Chapter 1: The Guitar and the Drum

Special Functions of Mathematical Physics

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Outline

Where we are going

Two instruments, one idea.

- A plucked **guitar string** and a struck **circular drum** are both described by the *wave equation*, which we build from a lattice of masses and springs (Hooke's law).
- **Separation of variables** turns each wave equation into ordinary differential equations for the spatial shape and the time evolution.
- For the string the spatial equation gives **sinusoids** (Fourier modes). For the drum it gives **Bessel's equation**, whose solutions are not built from elementary functions.
- Functions that arise this way but cannot be written in closed form with elementary functions are named: they are the **special functions** of the title.

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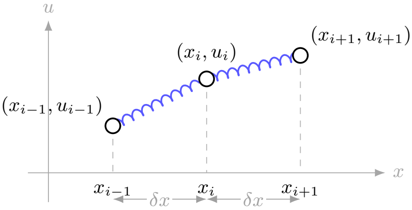
From masses and springs to the wave equation I

Model. We model a plucked string by its vertical displacement $u(x, t)$, a function of the horizontal position x along the string and the time t. Write $u_i = u(x_i, t)$ for the displacement at x_i. Partition the string into uniform mesh intervals and represent its distributed mass by small weights of mass m at nodes $\big(x_i,\ u_i\big)$, joined to their neighbors by massless springs. The nodes move only vertically, so the horizontal positions x_i are fixed with uniform spacing δx.



From masses and springs to the wave equation II

Force balance. In this scalar small-displacement model, each spring contributes a force proportional to the signed relative displacement (Hooke's law, stiffness κ). Newton's second law for node i balances its acceleration against the contributions from the springs to its left and right:

$$F = m\,\frac{d^2 u_i}{dt^2} = \kappa\,(u_{i+1} - u_i) - \kappa\,(u_i - u_{i-1}). \tag{1.1.1}$$

The two terms come from the two neighbors: node $i+1$ contributes $\kappa(u_{i+1} - u_i)$, while node $i-1$ contributes $\kappa(u_{i-1} - u_i) = -\kappa(u_i - u_{i-1})$.

Prepare the continuum limit. Use n mesh intervals across a string of length $L = n\,\delta x$, and hold fixed the total mass scale $M = nm$ and the end-to-end stiffness scale $K = \kappa/n$. We rewrite the right-hand side of (1.1.1) to expose a difference quotient. Combining the two spring terms,

$$\kappa(u_{i+1} - u_i) - \kappa(u_i - u_{i-1}) = \kappa\,(u_{i+1} - 2u_i + u_{i-1}).$$

From masses and springs to the wave equation III

Divide (1.1.1) by m and multiply and divide by $(\delta x)^2$ so the bracket becomes a centered second difference. Multiplying and dividing by $(\delta x)^2$ leaves the value unchanged, $\frac{\kappa}{m} = \frac{\kappa}{m}(\delta x)^2 \cdot \frac{1}{(\delta x)^2}$, so naming the constant prefactor $c^2 := \frac{\kappa}{m}(\delta x)^2$,

$$\frac{d^2 u_i}{dt^2} = \frac{\kappa}{m}\left(u_{i+1} - 2u_i + u_{i-1}\right) = \frac{\kappa}{m}(\delta x)^2\, \frac{u_{i+1} - 2u_i + u_{i-1}}{(\delta x)^2} = c^2\, \frac{u_{i+1} - 2u_i + u_{i-1}}{(\delta x)^2}. \tag{1.1.2}$$

Take the limit. To see that this prefactor is finite and independent of n, use $\kappa = Kn$, $m = M/n$, and $L = n\,\delta x$ (so $\delta x = L/n$):

$$c^2 = \frac{\kappa}{m}\,(\delta x)^2 = \frac{Kn}{M/n}\left(\frac{L}{n}\right)^2 = \frac{Kn^2}{M}\cdot\frac{L^2}{n^2} = \frac{KL^2}{M},$$

so $c^2 = KL^2/M$: the powers of n cancel, leaving a fixed material constant.

It remains to identify what the centered difference becomes in the limit. Write $h = \delta x$, hold t fixed, write $u_x, u_{xx}, \ldots$ for the partial derivatives with respect to x, and assume u has continuous derivatives through fourth order in x. Taylor's theorem about x_i, applied to the two neighboring nodes, gives

$$u(x_i \pm h,\ t) = u \pm h\,u_x + \frac{h^2}{2}\,u_{xx} \pm \frac{h^3}{6}\,u_{xxx} + O\!\left(h^4\right),$$

From masses and springs to the wave equation IV

every derivative on the right evaluated at (x_i, t), the $O(h^4)$ term being controlled by the fourth derivative, which is continuous and hence bounded near x_i. Adding the two expansions, the odd-order terms cancel in pairs because they enter with opposite signs:

$$u_{i+1} + u_{i-1} = 2u + h^2\, u_{xx} + O\big(h^4\big).$$

Subtracting $2u_i = 2u$ and dividing by h^2,

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\delta x)^2} = \frac{\partial^2 u}{\partial x^2}(x_i, t) + O\big((\delta x)^2\big),$$

the leading error being $\frac{1}{12}(\delta x)^2 u_{xxxx}$. It is proportional to $(\delta x)^2$ and so vanishes as $\delta x \to 0$ (equivalently $n \to \infty$), while the prefactor $c^2 = KL^2/M$ just computed does not depend on n and stays fixed. Passing to the limit in (1.1.2) therefore gives

$$\frac{\partial^2 u}{\partial t^2} = c^2\, \frac{\partial^2 u}{\partial x^2}. \tag{1.1.3}$$

This is the (one-dimensional) *wave equation*, and c is the *wave speed*.

From masses and springs to the wave equation V

Initial and boundary data. Equation (1.1.3) has two time derivatives and two space derivatives, so a unique solution needs two initial conditions and two boundary conditions. For a plucked string of length $L = 1$, clamped at both ends and released from rest in some initial shape $u_0(x)$:

$$\begin{aligned} \text{initial:} \quad & u(x,0) = u_0(x), \qquad && \frac{\partial}{\partial t} u(x,0) = 0, \\ \text{boundary:} \quad & u(0,t) = 0, && u(1,t) = 0. \end{aligned}$$

The string starts in the shape u_0 with zero velocity, and its endpoints never move.

Separation of variables for the string I

Step 1: separate shape from time. We look for a solution that factors into a pure shape times a pure time signal,

$$u(x, t) = X(x)\, T(t). \tag{1.1.4}$$

Substituting (1.1.4) into the wave equation (1.1.3), and writing a prime for differentiation with respect to a function's *independent* variable, gives $XT'' = c^2 TX''$. Dividing by XT separates the variables:

$$\frac{T''}{T} = c^2\, \frac{X''}{X}. \tag{1.1.5}$$

Step 2: introduce the separation constant. The left side depends only on t; the right side depends only on x. Two expressions in different variables can stay equal for all x and t only if both equal the same constant. Calling that constant λ,

$$\frac{1}{c^2}\, \frac{T''}{T} = \frac{X''}{X} = \lambda. \tag{1.1.6}$$

Step 3: pin down λ. Nothing so far forces λ to be a real number, so before testing signs we must rule out complex values. Multiply the spatial equation $X'' = \lambda X$ by the complex conjugate $\overline{X}$ and integrate across the string:

$$\int_0^1 \overline{X}\, X''\, dx = \lambda \int_0^1 |X|^2\, dx.$$

Separation of variables for the string II

Integrate the left-hand side by parts; the clamped ends $X(0) = X(1) = 0$ kill the boundary term, and $\overline{X'}\,X' = |X'|^2$:

$$\int_0^1 \overline{X}\,X''\,dx = \Big[\,\overline{X}\,X'\,\Big]_0^1 - \int_0^1 \overline{X'}\,X'\,dx = -\int_0^1 |X'|^2\,dx.$$

For a nontrivial mode $\int_0^1 |X|^2\,dx > 0$, so we may divide by it:

$$\lambda = -\frac{\displaystyle\int_0^1 |X'|^2\,dx}{\displaystyle\int_0^1 |X|^2\,dx}.$$

The right-hand side is a quotient of two real nonnegative numbers, so λ is real – a complex separation constant is impossible – and in fact $\lambda \leq 0$. The three real cases below therefore exhaust every possibility. It is still worth seeing concretely what goes wrong when $\lambda \geq 0$, so we now test the sign by writing $\lambda = \pm k^2$ for a real constant k.

Positive constant: $\lambda = +k^2$. Then $X'' = k^2 X$, with general solution

$$X(x) = A\,e^{kx} + B\,e^{-kx}.$$

Separation of variables for the string III

The clamped ends require $X(0) = X(1) = 0$. This is a homogeneous linear system for (A, B),

$$A + B = 0, \qquad Ae^k + Be^{-k} = 0,$$

whose coefficient determinant is $e^{-k} - e^k = -2\sinh k$. For any $k \neq 0$ this is nonzero, so the only solution is the trivial one: from $A = -B$ the second equation gives $B(e^{-k} - e^k) = 0$, hence $B = 0$ and then $A = 0$. Thus $X \equiv 0$: only the trivial (silent) string. We reject this case.

Zero constant: $\lambda = 0$. Then $X'' = 0$, so $X(x) = Ax + B$. The boundary conditions give $B = 0$ from $X(0) = 0$ and then $A = 0$ from $X(1) = 0$. This also gives only the silent solution, so the zero mode is rejected.

Separation of variables for the string IV

Negative constant: $\lambda = -k^2$. Then

$$X'' = -k^2 X, \qquad T'' = -c^2 k^2 T.$$

The spatial equation has oscillatory solutions

$$X(x) = A\cos kx + B\sin kx.$$

Imposing $X(0) = 0$ kills the cosine, $A = 0$. Imposing $X(1) = B\sin k = 0$ with $B \neq 0$ forces $\sin k = 0$, hence $k = n\pi$ for a positive integer $n = 1, 2, \ldots$. The value $n = 0$ is the zero mode already rejected, and negative n repeats the same sine shape up to a minus sign. The mode indexed by n fits n half-wavelengths along the string.

The time equation $T'' = -c^2 k^2 T$ has solution

$$T(t) = C\cos ckt + D\sin ckt.$$

The release-from-rest condition $\partial_t u(x, 0) = 0$ means $T'(0) = 0$. Since $T'(0) = ckD$, we need $D = 0$, leaving $T(t) = C\cos ckt$.

Multiplying the surviving factors and writing $k = n\pi$, each positive integer n yields an independent standing-wave mode

$$u_n(x, t) = a_n \sin(n\pi x)\cos(cn\pi t), \tag{1.1.7}$$

with $a_n = BC$ a single constant per mode.

Step 1: superpose the modes. The wave equation is linear, so every finite linear combination of the modes (1.1.7) is again a solution. Under the usual convergence hypotheses (or in the L^2 sense for the spatial expansion), the corresponding sine-mode series is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \cos(cn\pi t). \tag{1.1.8}$$

Here "in the L^2 sense" means that the mean-square error tends to zero: if $S_N(x)$ is the sum through $n = N$, then

$$\int_0^1 \left|u_0(x) - S_N(x)\right|^2 dx \longrightarrow 0.$$

Step 2: match the initial shape. The coefficients a_n are fixed by the remaining condition, the initial shape. Setting $t = 0$ (where every cos equals 1) gives

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = u_0(x). \tag{1.1.9}$$

Expression (1.1.9) is the *Fourier expansion* of $u_0(x)$, and each a_n is a *Fourier coefficient*. The pure sinusoids $\sin(n\pi x)$ are the special functions of the guitar string—except that here they are elementary, so we already know them. The drum will not be so kind.

The Fourier expansion of the string II

Step 3: sine orthogonality. To isolate one coefficient we need

$$I_{nm} := \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx.$$

Use the product-to-sum identity $2\sin A\sin B = \cos(A-B) - \cos(A+B)$ with $A = n\pi x$ and $B = m\pi x$:

$$\sin(n\pi x)\sin(m\pi x) = \frac{1}{2}\Big(\cos\big((n-m)\pi x\big) - \cos\big((n+m)\pi x\big)\Big).$$

Case $n \neq m$. Both frequencies $n-m$ and $n+m$ are nonzero integers, so

$$I_{nm} = \frac{1}{2}\int_0^1 \cos\big((n-m)\pi x\big)\,dx - \frac{1}{2}\int_0^1 \cos\big((n+m)\pi x\big)\,dx = 0,$$

because $\int_0^1 \cos(\ell\pi x)\,dx = \big[\sin(\ell\pi x)/(\ell\pi)\big]_0^1 = 0$ for every nonzero integer ℓ.

Case $n = m$. Then $\cos((n-m)\pi x) = \cos 0 = 1$ and $\cos((n+m)\pi x) = \cos(2n\pi x)$, so

$$I_{nn} = \frac{1}{2}\int_0^1 1\,dx - \frac{1}{2}\int_0^1 \cos(2n\pi x)\,dx = \frac{1}{2} - 0 = \frac{1}{2},$$

the second integral again vanishing as above. Summarizing,

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} \frac{1}{2}, & n = m, \\ 0, & n \neq m. \end{cases} \tag{1.1.10}$$

The Fourier expansion of the string III

Step 4: isolate one coefficient. Multiply (1.1.9) by $\sin(m\pi x)$ and integrate over $[0, 1]$. For a continuous u_0 (or, more generally, when the series converges in L^2), term-by-term integration is justified: the partial sums S_N converge to u_0 in mean square, and integrating against the bounded factor $\sin(m\pi x)$ passes to the limit. Thus

$$\int_0^1 u_0(x)\sin(m\pi x)\,dx = \sum_{n=1}^{\infty} a_n \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \frac{a_m}{2},$$

where the sum collapses by (1.1.10) to the single term $n = m$. Hence

$$a_m = 2\int_0^1 u_0(x)\sin(m\pi x)\,dx.$$

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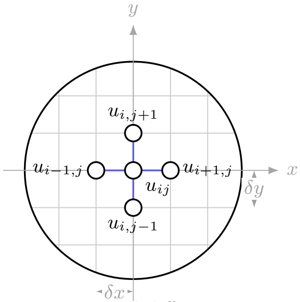
The two-dimensional wave equation I

Step 1: build the lattice model. Now model a circular drum head. We again build the equations of motion from Hooke's law, but this time the masses form a two-dimensional lattice connected by massless springs in both the x and y directions. Write $u_{ij}(t) = u(x_i, y_j, t)$ for the vertical displacement of the node at (x_i, y_j), let m be the mass carried by that node, let δx and δy be the mesh spacings, and let κ_x and κ_y be the spring stiffnesses along the two lattice directions. Node (i, j) now has four neighbors instead of two, and Newton's second law collects one Hooke term from each:

The two-dimensional wave equation II



two centered differences
from four nearest neighbors

The two-dimensional wave equation III

$$m\,\frac{d^2u_{ij}}{dt^2} = \kappa_x\big(u_{i+1,j}-u_{ij}\big)+\kappa_x\big(u_{i-1,j}-u_{ij}\big)+\kappa_y\big(u_{i,j+1}-u_{ij}\big)+\kappa_y\big(u_{i,j-1}-u_{ij}\big).$$

Each opposing pair combines into a centered second difference exactly as the two terms of (1.1.1) did,

$$m\,\frac{d^2u_{ij}}{dt^2} = \kappa_x\big(u_{i+1,j}-2u_{ij}+u_{i-1,j}\big)+\kappa_y\big(u_{i,j+1}-2u_{ij}+u_{i,j-1}\big).$$

Divide by m and, inside each bracket, multiply and divide by that direction's own spacing squared:

$$\frac{d^2u_{ij}}{dt^2} = \frac{\kappa_x}{m}(\delta x)^2\,\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{(\delta x)^2}+\frac{\kappa_y}{m}(\delta y)^2\,\frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{(\delta y)^2}.$$

Each prefactor is the same combination $\frac{\kappa}{m}(\delta x)^2$ that produced the wave speed in the one-dimensional case (1.1.2), once per direction, and exactly as there the stiffnesses and node masses are scaled with the mesh so that this combination stays fixed as the lattice is refined. Choosing the mesh and the spring scaling *isotropically* means precisely that the two prefactors agree,

$$c^2 := \frac{\kappa_x}{m}(\delta x)^2 = \frac{\kappa_y}{m}(\delta y)^2,$$

and only under that condition may the single constant c^2 be pulled in front of both terms:

$$\frac{d^2 u_{ij}}{dt^2} = c^2 \left[\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{(\delta x)^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{(\delta y)^2} \right], \tag{1.2.1}$$

Here c is the physical wave speed of the membrane. If the two lattice directions used different spacings or spring scalings, the limiting equation would carry different constants in the x and y directions. In the isotropic limit $\delta x, \delta y \to 0$, each bracketed term becomes a second partial derivative,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]. \tag{1.2.2}$$

Writing ∇^2 for the Laplacian (the sum of the unmixed second derivatives), this is more succinctly

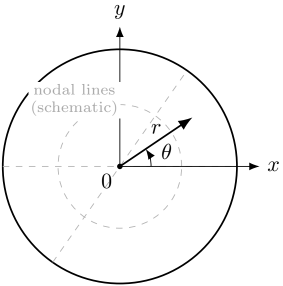
$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \nabla^2 u. \tag{1.2.3}$$

The time derivative is *second* order: the drum, like the string, obeys a genuine wave equation, not a diffusion equation.

The two-dimensional wave equation V

Step 2: use polar coordinates. The boundary of the drum is a circle, so polar coordinates (r, θ) make the boundary condition simple. Away from the coordinate singularity at $r = 0$, the Laplacian is

$$\nabla^2 u = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2}. \tag{1.2.4}$$



rim: $u(1, \theta, t) = 0$

The two-dimensional wave equation VI

Expanding the chain-rule calculation skipped in the book, invert $x = r\cos\theta$ and $y = r\sin\theta$ into $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$. Differentiating the first of these once in each variable,

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{r}.$$

For the angle, differentiate $\tan\theta = y/x$ implicitly (working where $x \neq 0$; the resulting formulas then extend to the rest of the punctured plane by continuity), holding y fixed in the first case and x fixed in the second:

$$\sec^2\theta\,\frac{\partial\theta}{\partial x} = -\frac{y}{x^2}, \qquad \sec^2\theta\,\frac{\partial\theta}{\partial y} = \frac{1}{x}.$$

The factor $\sec^2\theta$ is where the radius reappears: from $\tan\theta = y/x$,

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{y^2}{x^2} = \frac{x^2+y^2}{x^2} = \frac{r^2}{x^2},$$

so dividing by $\sec^2\theta$ is the same as multiplying by x^2/r^2:

$$\frac{\partial\theta}{\partial x} = \frac{x^2}{r^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}, \qquad \frac{\partial\theta}{\partial y} = \frac{x^2}{r^2}\cdot\frac{1}{x} = \frac{x}{r^2}.$$

The two-dimensional wave equation VII

Using $x = r\cos\theta$ and $y = r\sin\theta$ to rewrite these right-hand sides gives the four partials

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta,$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

The chain rule $\partial_x = \dfrac{\partial r}{\partial x}\,\partial_r + \dfrac{\partial \theta}{\partial x}\,\partial_\theta$ (and likewise for ∂_y) then gives

$$\partial_x = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta, \qquad \partial_y = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\,\partial_\theta.$$

Because these coefficients themselves depend on position, applying ∂_x a second time needs the product rule—this is what generates the first-order $\frac{1}{r}u_r$ term. Taking $\partial_{xx}u = \partial_x(\partial_x u)$ as the representative case,

$$\partial_{xx}u = \left(\cos\theta\,\partial_r - \tfrac{\sin\theta}{r}\partial_\theta\right)\left(\cos\theta\,u_r - \tfrac{\sin\theta}{r}u_\theta\right).$$

Introduce the first derivative as a scalar field $\varphi := \partial_x u$, so

$$\varphi = \cos\theta\,u_r - \tfrac{\sin\theta}{r}u_\theta, \qquad \partial_{xx}u = \cos\theta\,\varphi_r - \tfrac{\sin\theta}{r}\varphi_\theta.$$

The two-dimensional wave equation VIII

Using $\partial_r \cos\theta = 0$, $\partial_r(\sin\theta/r) = -\sin\theta/r^2$, $\partial_\theta \cos\theta = -\sin\theta$, and $\partial_\theta \sin\theta = \cos\theta$,

$$\varphi_r = \cos\theta\, u_{rr} - \tfrac{\sin\theta}{r} u_{r\theta} + \tfrac{\sin\theta}{r^2} u_\theta,$$
$$\varphi_\theta = -\sin\theta\, u_r + \cos\theta\, u_{r\theta} - \tfrac{\cos\theta}{r} u_\theta - \tfrac{\sin\theta}{r} u_{\theta\theta}.$$

Substitute into $\cos\theta\,\varphi_r - (\sin\theta/r)\varphi_\theta$ and collect terms:

$$\partial_{xx} u = \cos^2\theta\, u_{rr} - \tfrac{2\sin\theta\cos\theta}{r} u_{r\theta} + \tfrac{\sin^2\theta}{r^2} u_{\theta\theta} + \tfrac{\sin^2\theta}{r} u_r + \tfrac{2\sin\theta\cos\theta}{r^2} u_\theta.$$

The companion $\partial_{yy} u = \partial_y(\partial_y u)$ is obtained the same way from $\partial_y = \sin\theta\,\partial_r + (\cos\theta/r)\partial_\theta$. Swapping $\cos^2\theta \leftrightarrow \sin^2\theta$ in the u_{rr}, $u_{\theta\theta}$, and u_r terms and flipping the signs of the two cross-terms gives

$$\partial_{yy} u = \sin^2\theta\, u_{rr} + \tfrac{2\sin\theta\cos\theta}{r} u_{r\theta} + \tfrac{\cos^2\theta}{r^2} u_{\theta\theta} + \tfrac{\cos^2\theta}{r} u_r - \tfrac{2\sin\theta\cos\theta}{r^2} u_\theta.$$

Adding $\partial_{xx} u + \partial_{yy} u$, the cross-terms cancel:

$$\left(-\tfrac{2\sin\theta\cos\theta}{r} + \tfrac{2\sin\theta\cos\theta}{r}\right) u_{r\theta} + \left(\tfrac{2\sin\theta\cos\theta}{r^2} - \tfrac{2\sin\theta\cos\theta}{r^2}\right) u_\theta = 0,$$

while $\cos^2\theta + \sin^2\theta = 1$ collapses the remaining coefficients,

$$\partial_{xx} u + \partial_{yy} u = (\cos^2\theta + \sin^2\theta)\, u_{rr} + \tfrac{\sin^2\theta + \cos^2\theta}{r}\, u_r + \tfrac{\sin^2\theta + \cos^2\theta}{r^2}\, u_{\theta\theta}$$
$$= u_{rr} + \tfrac{1}{r}\, u_r + \tfrac{1}{r^2}\, u_{\theta\theta},$$

The two-dimensional wave equation IX

which is exactly (1.2.4). Inserting (1.2.4) into (1.2.3) gives the polar wave equation for $u(r, \theta, t)$:

$$\frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2}. \tag{1.2.5}$$

The two-dimensional wave equation X

Step 3: state the drum data. For a drum head of radius 1, idealized with prescribed initial shape $u_0(r,\theta)$ and zero initial velocity, the conditions are

$$\begin{aligned} \text{initial:}\;\; & u(r,\theta,0) = u_0(r,\theta), \qquad \tfrac{\partial}{\partial t}u(r,\theta,0) = 0, \\ \text{boundary:}\;\; & u(1,\theta,t) = 0, \qquad u(r,0,t) = u(r,2\pi,t), \\ & \tfrac{\partial}{\partial\theta}u(r,0,t) = \tfrac{\partial}{\partial\theta}u(r,2\pi,t), \\ \text{center:}\;\; & u(0,\theta,t) \text{ is finite and independent of } \theta. \end{aligned}$$

The rim is clamped ($r = 1$ at the edge gives zero displacement there), the center condition is regularity at the coordinate singularity $r = 0$, and the periodic lines say the solution is single-valued as θ wraps around the circle: $\theta = 0$ and $\theta = 2\pi$ are the same physical point. For a sharply struck drum, the complementary idealization is also common: one takes $u(r,\theta,0) = 0$ and prescribes a nonzero initial velocity $\partial_t u(r,\theta,0) = v_0(r,\theta)$.

Separation in time: the Helmholtz equation I

Step 1: separate time from space. As before, look for a separable solution—now the time part splits off first:

$$u(r,\theta,t) = H(r,\theta)\,T(t). \tag{1.2.6}$$

Substituting into the polar wave equation (1.2.5) gives $\frac{1}{c^2}HT'' = T\,\nabla^2 H$. Dividing by HT,

$$\frac{1}{c^2}\,\frac{T''}{T} = \frac{\nabla^2 H}{H}. \tag{1.2.7}$$

Step 2: determine the sign. The left side depends only on t; the right only on (r,θ). Both must equal one constant λ. As for the string, only $\lambda < 0$ can produce a nontrivial clamped mode. Nothing so far forces λ to be real – it entered only as a separation constant – so we prove reality and negativity in one stroke. Multiply $\nabla^2 H = \lambda H$ by the complex conjugate $\overline{H}$ and integrate over the disk D:

$$\int_D \overline{H}\,\nabla^2 H\,dA = \lambda \int_D |H|^2\,dA.$$

The divergence theorem on the disk D states that for a smooth vector field $\mathbf{F}$,

$$\int_D \operatorname{div}\mathbf{F}\,dA = \int_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds$$

Separation in time: the Helmholtz equation II

(for a complex field this is just the theorem applied to its real and imaginary parts separately). Take $\mathbf{F} = \overline{H}\,\nabla H$. The product rule gives

$$\operatorname{div}(\overline{H}\nabla H) = \nabla\overline{H}\cdot\nabla H + \overline{H}\,\nabla^2 H = |\nabla H|^2 + \overline{H}\,\nabla^2 H,$$

where each term of $\nabla\overline{H}\cdot\nabla H$ is a number times its own conjugate, so $|\nabla H|^2 = |\partial_x H|^2 + |\partial_y H|^2 \geq 0$ is real. The boundary integrand is $\mathbf{F}\cdot\mathbf{n} = \overline{H}\,(\nabla H\cdot\mathbf{n}) = \overline{H}\,\partial_n H$. Therefore

$$\int_D \left(|\nabla H|^2 + \overline{H}\,\nabla^2 H\right) dA = \int_{\partial D} \overline{H}\,\partial_n H\, ds,$$

and rearranging yields the Green identity

$$\int_D \overline{H}\,\nabla^2 H\, dA = \int_{\partial D} \overline{H}\,\partial_n H\, ds - \int_D |\nabla H|^2\, dA.$$

The rim term vanishes because $H = 0$ on the clamped boundary, so

$$\lambda \int_D |H|^2\, dA = -\int_D |\nabla H|^2\, dA.$$

Separation in time: the Helmholtz equation III

For a nonzero mode $\int_D |H|^2\, dA > 0$, so we may divide by it:

$$\lambda = -\frac{\displaystyle\int_D |\nabla H|^2\, dA}{\displaystyle\int_D |H|^2\, dA}.$$

Both integrals are real and nonnegative, so λ is real and $\lambda \leq 0$. If $\lambda = 0$ then $\int_D |\nabla H|^2\, dA = 0$, hence $\nabla H \equiv 0$ and H is constant; a constant that vanishes on the rim is $H \equiv 0$, the rejected zero mode. Thus every nontrivial clamped mode has $\lambda < 0$. Only now, with λ known to be real, may we take H real from here on: the real and imaginary parts of a complex mode then separately satisfy $\nabla^2 H = \lambda H$, so nothing is lost. Write $\lambda = -k^2$ with $k > 0$. This yields two equations:

$$T'' = -c^2 k^2\, T, \tag{1.2.8}$$

$$\nabla^2 H = -k^2 H. \tag{1.2.9}$$

Separation in time: the Helmholtz equation IV

The time equation (1.2.8) solves to $T(t) = C\cos ckt + D\sin ckt$. The release-from-rest condition $\partial_t u = 0$ at $t = 0$ forces $D = 0$ (since $T'(0) = ckD$), leaving

$$T(t) = A\cos ckt. \tag{1.2.10}$$

Thus the angular frequency is $\omega = ck$, set by the wave speed times the spatial wavenumber.

Separation in time: the Helmholtz equation V

Step 3: separate the spatial variables. Equation (1.2.9) is the *Helmholtz equation*, usually written

$$\nabla^2 H + k^2 H = 0. \tag{1.2.11}$$

In polar coordinates, using (1.2.4),

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial H}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 H}{\partial \theta^2} + k^2 H = 0. \tag{1.2.12}$$

We separate the spatial part once more, $H(r,\theta) = R(r)\,\Theta(\theta)$. Substituting and writing $(rR')' = \frac{d}{dr}(rR')$,

$$\Theta\,\frac{1}{r}(rR')' + R\,\frac{1}{r^2}\Theta'' + k^2 R\Theta = 0.$$

Multiply through by $r^2/(R\Theta)$:

$$\frac{r(rR')'}{R} + \frac{\Theta''}{\Theta} + k^2 r^2 = 0,$$

and move the θ-term to the other side,

$$r\,\frac{(rR')'}{R} + k^2 r^2 = -\frac{\Theta''}{\Theta}. \tag{1.2.13}$$

Separation in time: the Helmholtz equation VI

The left side is a function of r only, the right of θ only, so both equal a constant μ:

$$r\,\frac{(rR')'}{R} + k^2 r^2 = -\frac{\Theta''}{\Theta} = \mu. \tag{1.2.14}$$

This gives the angular and radial equations

$$r\,(rR')' + (k^2 r^2 - \mu)R = 0, \tag{1.2.15}$$

$$\Theta'' + \mu\,\Theta = 0. \tag{1.2.16}$$

The angular equation and periodicity I

Step 1: impose periodicity. Start with the angular part (1.2.16), $\Theta'' + \mu\Theta = 0$.

If $\mu < 0$, write $\mu = -\kappa^2$ with $\kappa = \sqrt{|\mu|} > 0$. The general solution is

$$\Theta(\theta) = A\,e^{\kappa\theta} + B\,e^{-\kappa\theta} \qquad \big(\text{equivalently } A\cosh(\kappa\theta) + B\sinh(\kappa\theta)\big).$$

Periodicity $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$ becomes the linear system

$$A + B = A\,e^{2\pi\kappa} + B\,e^{-2\pi\kappa},$$
$$A - B = A\,e^{2\pi\kappa} - B\,e^{-2\pi\kappa},$$

i.e.

$$A\big(1 - e^{2\pi\kappa}\big) + B\big(1 - e^{-2\pi\kappa}\big) = 0,$$
$$A\big(1 - e^{2\pi\kappa}\big) - B\big(1 - e^{-2\pi\kappa}\big) = 0.$$

Adding and subtracting these two equations gives $2A(1 - e^{2\pi\kappa}) = 0$ and $2B(1 - e^{-2\pi\kappa}) = 0$. Since $\kappa > 0$, neither factor $1 - e^{\pm 2\pi\kappa}$ vanishes, so $A = B = 0$: only the trivial solution is 2π-periodic. (Equivalently, in the cosh / sinh basis the coefficient matrix has determinant
$(1 - \cosh(2\pi\kappa))^2 - \sinh^2(2\pi\kappa) = -4\sinh^2(\pi\kappa) \neq 0$.)

If $\mu = 0$, the general solution is $\Theta(\theta) = A + B\theta$. Then $\Theta(0) = A$ and $\Theta(2\pi) = A + 2\pi B$, so $\Theta(0) = \Theta(2\pi)$ forces $B = 0$; the derivative condition $\Theta' = B$ is then automatic. Thus $\mu = 0$ yields only the constant angular mode $\Theta \equiv A$.

The angular equation and periodicity II

For a genuine oscillation around the rim we need $\mu > 0$; write the solution as

$$\Theta(\theta) = A\cos\big(\sqrt{\mu}\,\theta\big) + B\sin\big(\sqrt{\mu}\,\theta\big), \tag{1.2.17}$$

for coefficients A and B. The drum head is one connected surface, so Θ must return to its starting value after a full turn: $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. As in the two cases just treated, what has to be decided is for which $\mu > 0$ *some* nonzero pair (A, B) survives both conditions. With $\Theta(0) = A$ and $\Theta(2\pi) = A\cos(2\pi\sqrt{\mu}) + B\sin(2\pi\sqrt{\mu})$, the first condition reads

$$A\big(1 - \cos(2\pi\sqrt{\mu})\big) - B\sin(2\pi\sqrt{\mu}) = 0.$$

With $\Theta'(0) = \sqrt{\mu}\,B$ and $\Theta'(2\pi) = \sqrt{\mu}\big(-A\sin(2\pi\sqrt{\mu}) + B\cos(2\pi\sqrt{\mu})\big)$, the second reads $\sqrt{\mu}\,B = \sqrt{\mu}\big(-A\sin(2\pi\sqrt{\mu}) + B\cos(2\pi\sqrt{\mu})\big)$, and dividing by $\sqrt{\mu} > 0$,

$$A\sin(2\pi\sqrt{\mu}) + B\big(1 - \cos(2\pi\sqrt{\mu})\big) = 0.$$

This is a homogeneous linear system for (A, B) with coefficient matrix

$$M = \begin{pmatrix} 1 - \cos(2\pi\sqrt{\mu}) & -\sin(2\pi\sqrt{\mu}) \\ \sin(2\pi\sqrt{\mu}) & 1 - \cos(2\pi\sqrt{\mu}) \end{pmatrix},$$

The angular equation and periodicity III

and a nonzero (A, B) exists exactly when $\det M = 0$. Expanding the determinant, then using $\cos^2 + \sin^2 = 1$ and the half-angle identity $1 - \cos 2x = 2\sin^2 x$ with $x = \pi\sqrt{\mu}$,

$$\det M = \left(1 - \cos(2\pi\sqrt{\mu})\right)^2 + \sin^2(2\pi\sqrt{\mu}) = 2 - 2\cos(2\pi\sqrt{\mu}) = 4\sin^2\left(\pi\sqrt{\mu}\right).$$

So $\det M = 0$ if and only if $\sin(\pi\sqrt{\mu}) = 0$, that is, if and only if $\sqrt{\mu}$ is a whole number. This settles both directions at once: for non-integer $\sqrt{\mu}$ the determinant is nonzero and only $A = B = 0$ survives, so no such μ is admissible; while for $\sqrt{\mu} = m$ we have $\cos(2\pi m) = 1$ and $\sin(2\pi m) = 0$, so M is the zero matrix and *both* A and B are free. Here $\mu > 0$ gives $m \geq 1$, and $m = 0$ is exactly the constant mode already found in the $\mu = 0$ case, so the two cases combine into

$$\sqrt{\mu} = m \quad \text{for some nonnegative integer } m, \qquad \text{i.e.} \qquad \mu = m^2. \tag{1.2.18}$$

The integer m counts how many full angular oscillations fit around the drum; $m = 0$ is the radially symmetric mode.

The angular equation and periodicity IV

Step 2: rescale the radial equation. Substituting $\mu = m^2$ into the radial equation (1.2.15),

$$r\,(rR')' + (k^2r^2 - m^2)R = 0. \qquad (1.2.19)$$

The factor k^2 is a nuisance; remove it by rescaling the radius. Introduce the dimensionless radius $s = kr$ and let $\widetilde{R}(s) = R(r)$ be the same displacement read off against s. By the chain rule each r-derivative brings down a factor k,

$$\frac{dR}{dr} = k\,\frac{d\widetilde{R}}{ds}, \qquad \frac{d^2R}{dr^2} = k^2\,\frac{d^2\widetilde{R}}{ds^2},$$

and $r = s/k$, so $k^2r^2 = s^2$. Substituting into the expanded form $r^2R'' + rR' + (k^2r^2 - m^2)R = 0$ of (1.2.19),

$$\frac{s^2}{k^2}\,k^2\widetilde{R}'' + \frac{s}{k}\,k\widetilde{R}' + (s^2 - m^2)\widetilde{R} = 0,$$

and the powers of k cancel termwise. Renaming s back to r and $\widetilde{R}$ to R, the rescaled equation is

$$r\,(rR')' + (r^2 - m^2)R = 0. \qquad (1.2.20)$$

Expanding the derivative $r(rR')' = r(R' + rR'') = r^2R'' + rR'$ gives the standard form

$$r^2R'' + rR' + (r^2 - m^2)R = 0. \qquad (1.2.21)$$

This is *Bessel's equation* of order m.

Bessel's equation and the drum solution I

Step 1: choose the regular radial solution. The solution of Bessel's equation (1.2.21) cannot be written as a finite product and sum of elementary functions. The two independent radial solutions are the Bessel functions of the first and second kind, denoted $J_m(r)$ and $Y_m(r)$. The function Y_m is singular at the center (Y_0 has a logarithmic singularity, and Y_m for $m \geq 1$ blows up like r^{-m}), so the boundedness and smoothness condition at $r = 0$ removes it. The regular solution is the *Bessel function* $J_m(r)$, normalized so that $J_m(r) \big/ \big((r/2)^m/m!\big) \to 1$ as $r \to 0$. Naming it lets us write the answer in closed form even though no elementary formula exists.

Step 2: assemble one drum mode. Assembling the pieces—radial factor $J_m(kr)$ (undoing the rescaling), angular factors $\cos(m\theta)$ and $\sin(m\theta)$, and time factor $\cos(ckt)$ from (1.2.10)—and superposing over all admissible angular integers m and all allowed radial constants k, the displacement of the drum is

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} \Big[A_{jm}\cos(m\theta) + B_{jm}\sin(m\theta)\Big]\, J_m(k_{jm} r)\, \cos(c k_{jm} t). \qquad (1.2.22)$$

For $m = 0$ the sine term is zero and may be dropped. For each j, the allowed value is $k_{jm} = \alpha_{m,j}$, where $\alpha_{m,j}$ is the jth positive zero of J_m, so that the rim condition $u(1, \theta, t) = 0$ holds. **Step 3: coefficients from orthogonality (preview).** The

Bessel's equation and the drum solution II

constants A_{jm} and B_{jm} are fixed by the initial shape u_0, just as the Fourier coefficients a_n were fixed for the string. The needed tools are angular sine/cosine orthogonality on $[0, 2\pi]$ together with a *radial* orthogonality for the family $\{J_m(\alpha_{m,j} r)\}_{j\geq 1}$ on $[0, 1]$ with weight r (the polar area factor from $dA = r\,dr\,d\theta$). Both will be proved later, once Bessel functions are developed (Chapters 5–6). For this chapter it is enough to remember the mode form (1.2.22) and the fact that the allowed wavenumbers k_{jm} are set by the positive zeros of J_m.

Table of contents

Named special functions

The shape of the boundary decides which functions appear.

- **Guitar string** (interval): the modes are sinusoids; the expansion is a *Fourier* series.
- **Circular drum**: the radial modes are *Bessel functions* J_m.
- **Rectangular drum**: the solution would again be built from sinusoidal Fourier modes, like the string.
- **Elliptical drum**: the solutions are *Mathieu functions*.

Many differential equations of physics cannot be solved with elementary functions, but their solutions arise so often that we assign them proper names. This class of named solutions is the family of *special functions*.

The purpose of the course is to study several of the most important special functions—the Bessel function, the gamma function and its relatives, and the orthogonal polynomials. To analyze them comfortably we will first develop some tools of complex analysis.