psi$ depends on ψ, and $z_0 > 0$, so u decreases strictly as $|\psi|$ increases on $[-\pi, \pi]$ and attains its unique maximum at $\psi = 0$, i.e. at $z = z_0$. The saddle

Worked example: large binomial coefficient III

therefore dominates, as Step 4 requires. The counterclockwise tangent to the circle at $z = z_0$ is $+i$, which is the descent direction just found, matching $\alpha = i$ in (2.9.4).

Apply (2.9.4) **with $s = n$.**

$$\binom{n}{m} \sim \frac{1}{2\pi i z_0}\, e^{nf(z_0)}\, i\sqrt{\frac{2\pi}{nf''(z_0)}} = \frac{1}{2\pi z_0}\, e^{nf(z_0)}\sqrt{\frac{2\pi}{nf''(z_0)}}.$$

Substitute $1/z_0 = (1-t)/t$ and $1/f''(z_0) = t/(1-t)^3$:

$$\frac{1}{2\pi z_0}\sqrt{\frac{2\pi}{nf''(z_0)}} = \frac{1-t}{2\pi t}\sqrt{\frac{2\pi t}{n(1-t)^3}} = \frac{1}{\sqrt{2\pi n t(1-t)}}.$$

Also $e^{nf(z_0)} = \exp\Big[-n\Big(t\log t + (1-t)\log(1-t)\Big)\Big]$, so

$$\binom{n}{m} \sim \frac{\exp\Big[-n\Big(t\log t + (1-t)\log(1-t)\Big)\Big]}{\sqrt{2\pi n t(1-t)}}. \tag{2.9.7}$$

Worked example: large binomial coefficient IV

Undoing $t = m/n$ turns this into powers of m, $n-m$ and n. Since $nt = m$ and $n(1-t) = n-m$,

$$e^{-nt\log t} = t^{-nt} = \left(\frac{m}{n}\right)^{-m} = \frac{n^m}{m^m},$$

$$e^{-n(1-t)\log(1-t)} = (1-t)^{-n(1-t)} = \left(\frac{n-m}{n}\right)^{-(n-m)} = \frac{n^{\,n-m}}{(n-m)^{\,n-m}},$$

and multiplying the two, $e^{nf(z_0)} = n^n \big/ \big(m^m (n-m)^{\,n-m}\big)$. For the denominator,

$$\sqrt{2\pi n\, t(1-t)} = \sqrt{2\pi n \cdot \frac{m}{n} \cdot \frac{n-m}{n}} = \sqrt{\frac{2\pi m(n-m)}{n}}.$$

Dividing, and absorbing $\sqrt{n}$, $1/\sqrt{m}$ and $1/\sqrt{n-m}$ into the three powers,

$$\binom{n}{m} \sim \frac{n^n}{m^m (n-m)^{\,n-m}} \sqrt{\frac{n}{2\pi m(n-m)}} = \frac{1}{\sqrt{2\pi}}\, \frac{n^{\,n+1/2}}{m^{\,m+1/2}(n-m)^{\,n-m+1/2}},$$

the same leading approximation obtained by applying $k! \sim \sqrt{2\pi k}\,(k/e)^k$ to the three factorials in $\binom{n}{m}$.

Summary I

- Complex differentiability requires one direction-independent difference-quotient limit. With continuous first partial derivatives, this is equivalent to the Cauchy–Riemann equations $u_x = v_y$, $u_y = -v_x$.
- Cauchy's theorem implies path independence on simply connected domains. Cauchy's integral formula yields all derivatives and the local Taylor expansion.
- A Laurent series is valid on an annulus. Its principal part classifies an isolated singularity as removable, a pole, or essential. Branch points are not isolated and do not possess ordinary residues.
- The residue is the coefficient a_{-1}. The residue theorem evaluates a closed contour integral as $2\pi i$ times the sum of enclosed residues.
- Three real-integral templates: semicircle (rational/exponential), keyhole (powers and branch cuts, as in $\int_0^\infty x^a/(1+x)^2\,dx$), and Jordan's indented semicircle (oscillatory integrals such as $\int \sin x/x\,dx$). Always check orientation and vanishing arcs before taking limits.
- **Steepest descent** (2.9.4) is a local Gaussian approximation at a nondegenerate dominant saddle. Real special cases: Laplace method (2.9.5) (Stirling in Ch. 3) and stationary phase (2.9.6) (large-x Bessel in Ch. 6). Orientation and the phase of the square root are part of the answer.