Mathematical Physics: Complex Analysis and Special Functions

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Abstract

These lecture notes develop the mathematical methods most useful in theoretical physics, centered on complex analysis and the classical special functions. The first two sections build complex analysis up to the method of steepest descent. The middle sections treat the Gamma, Bessel, Legendre, Hermite, Laguerre, Chebyshev, and hypergeometric families from a complex-analytic viewpoint: each function is introduced through its generating function or integral representation, with series expansions and differential equations emerging as consequences rather than as definitions. The final section develops the calculus of variations, including Euler–Lagrange equations, constraints, Noether's theorem, and classical applications. The main derivations are written out step by step, with standard background results cited explicitly where we use them, and each major result is accompanied by at least one worked example.

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1 Complex Analysis I — Foundations

Reader's analysis vocabulary. A few analysis terms recur throughout the notes. In the complex plane, a *disk* of radius $r > 0$ around a is the set of points with distance less than r from a:

$$D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}.$$

On the real line, the analogous object is the interval $(a-r, a+r)$. A *neighborhood* of a point means any set that contains some disk or interval around that point. A set is *open* if every point in it has a small neighborhood still lying completely inside the set. This is why, on an open set, we may perturb a point slightly without leaving the set. The word *local* means "after restricting attention to a sufficiently small neighborhood."

A point b is a *boundary point* of a set S if every neighborhood of b touches both S and the outside of S. Points of S that are not on the boundary are *interior points*. A set is *closed* if it contains all its boundary points. For example, $[0,1]$ is closed, while $(0,1)$ is not because the boundary points 0 and 1 are missing. Equivalently, a closed set contains the limit of every convergent sequence of its own points, meaning every list of points of the set that approaches a limiting value has that limiting value still in the set. A set is *bounded* if it fits inside some disk or interval of finite radius. In these notes, *compact* means closed and bounded. A function is *continuous* if nearby inputs give nearby outputs; continuous functions on compact sets attain maximum and minimum values.

A set is *connected* if it is all one piece, rather than two separated pieces. A *domain* is a connected open set. A *limit point* of a set S is a point that can be approached by other points of S; every neighborhood of a limit point contains some point of S different from the limit point itself.

The symbol $\sup S$ means the *supremum* of a set S: the least number that is at least every element of S; when a maximum exists, the supremum equals that maximum. We write $\|g\|_\infty = \sup|g|$ for the largest size of a function on the set under discussion.

A statement holds *uniformly* on a set if the same error bound works for every point of the set, not separately point by point. A convergence statement is *locally uniform* on an open set if it is uniform after restricting to any closed bounded piece that stays inside that open set.

We use *Big-O* and *little-o* notation for error terms. The statement $R(h) = O(A(h))$ means $|R(h)| \le C|A(h)|$ near the limiting point for some constant C. The statement $R(h) = o(A(h))$ means $R(h)/A(h) \to 0$ at the limiting point, so R is smaller than A in the limit.

The *support* of a function is the place where the function lives, including edge points: a point belongs to the support if every neighborhood of that point contains at least one point where the function is nonzero. Compact support means the function is zero outside a closed bounded set. L^1 means integrable and L^2 means square-integrable; a set of functions is *dense* in L^2 if every L^2 function can be approximated arbitrarily well by finite combinations from that set. The phrase *dominated convergence* refers to the standard theorem saying that if functions converge pointwise and are all bounded in absolute value by one L^1 function, then the limit may be moved through the integral.

Prerequisites. This section assumes real multivariable calculus (partial derivatives, line integrals, the change-of-variables formula) and basic real analysis at the level of epsilon–delta limits, uniform convergence of series of continuous functions, and the interchange of limit and integral under uniform convergence. No measure theory is required; Riemann integration suffices throughout.

Much of mathematical physics — wave propagation, scattering, quantum amplitudes, and large-parameter expansions of special functions — is most transparent when the real variables on which physical quantities nominally depend are promoted to *complex* variables. The theory introduced below is unreasonably rigid: once the complex derivative exists throughout an open region, the function is automatically infinitely differentiable, locally given by a convergent power series, and determined on a one-piece open region by its values on any small open patch. This section builds the core machinery for integrating over curves, expanding functions near exceptional points, and extracting coefficients; later sections use this machinery repeatedly.

1.1 Complex differentiability and the Cauchy–Riemann equations

Ordinary differentiability at a real point x_0 says the graph of f has a tangent line there. When we replace $x \in \mathbb{R}$ with $z \in \mathbb{C}$, the point z_0 can be approached from infinitely many directions in the plane, not only from left and right. Demanding that the difference quotient have the *same* limit along every direction is much stronger than having ordinary partial derivatives in the two variables (x, y): it couples the real and imaginary parts of f through the Cauchy–Riemann equations. These equations are the entry point to everything that follows.

Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$. Here "open" means that every point of Ω has a small disk around it that still lies inside Ω. This condition lets us perturb a point $z_0 \in \Omega$ by all sufficiently small complex numbers h without leaving the input set of f.

We write

$$D(a,r) := \{z \in \mathbb{C} : |z-a| < r\}, \qquad \overline{D(a,r)} := \{z \in \mathbb{C} : |z-a| \le r\} \tag{1.1}$$

for the open and closed disks of radius r centered at a. The boundary circle is $\{|z-a| = r\}$ and the arclength is $2\pi r$.

When an open set is all one piece, we call it *connected*; many books call a connected open set a *domain*.

Write

$$z = x + iy, \qquad f(z) = u(x,y) + iv(x,y),$$

with $x, y, u(x,y), v(x,y) \in \mathbb{R}$. Thus the same function has two equivalent descriptions:

$$f : \Omega \subset \mathbb{C} \to \mathbb{C}, \qquad (x,y) \longmapsto (u(x,y), v(x,y)).$$

The second description treats f as a real two-variable map from a region in $\mathbb{R}^2$ to $\mathbb{R}^2$. We use this viewpoint whenever we speak about partial derivatives.

Definition 1.1 (Holomorphic function). *f is* complex-differentiable *at $z_0 \in \Omega$ if the limit*

$$f'(z_0) = \lim_{h \to 0,\, h \in \mathbb{C}\setminus\{0\}} \frac{f(z_0+h) - f(z_0)}{h} \tag{1.2}$$

exists as a complex number, independent of the direction from which $h \to 0$. The function f is holomorphic *on Ω if it is complex-differentiable at every point of Ω. Equivalently, f is holomorphic at z_0 if it is holomorphic on some open neighborhood of z_0. In these notes,* analytic *means holomorphic on an open set. A function holomorphic on all of $\mathbb{C}$ is called* entire.

Remark 1.2. *Equation* (1.2) *looks identical to the real derivative, but the constraint "independent of direction" is strong. Approaching along $h = \Delta x$ (real axis) and along $h = i\Delta y$ (imaginary axis) must give the same number, and we will see this forces a PDE on u, v.*

Theorem 1.3 (Cauchy–Riemann). *Let $f = u + iv$ be defined on an open set Ω. If f is holomorphic at $z_0 = x_0 + iy_0$, then the partial derivatives below exist at (x_0, y_0) and*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{1.3}$$

Conversely, if u, v are C^1 (continuously differentiable: their first partial derivatives exist and are continuous) on a neighborhood of (x_0, y_0) and (1.3) holds there, then f is holomorphic at z_0.

Proof. Forward direction. Write $f'(z_0) = A = \alpha + i\beta$. Approach z_0 first along the real direction, so $h = \Delta x$:

$$\frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \longrightarrow A \qquad (\Delta x \to 0).$$

Writing $f = u + iv$ and $z_0 = x_0 + iy_0$, the same quotient is

$$\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}.$$

Therefore $u_x = \alpha$ and $v_x = \beta$ at (x_0, y_0).

Next approach z_0 vertically, so $h = i\Delta y$. The same complex derivative must still be A:

$$\frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \longrightarrow A \qquad (\Delta y \to 0).$$

Expand the numerator:

$$f(z_0 + i\Delta y) - f(z_0) = \big(u(x_0, y_0 + \Delta y) - u(x_0, y_0)\big) + i\big(v(x_0, y_0 + \Delta y) - v(x_0, y_0)\big).$$

Dividing by $i\Delta y$ is the same as multiplying by $-i/\Delta y$, so

$$\frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} = \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}.$$

Thus $v_y = \alpha$ and $u_y = -\beta$. Combining the two directions gives $u_x = v_y$ and $u_y = -v_x$.

Converse. Now assume $u, v \in C^1$ near (x_0, y_0) and (1.3) holds there. The first-order Taylor approximation for a two-variable C^1 function $\phi(x, y)$ at (x_0, y_0) reads $\phi(x_0 + \Delta x, y_0 + \Delta y) = \phi(x_0, y_0) + \phi_x\,\Delta x + \phi_y\,\Delta y + o(|h|)$, with $|h| = \sqrt{\Delta x^2 + \Delta y^2}$ and partials evaluated at (x_0, y_0). Applying this to u and v separately and combining into $f = u + iv$,

$$f(z_0 + h) - f(z_0) = (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + o(|h|),$$

with all partials evaluated at (x_0, y_0). We want to combine the Δx and Δy terms into a single multiple of $h = \Delta x + i\Delta y$. This requires the coefficient of Δy to be i times the coefficient of Δx. By (1.3), $u_y = -v_x$ and $v_y = u_x$, so

$$u_y + iv_y = -v_x + iu_x = i(u_x + iv_x),$$

where the last step is $i(u_x + iv_x) = iu_x + i^2 v_x = -v_x + iu_x$. Hence

$$f(z_0 + h) - f(z_0) = (u_x + iv_x)(\Delta x + i\Delta y) + o(|h|) = (u_x + iv_x)h + o(|h|).$$

After division by h, the error term tends to 0. Thus $f'(z_0) = u_x + iv_x$. □

Corollary 1.4 (Harmonicity). *If $f = u + iv$ is holomorphic on Ω and $u, v \in C^2$, then*

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad \Delta v = 0. \tag{1.4}$$

That is, the real and imaginary parts of a holomorphic function are harmonic.

Proof. From (1.3), differentiate $u_x = v_y$ with respect to x:

$$u_{xx} = v_{yx}. \tag{1.5}$$

Differentiate $u_y = -v_x$ with respect to y:

$$u_{yy} = -v_{xy}. \tag{1.6}$$

Adding,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0, \tag{1.7}$$

where the final equality uses Clairaut's theorem (equality of mixed partials for C^2 functions). For the imaginary part, use the same Cauchy–Riemann equations in the other order: from $v_x = -u_y$,

$$v_{xx} = -u_{yx},$$

and from $v_y = u_x$,

$$v_{yy} = u_{xy}.$$

Adding gives $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$. □

This is already a useful payoff: the steady-state temperature distributions, electrostatic potentials, and incompressible irrotational flows in the plane — all governed by Laplace's equation $\Delta\phi = 0$ — can be manufactured as real parts of holomorphic functions.

Example 1.5 ($f(z) = z^2$). *Expand $z^2 = (x + iy)^2 = x^2 + 2ixy + (iy)^2 = (x^2 - y^2) + i(2xy)$, so $u = x^2 - y^2$ and $v = 2xy$. Compute partials:*

$$u_x = 2x, \quad u_y = -2y, \quad v_x = 2y, \quad v_y = 2x. \tag{1.8}$$

Then $u_x = 2x = v_y$ and $u_y = -2y = -v_x$: Cauchy–Riemann holds at every $(x, y) \in \mathbb{R}^2$, so f is entire. The derivative is

$$f'(z) = u_x + iv_x = 2x + i(2y) = 2(x + iy) = 2z, \tag{1.9}$$

matching the formal answer obtained by treating z as a single variable. Also observe that the level curves $u = x^2 - y^2 =$ const and $v = 2xy =$ const are two orthogonal families of hyperbolas: at every $z \neq 0$ the two level lines meet at right angles. This orthogonality is not a coincidence; it is the general conformal *property of holomorphic maps, recorded in the next remark.*

Example 1.6 ($f(z) = \bar{z}$ is nowhere holomorphic). *Here $\bar{z} = x - iy$, so $u = x$ and $v = -y$. Then $u_x = 1$ but $v_y = -1$, so $u_x \neq v_y$ at every point. Cauchy–Riemann fails everywhere; conjugation is* not *complex-differentiable despite being smooth (indeed linear) in (x, y). This example shows that smoothness in the real sense is strictly weaker than holomorphicity.*

Remark 1.7 (Geometric meaning: conformality). *Read f as a map of the plane to itself, $(x,y)\mapsto(u,v)$. Its real Jacobian is*

$$Df=\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}.$$

The Cauchy–Riemann equations (1.3) say $u_x=v_y$ and $u_y=-v_x$, so this matrix has the form $\left(\begin{smallmatrix}a & -b\\ b & a\end{smallmatrix}\right)$ with $a=u_x$ and $b=v_x$. Such a matrix acts on $\mathbb{R}^2$ as a rotation by angle $\theta=\arg(a+ib)$ composed with a uniform scaling by $\rho=\sqrt{a^2+b^2}$: this is the matrix representation of multiplication by the complex number $a+ib=f'(z)$. Wherever $f'(z)\neq 0$, the map therefore preserves angles between curves at z and scales infinitesimal lengths uniformly by $|f'(z)|$. This is the conformal *property; the orthogonal hyperbolas in Example 1.5 illustrate it. By contrast, conjugation $\bar{z}$ (Example 1.6) has Jacobian $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, a reflection: it preserves angle magnitudes but reverses their sign, which is why it fails Cauchy–Riemann.*

1.2 Contour integrals

To probe a holomorphic function globally we integrate it along curves. The key discovery of the next two subsections is that these integrals are largely insensitive to the exact curve: for holomorphic integrands, what matters is which *singularities* the path winds around and how many times. A singularity is a point where the formula is not holomorphic, often because the function is not defined there or blows up there. Before we can prove that curve-insensitivity, we need the definition.

A *contour* γ in $\mathbb{C}$ is a piecewise C^1 map $\gamma:[a,b]\to\mathbb{C}$: it is continuous, and the interval $[a,b]$ can be split into finitely many subintervals on which γ is C^1. A contour is *closed* if $\gamma(a)=\gamma(b)$, and it is *simple* if it has no self-intersections except for this allowed closing point. For f continuous on the image of γ, define the *contour integral*

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt. \tag{1.10}$$

The right-hand side is a complex-valued Riemann integral in the real variable t; equivalently, integrate its real and imaginary parts as ordinary Riemann integrals. A change of variables $t=\phi(s)$ with $\phi'>0$ shows that the value is independent of orientation-preserving reparametrization, and reversing orientation ($t\mapsto a+b-t$) flips its sign. We write $\oint$ for integrals over closed contours. For a simple closed contour, *positive orientation* means counterclockwise around its interior; negative orientation means clockwise.

When a contour is written as $z=x+iy$, the shorthand $dz=dx+i\,dy$ means

$$dx=x'(t)\,dt,\qquad dy=y'(t)\,dt,\qquad dz=(x'(t)+iy'(t))\,dt.$$

Thus, if $f=u+iv$ along the contour, then

$$f\,dz=(u+iv)(dx+i\,dy)=(u\,dx-v\,dy)+i(v\,dx+u\,dy).$$

This is just the contour integral (1.10) written in real and imaginary parts; we use exactly this bookkeeping in the Green's theorem proof of Cauchy's theorem.

Example 1.8 ($\oint_{|z|=1} z^n\,dz$). *Parametrize the unit circle by $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then $\gamma'(t) = ie^{it}$, so $dz = ie^{it}\,dt$, and for $n \in \mathbb{Z}$,*

$$\begin{aligned}\oint_{|z|=1} z^n\,dz &= \int_0^{2\pi} (e^{it})^n \cdot ie^{it}\,dt \\ &= i\int_0^{2\pi} e^{int}\,e^{it}\,dt \\ &= i\int_0^{2\pi} e^{i(n+1)t}\,dt.\end{aligned}$$

using $(e^{it})^n = e^{int}$ (well-defined integer power) and the law $e^{int}e^{it} = e^{i(n+1)t}$. For $n \neq -1$ the exponent is nonzero; the antiderivative of $e^{i(n+1)t}$ with respect to t is $e^{i(n+1)t}/[i(n+1)]$, and

$$i\int_0^{2\pi} e^{i(n+1)t}\,dt = i \cdot \frac{e^{i(n+1)\cdot 2\pi} - e^0}{i(n+1)} = \frac{1-1}{n+1} = 0, \tag{1.11}$$

since $e^{i(n+1)\cdot 2\pi} = 1$ for integer $n+1$. For $n = -1$ the integrand reduces to the constant 1, and

$$i\int_0^{2\pi} 1\,dt = 2\pi i. \tag{1.12}$$

Summarizing,

$$\oint_{|z|=1} z^n\,dz \;=\; \begin{cases} 2\pi i, & n = -1, \\ 0, & n \in \mathbb{Z},\ n \neq -1. \end{cases} \tag{1.13}$$

This single computation is the seed of residue theory: the only power of z around the unit circle that survives is z^{-1}, with a universal coefficient $2\pi i$.

The basic estimate for contour integrals is the same idea as "area under a bound" in real-variable calculus: maximum size of the integrand times length of the path.

Lemma 1.9 (ML inequality)**.** *If f is continuous on the image of γ with $|f(z)| \leq M$ there, and γ has arclength $L = \int_a^b |\gamma'(t)|\,dt$, then*

$$\left|\int_\gamma f(z)\,dz\right| \leq M \cdot L. \tag{1.14}$$

Proof. The proof is the triangle inequality, followed by the bound $|f| \leq M$ along the curve. By (1.10),

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \\ &\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \\ &\leq M\int_a^b |\gamma'(t)|\,dt \\ &= ML. \end{aligned} \tag{1.15}$$

The first equality is the definition (1.10). The second step is the triangle inequality for the complex Riemann integral ($|\int g\,dt| \leq \int |g|\,dt$ for any continuous $g : [a,b] \to \mathbb{C}$). The third uses $|f(\gamma(t))| \leq M$ pointwise. The final equality is the definition of arclength. □

1.3 Cauchy's theorem

Example 1.8 already hints that closed-contour integrals of "well-behaved" functions might vanish: every power z^n with $n \geq 0$ is a polynomial, hence entire, and its integral around the unit circle is 0. Cauchy's theorem turns this computation into a principle. The domain must have no holes in the following precise sense: every closed curve inside it can be shrunk continuously to a point without leaving the domain. Such a domain is called *simply connected*. If the function is holomorphic and the domain is simply connected, then every closed-contour integral vanishes. Everything downstream — path independence, the integral formula, Taylor expansions, and later coefficient formulas — comes from this fact.

Theorem 1.10 (Cauchy–Goursat)**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected open set and $f : \Omega \to \mathbb{C}$ holomorphic. Then for every closed contour $\gamma \subset \Omega$,*

$$\oint_\gamma f(z)\,dz = 0. \tag{1.16}$$

Proof via Green's theorem (assuming f' continuous). We give the short proof first for a positively oriented simple closed contour, under the additional assumption that f' is continuous on Ω. A general closed contour follows by decomposing polygonal approximations into simple loops; Remark 1.11 explains why the continuity assumption on f' can also be removed.

Step 1 (Split into real and imaginary parts). Write $f = u + iv$ with $u, v : \Omega \to \mathbb{R}$, and $dz = dx + i\,dy$. Then

$$\begin{aligned} \oint_\gamma f\,dz &= \oint_\gamma (u+iv)(dx + i\,dy) \\ &= \underbrace{\oint_\gamma (u\,dx - v\,dy)}_{\text{real part}} + i\underbrace{\oint_\gamma (v\,dx + u\,dy)}_{\text{imaginary part}}. \end{aligned} \tag{1.17}$$

Step 2 (Apply Green's theorem). In the simple-contour case, simple connectivity ensures that γ bounds a region $D \subset \Omega$. Green's theorem from multivariable calculus states that for C^1 functions P, Q on D,

$$\oint_{\partial D} (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

(This is where continuity of f', equivalently continuity of u_x, u_y, v_x, v_y, is used: Green's theorem requires $P, Q \in C^1$.)

Apply this to each line integral in (1.17). For the real part, $P = u$, $Q = -v$:

$$\oint_\gamma (u\,dx - v\,dy) = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA. \tag{1.18}$$

For the imaginary part, $P = v$, $Q = u$:

$$\oint_\gamma (v\,dx + u\,dy) = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA. \tag{1.19}$$

Step 3 (Cauchy–Riemann makes both vanish). Since f is holomorphic, u and v satisfy the Cauchy–Riemann equations (Theorem 1.3):

$$u_x = v_y, \qquad u_y = -v_x.$$

Substituting into (1.18): $-v_x - u_y = -(-u_y) - u_y = 0$. Substituting into (1.19): $u_x - v_y = v_y - v_y = 0$. Both double integrals vanish, so $\oint_\gamma f\,dz = 0$. □

Remark 1.11 (Goursat's refinement: removing the continuity assumption)**.** *The proof above uses Green's theorem, which requires f' to be continuous ($u, v \in C^1$). The definition of holomorphicity (Definition 1.1) only asks that $f'(z)$ exist at each point, not that it be continuous. Goursat showed that the theorem holds under this weaker hypothesis alone, using a purely complex-variable argument that avoids Green's theorem entirely.*

Sketch of Goursat's approach. *The idea is a quadrisection argument for triangles. Connect the midpoints of a triangle's sides to cut it into four congruent sub-triangles; the boundary integrals add up to the parent's integral, so at least one sub-triangle carries $\geq 1/4$ of the total. Iterating produces nested triangles shrinking to a single point z_*. At z_*, holomorphicity gives $f(z) = f(z_*) + f'(z_*)(z - z_*) + \eta(z)(z - z_*)$ with $\eta \to 0$. The polynomial part integrates to zero (it has an antiderivative), and the error term is squeezed to zero by the ML inequality. This proves the triangle case; polygons follow by triangulation, and smooth contours by polygonal approximation.*

In practice, the distinction rarely matters: one can show (using Cauchy's integral formula, proved downstream) that any holomorphic function automatically has a continuous derivative — indeed, derivatives of all orders. So the C^1 assumption used in the Green's theorem proof is ultimately redundant, but the logical point is that Goursat's argument establishes Cauchy's theorem without relying on it.

The hypothesis of simple connectivity is essential: Example 1.8 shows $\oint_{|z|=1} z^{-1}\, dz = 2\pi i \neq 0$, even though $1/z$ is holomorphic on the punctured plane $\mathbb{C} \setminus \{0\}$. Simple connectivity fails because the unit circle cannot be shrunk to a point without crossing the missing point 0.

Corollary 1.12 (Path independence)**.** *Under the hypotheses of Thm. 1.10, $\int_\gamma f\, dz$ depends only on the endpoints of γ, not on the path taken within Ω.*

Proof. Let γ_1, γ_2 be two contours in Ω sharing the same initial and final endpoints. The concatenation γ_1 followed by the reverse of γ_2 is a closed contour in Ω. Here γ_2^{-1} means the same path traversed backward, and $*$ means "follow the first path, then the second." Thus Thm. 1.10 gives

$$\int_{\gamma_1} f\, dz - \int_{\gamma_2} f\, dz = \oint_{\gamma_1 * \gamma_2^{-1}} f\, dz = 0.$$

□

1.4 Cauchy's integral formula

Cauchy's theorem says $\oint f = 0$ when f is holomorphic inside. If instead f has one controlled singularity inside — specifically, the singularity introduced by dividing by $(w - z)$ — the integral picks up exactly the value of f at z. This reproducing property shows that the values of f inside a disk are completely determined by its values on the boundary circle. Nothing like this happens in real analysis.

Theorem 1.13 (Cauchy integral formula)**.** *Let f be holomorphic on an open set Ω, and let $a \in \Omega$ and $r > 0$ be such that the closed disk $\overline{D(a,r)}$ is contained in Ω. Then for every z in the open disk $D(a,r)$,*

$$f(z) = \frac{1}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{w - z}\, dw, \tag{1.20}$$

where the contour is the boundary circle $|w - a| = r$ traversed counterclockwise.

Proof. Intuition. The integrand $f(w)/(w - z)$ is holomorphic on Ω except at the single point $w = z$, where it has a simple pole. Cauchy's theorem allows us to deform the contour from the large circle $|w-a|=r$ to a tiny circle around z without changing the integral. On that tiny circle, $f(w) \approx f(z)$ (continuity) and the remaining factor $1/(w-z)$ integrates to $2\pi i$, reproducing $f(z)$.

Fix $z \in D(a,r)$. The function $w \mapsto f(w)/(w-z)$ is holomorphic on $\Omega \setminus \{z\}$ (quotient of holomorphic functions with nonvanishing denominator). Pick $\varepsilon > 0$ small enough that $\overline{D(z,\varepsilon)} \subset D(a,r)$. The open region

$$A = \{w : |w-a| < r\} \setminus \overline{D(z,\varepsilon)}$$

is not simply connected, but its boundary (outer circle $|w-a|=r$ traversed counterclockwise, plus inner circle $|w-z|=\varepsilon$ traversed clockwise) bounds a region on which $f(w)/(w-z)$ is holomorphic. We convert A to a simply connected region by the following standard *slit construction*, which we will reuse below. Pick a straight segment σ (the slit) joining the outer circle to the inner circle, meeting each transversely and otherwise staying inside A. The set $A \setminus \sigma$ is simply connected, and its oriented boundary consists of the outer circle (counterclockwise), the two sides of σ, and the inner circle (clockwise). Because the two sides of σ are traversed in opposite directions with the same continuous integrand, their contour-integral contributions cancel.

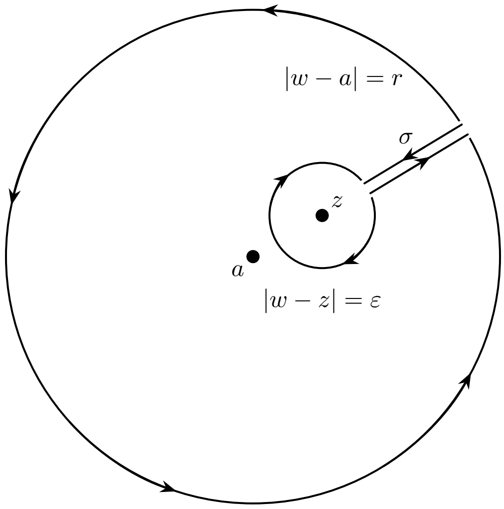

Figure 1: Slit construction used in the proof of Cauchy's integral formula (Thm. 1.13). The outer circle $|w-a|=r$ and inner circle $|w-z|=\varepsilon$ are connected by a radial slit σ. Traversing the boundary of $A \setminus \sigma$ makes the two sides of σ cancel, leaving the outer circle equal to the inner circle.

Applying Cauchy's theorem (1.16) to $A \setminus \sigma$ therefore gives

$$\oint_{|w-a|=r} \frac{f(w)}{w-z}\, dw = \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\, dw, \tag{1.21}$$

with both circles positively oriented.

On the small circle, parametrize $w = z+\varepsilon e^{i\theta}$, $\theta \in [0,2\pi]$, so $dw = i\varepsilon e^{i\theta}\, d\theta$ and $w-z = \varepsilon e^{i\theta}$:

$$\begin{aligned} \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\, dw &= \int_0^{2\pi} \frac{f(z+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \cdot i\varepsilon e^{i\theta}\, d\theta \\ &= i \int_0^{2\pi} f(z+\varepsilon e^{i\theta})\, d\theta. \end{aligned} \tag{1.22}$$

The first equality substitutes the parametrization; the second cancels $\varepsilon e^{i\theta}$.

Now let $\varepsilon \to 0$. Since f is continuous at z, every point on the circle $|w-z| = \varepsilon$ satisfies $|f(w)-f(z)| \to 0$ as $\varepsilon \to 0$; moreover the convergence is uniform in θ because $|z+\varepsilon e^{i\theta}-z| = \varepsilon$ is independent of θ. Uniform convergence allows interchanging limit and integral:

$$\lim_{\varepsilon\to 0} i \int_0^{2\pi} f(z+\varepsilon e^{i\theta})\, d\theta = i \int_0^{2\pi} f(z)\, d\theta = 2\pi i f(z).$$

The left-hand side of (1.21) does not depend on ε (by Cauchy's theorem it equals the small-circle integral for every admissible ε), so it equals $2\pi i f(z)$. Dividing by $2\pi i$ yields (1.20). □

Corollary 1.14 (Derivatives of all orders). *Under the hypotheses of Thm. 1.13, f has complex derivatives of all orders at every $z \in D(a,r)$, and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{(w-z)^{n+1}}\, dw, \qquad n \in \mathbb{N}. \tag{1.23}$$

Proof. Differentiate (1.20) n times with respect to z under the integral sign, by the standard Leibniz rule for parameter-dependent Riemann integrals with continuous partials. The chain rule gives $\frac{d}{dz}(w-z)^{-1} = (-1)(w-z)^{-2}\cdot(-1) = (w-z)^{-2}$, where the inner factor -1 comes from $\frac{d}{dz}(w-z) = -1$. Inductively, n differentiations give $n!(w-z)^{-n-1}$. To justify differentiation under the integral, note that on the contour $|w-a| = r$, the quantity $|w-z| \geq r - |z-a| > 0$ is bounded below uniformly in w for each fixed $z \in D(a,r)$ (and, locally, uniformly in z). Hence the integrand and its z-derivatives of all orders are continuous in (z,w) on the compact contour, and the Leibniz rule applies. □

This is the rigidity of complex analysis in one line: *one complex derivative implies infinitely many*. In particular, the C^2 hypothesis in Cor. 1.4 is automatic for holomorphic functions.

Corollary 1.15 (Cauchy estimates). *If f is holomorphic on an open neighborhood of $\overline{D(a,R)}$ and $M_R = \sup_{|w-a|=R} |f(w)|$, then*

$$|f^{(n)}(a)| \leq \frac{n!\, M_R}{R^n}, \qquad n \in \mathbb{N}. \tag{1.24}$$

Proof. Apply (1.23) at $z = a$ with contour $|w-a| = R$. On this contour, $|f(w)| \leq M_R$ and $|w-a| = R$, so the integrand is bounded in modulus by M_R/R^{n+1}. The arclength is $2\pi R$, and the prefactor has modulus $n!/(2\pi)$:

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n!\, M_R}{R^n}.$$

The inequality is the ML bound (Lem. 1.9). □

The Cauchy estimates are the quantitative form of the rigidity: growth of f at a distance R controls all derivatives at the center, with the n-th derivative decaying like R^{-n} when f is bounded far away.

Corollary 1.16 (Taylor series). *If f is holomorphic on the open disk $D(a,R)$, then for all $z \in D(a,R)$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n, \tag{1.25}$$

and the series converges absolutely and locally uniformly on $D(a,R)$.

Proof. Fix $z \in D(a,R)$ and choose r with $|z-a| < r < R$. Then $\overline{D(a,r)} \subset D(a,R)$, so (1.20) applies. The key step is to expand the kernel $1/(w-z)$ in powers of $z-a$. For w on the contour $|w-a| = r$,

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a} \cdot \frac{1}{1-\frac{z-a}{w-a}},$$

using the algebraic identity $(w-a)-(z-a) = w-z$ and factoring. Since $|(z-a)/(w-a)| = |z-a|/r < 1$, the geometric series converges:

$$\frac{1}{1-\frac{z-a}{w-a}} = \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n. \tag{1.26}$$

Uniform convergence on the contour. For every w on the circle $|w-a| = r$ the ratio $|(z-a)/(w-a)| = |z-a|/r =: q < 1$, independent of w. Hence the n-th term satisfies $|((z-a)/(w-a))^n| \le q^n$. The geometric majorant $\sum q^n$ converges, so the Weierstrass M-test gives uniform convergence of (1.26) for w on the contour.

Multiply by $f(w)/(w-a)$, which is bounded on the compact contour by some $K = \sup_{|w-a|=r} |f(w)|/r$:

$$\frac{f(w)}{w-z} = \sum_{n=0}^{\infty} \frac{f(w)}{(w-a)^{n+1}}(z-a)^n,$$

and this series also converges uniformly in w (each term is bounded by $K \cdot q^n$). Uniform convergence justifies interchanging sum and integral:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{w-z}\, dw \\ &= \frac{1}{2\pi i} \oint_{|w-a|=r} \sum_{n=0}^{\infty} \frac{f(w)}{(w-a)^{n+1}}(z-a)^n\, dw \\ &= \sum_{n=0}^{\infty} (z-a)^n \cdot \frac{1}{2\pi i} \oint_{|w-a|=r} \frac{f(w)}{(w-a)^{n+1}}\, dw. \end{aligned}$$

By Cor. 1.14 applied at the center a, the inner integral is exactly $f^{(n)}(a)/n!$. Local uniform convergence on any compact subdisk $\overline{D(a,r_1)}$ with $r_1 < r$ follows from the Cauchy estimate $|f^{(n)}(a)/n!| \le K/r^{n-1}$ (with K as above), which gives $\sum_{n\ge 0} |f^{(n)}(a)/n!|\,|z-a|^n \le \sum_{n\ge 0} Kr(r_1/r)^n = Kr/(1-r_1/r) < \infty$. □

Example 1.17 (Radius of convergence of $1/(1-z)$)**.** *Take $f(z) = 1/(1-z)$, holomorphic on $\mathbb{C} \setminus \{1\}$. Expanding around $a = 0$: $f^{(n)}(0) = n!$, so $f^{(n)}(0)/n! = 1$, giving $f(z) = \sum_{n\ge 0} z^n$ on $|z| < 1$. The radius of convergence is 1, which equals the distance from the center $a = 0$ to the nearest singularity $z = 1$. This is the general fact recorded below.*

Corollary 1.18 (Radius of convergence equals distance to nearest singularity)**.** *Let f be holomorphic on a domain Ω and $a \in \Omega$. Let*

$$R := \sup\{r > 0 : D(a,r) \subset \Omega\}$$

be the distance from a to the boundary of Ω (with $R = \infty$ if $\Omega = \mathbb{C}$). Then the Taylor series $\sum f^{(n)}(a)(z-a)^n/n!$ has radius of convergence at least R, and equals R exactly when f has a true singularity on the boundary circle $|z-a| = R$ (so it cannot be holomorphically extended through any point of that circle).

Proof. Cor. 1.16 shows the series converges on $D(a,R)$, so the radius of convergence is at least R. If the radius were strictly larger, the same series would define a holomorphic extension of f to a larger disk, contradicting the assumed singularity on $|z-a|=R$. □

This is the engineering rule of thumb: *a Taylor expansion gives away the nearest pole*. The geometric series above sees the singularity at 1; expanding $1/(1-z)$ around $a=2$ instead would have radius $|2-1|=1$ and capture the same singularity from the other side.

Corollary 1.19 (Liouville)**.** *A bounded entire function is constant.*

Proof. Suppose $|f| \le M$ on $\mathbb{C}$. Fix $z \in \mathbb{C}$ and apply (1.23) with $n=1$ on the circle $|w-z|=R$ (whose arclength is $2\pi R$):

$$|f'(z)| \le \frac{1!}{2\pi} \cdot \sup_{|w-z|=R} \frac{|f(w)|}{|w-z|^2} \cdot 2\pi R \le \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}.$$

The first inequality is Lem. 1.9 applied to the integral representation (1.23). Let $R \to \infty$: $|f'(z)| \to 0$, so $f'(z) = 0$ for every z. A function with vanishing derivative on the connected set $\mathbb{C}$ is constant. □

Corollary 1.20 (Fundamental theorem of algebra)**.** *Every non-constant polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ (with $n \ge 1$, $a_n \ne 0$) over $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Suppose for contradiction that $p(z) \ne 0$ for every $z \in \mathbb{C}$. Then $g(z) = 1/p(z)$ is entire (ratio of holomorphic functions with nonvanishing denominator). We show g is bounded on $\mathbb{C}$, then apply Liouville (Cor. 1.19).

Bound at infinity. For $|z|$ large, factor the leading term:

$$p(z) = a_n z^n \left(1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + \cdots + \frac{a_0}{a_n z^n}\right).$$

Set $A = |a_{n-1}/a_n| + |a_{n-2}/a_n| + \cdots + |a_0/a_n| \ge 0$. If $A=0$, then $p(z) = a_n z^n$, so $p(0)=0$, contradicting our assumption that p has no roots. Thus $A > 0$. For $|z| \ge R_0 := \max(1, 2A)$, every denominator $|z|^j \ge |z|$ for $j \ge 1$, so

$$\left|\frac{a_{n-1}}{a_n z} + \cdots + \frac{a_0}{a_n z^n}\right| \le \sum_{j=1}^{n} \frac{|a_{n-j}/a_n|}{|z|^j} \le \frac{A}{|z|} \le \frac{1}{2}.$$

By the *reverse triangle inequality* $|1+w| \ge 1 - |w|$ (valid for any $w \in \mathbb{C}$ with $|w| \le 1$; apply $|a| \le |a+b| + |b|$ with $a=1$, $b=w$ and rearrange), the factor in parentheses has modulus at least $1 - 1/2 = 1/2$. Therefore

$$|p(z)| \ge |a_n| \cdot |z|^n \cdot \tfrac{1}{2} \ge \tfrac{1}{2}|a_n| R_0^n, \qquad |z| \ge R_0,$$

which gives $|g(z)| = 1/|p(z)| \le 2/(|a_n| R_0^n)$ on $\{|z| \ge R_0\}$.

Bound on the compact disk. On $\overline{D(0,R_0)}$, $|p|$ is continuous and never zero, so it attains a positive minimum $m > 0$ there (extreme-value theorem); hence $|g| \le 1/m$.

Combining, $|g|$ is bounded on all of $\mathbb{C}$. By Cor. 1.19 g is constant, so p is constant, contradicting $n \ge 1$ with $a_n \ne 0$. □

1.5 Laurent series and singularities

Taylor series (Cor. 1.16) describe holomorphic functions on disks. Many functions of interest — $1/z$, $e^{1/z}$, $\sin(z)/z^3$ — are not holomorphic at a point but are holomorphic on an annulus, meaning a ring-shaped region around that point. To describe such functions locally, we need a two-sided series with negative powers allowed.

Theorem 1.21 (Laurent expansion). *If f is holomorphic on the annulus $A = \{z : r_1 < |z-a| < r_2\}$ with $0 \le r_1 < r_2 \le \infty$, then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad c_n = \frac{1}{2\pi i} \oint_{|w-a|=\rho} \frac{f(w)}{(w-a)^{n+1}}\, dw, \tag{1.27}$$

for any $\rho \in (r_1, r_2)$. The series converges absolutely and locally uniformly on A, and the coefficients c_n are unique.

Proof. Intuition. When z lies in an annulus where f is holomorphic, we trap z between two concentric circles. Cauchy's integral formula applied to the region between them expresses $f(z)$ as the difference of two contour integrals: one over the outer circle (giving non-negative powers of $z-a$, as in the Taylor proof) and one over the inner circle (giving negative powers, by expanding $1/(w-z)$ in powers of $(w-a)/(z-a)$ instead).

Fix $z \in A$ and pick radii $r_1 < r_1' < |z-a| < r_2' < r_2$. Apply Cauchy's theorem to the annular region between the two circles, with a small circle around z removed. The region is made simply connected by cutting radial slits from the inner circle out to the outer circle (the same slit construction used in the proof of Thm. 1.13; see Fig. 1). The two sides of each slit are traversed in opposite directions, so their integrals cancel. What remains is

$$f(z) = \frac{1}{2\pi i} \oint_{|w-a|=r_2'} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \oint_{|w-a|=r_1'} \frac{f(w)}{w-z}\, dw.$$

On the outer circle, $|z-a| < |w-a|$, so

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n.$$

Thus the outer contribution is

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{|w-a|=r_2'} \frac{f(w)}{(w-a)^{n+1}}\, dw \right) (z-a)^n.$$

On the inner circle, $|z-a| > |w-a|$, so use the other geometric expansion:

$$\frac{1}{w-z} = -\frac{1}{(z-a)-(w-a)} = -\frac{1}{z-a} \sum_{n=0}^{\infty} \left(\frac{w-a}{z-a}\right)^n.$$

Remember that the inner integral is subtracted in the annular formula above. The minus sign there cancels the minus sign in this expansion, so the inner contribution becomes

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{|w-a|=r_1'} f(w)(w-a)^n\, dw \right) (z-a)^{-n-1}.$$

This is the source of the negative powers of $z-a$. If $k = -n-1$, then $f(w)(w-a)^n = f(w)/(w-a)^{k+1}$, so the same coefficient formula covers both positive and negative powers.

Uniform convergence on each contour justifies interchanging sum and integral. The coefficient integral may be taken on any circle $|w-a| = \rho$ inside the annulus, because $f(w)/(w-a)^{k+1}$ is holomorphic in the annular region swept out when one such circle is moved to another. This gives (1.27). Uniqueness follows from $\oint (w-a)^m\, dw = 2\pi i\, \delta_{m,-1}$, where $\delta_{m,-1}$ is the *Kronecker delta*: it equals 1 if $m = -1$ and 0 otherwise. Ex. 1.8 proves this for the unit circle, and translation plus rescaling gives the same identity for any circle centered at a. Multiplying the Laurent series by $(w-a)^{-k-1}$ and integrating then extracts exactly c_k. □

Definition 1.22 (Classification of isolated singularities)**.** *Let a be an isolated singularity of f, meaning that f is holomorphic on some punctured disk $0 < |z-a| < r$ (a disk with its center point removed) but is not assumed holomorphic at a itself. Write its Laurent expansion as $f(z) = \sum_{n\in\mathbb{Z}} c_n (z-a)^n$. The singularity is:*

- Removable *if $c_n = 0$ for all $n < 0$.*
- *A* pole of order $m \geq 1$ *if $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$. A pole of order 1 is called* simple.
- Essential *if $c_n \neq 0$ for infinitely many negative n.*

Example 1.23 (Three singularities at $z = 0$)**.** *Using the Taylor series* $\sin z = z - z^3/3! + z^5/5! - \cdots$, $\cos z = 1 - z^2/2! + z^4/4! - \cdots$, *and* $e^w = \sum w^n/n!$:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \qquad \textit{removable (all } c_n = 0, n < 0\textit{)}$$

$$\frac{\cos z}{z^3} = \frac{1}{z^3} - \frac{1}{2!\,z} + \frac{z}{4!} - \cdots \qquad \textit{pole of order } 3\ (c_{-3} = 1)$$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} \qquad \textit{essential (infinitely many } c_n \neq 0,\ n < 0\textit{)}$$

Each Laurent series here is obtained by substituting the Taylor series for sin, cos, *or* exp *and then simplifying powers of z. The substitution is valid because these three functions are entire and their Taylor series converge absolutely everywhere.*

1.6 The residue theorem

The Laurent coefficient c_{-1} is special: it is the *only* one that survives integration around a small circle enclosing the singularity (recall Ex. 1.8). This makes it the unique quantity one needs to know to compute closed-contour integrals.

Definition 1.24 (Residue)**.** *The* residue *of f at an isolated singularity a is the coefficient c_{-1} in the Laurent expansion* (1.27)*:*

$$\operatorname*{Res}_{z=a} f \;:=\; c_{-1} \;=\; \frac{1}{2\pi i} \oint_{|w-a|=\rho} f(w)\, dw \tag{1.28}$$

for every $\rho > 0$ small enough that the circle $|w-a| = \rho$ lies in a punctured neighborhood of a on which f is holomorphic.

Theorem 1.25 (Residue theorem)**.** *Let Ω be a simply connected open set and f holomorphic on Ω except at finitely many isolated singularities $a_1, \ldots, a_N \in \Omega$. Let $\gamma \subset \Omega \setminus \{a_1, \ldots, a_N\}$ be a* *positively oriented simple closed contour, and let $I \subseteq \{1,\dots,N\}$ index the singularities that lie inside γ (the remaining a_k may lie outside γ). Then*

$$\oint_\gamma f(z)\,dz \;=\; 2\pi i \sum_{k\in I} \operatorname*{Res}_{z=a_k} f. \tag{1.29}$$

Proof. Intuition. Cauchy's theorem says the integral around a closed contour vanishes if f is holomorphic inside. With singularities present, we isolate each one by a small circle, connect each circle to the outer contour with slits, and apply Cauchy's theorem to the resulting simply connected region. The slits cancel pairwise, leaving the outer contour equal to the sum of integrals around each small circle — each of which is $2\pi i$ times the residue.

Only the singularities inside γ matter. Pick $\varepsilon > 0$ small enough that the closed disks $\overline{D(a_k,\varepsilon)}$ for $k \in I$ are pairwise disjoint and contained in the interior of γ, which we denote $\operatorname{int}(\gamma)$ (the bounded component of $\mathbb{C} \setminus \gamma$). Applying the slit construction from the proof of Thm. 1.13 once per interior singularity, choose pairwise disjoint polygonal slits σ_k joining each circle $|z - a_k| = \varepsilon$ to γ, meeting γ and the small circle transversely and otherwise staying inside $\operatorname{int}(\gamma) \setminus \bigcup_{j\in I} \overline{D(a_j,\varepsilon)}$. Remove both the disks and the slits, and set

$$U := \operatorname{int}(\gamma) \setminus \left(\bigcup_{k\in I} \overline{D(a_k,\varepsilon)} \cup \bigcup_{k\in I} \sigma_k \right).$$

The region U is simply connected, and f is holomorphic on a neighborhood of $\overline{U}$. Hence Cauchy's theorem applies to ∂U:

$$\oint_{\partial U} f(z)\,dz = 0.$$

Traverse ∂U positively. Its boundary consists of the outer contour γ, the two sides of each slit, and the small circles $|z - a_k| = \varepsilon$ for $k \in I$ with negative orientation. The two sides of a slit are traversed in opposite directions with the same integrand, so their contributions cancel pairwise. Therefore

$$\oint_\gamma f(z)\,dz - \sum_{k\in I} \oint_{|z-a_k|=\varepsilon} f(z)\,dz = 0.$$

Each small-circle integral equals $2\pi i \operatorname{Res}_{z=a_k} f$ by Definition 1.24. Rearranging gives (1.29). □

Proposition 1.26 (Residue at a pole). *If f has a pole of order $m \ge 1$ at a, then*

$$\operatorname*{Res}_{z=a} f \;=\; \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\big[(z-a)^m f(z)\big]. \tag{1.30}$$

In particular for a simple pole ($m = 1$), $\operatorname{Res}_{z=a} f = \lim_{z\to a}(z-a)f(z)$. If $f = g/h$ with g, h holomorphic near a, $g(a) \neq 0$, $h(a) = 0$, $h'(a) \neq 0$, then $\operatorname{Res}_{z=a}(g/h) = g(a)/h'(a)$.

Proof. Write the Laurent expansion $f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n$. Multiplication by $(z-a)^m$ removes the negative powers:

$$(z-a)^m f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^{n+m} = \sum_{k=0}^{\infty} c_{k-m} (z-a)^k,$$

where $k = n + m$. This is a Taylor series in $(z - a)$. Differentiating $m - 1$ times, the k-th term $(z-a)^k$ differentiates to $k(k-1)\cdots(k-m+2)(z-a)^{k-m+1}$. Setting $z = a$ kills every term except

$k = m-1$, where the factor $(z-a)^0 = 1$ survives. The coefficient of that term is $c_{(m-1)-m} = c_{-1}$, and the prefactor is $(m-1)!$:

$$\frac{d^{m-1}}{dz^{m-1}}\Big[(z-a)^m f(z)\Big]\bigg|_{z=a} = (m-1)!\, c_{-1}.$$

Dividing by $(m-1)!$ gives (1.30). The limit notation in (1.30) means the continuous value at a of the displayed derivative.

For the g/h special case with $m=1$: near a, $h(z) = h'(a)(z-a) + O((z-a)^2)$, so

$$(z-a)\frac{g(z)}{h(z)} = \frac{g(z)}{h'(a) + O(z-a)} \xrightarrow{z\to a} \frac{g(a)}{h'(a)}.$$

□

Example 1.27 (Residue at a higher-order pole). *Let $f(z) = e^z/z^3$. The origin is a pole of order $m = 3$. By Prop. 1.26,*

$$\operatorname*{Res}_{z=0} \frac{e^z}{z^3} = \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^2}{dz^2}\Big[z^3 \cdot e^z/z^3\Big] = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} e^z = \frac{1}{2}\cdot e^0 = \frac{1}{2}.$$

The formula says to cancel $z^3 \cdot z^{-3} = 1$, differentiate e^z twice, evaluate at 0, and divide by 2!. The Laurent series gives the same answer:

$$\frac{e^z}{z^3} = z^{-3} + z^{-2} + \frac{1}{2}z^{-1} + \frac{1}{6} + \cdots,$$

so the coefficient of z^{-1} is 1/2.

Example 1.28 (Residues of $\cot z$). $\cot z = \cos z/\sin z$ *has simple poles precisely at the zeros of $\sin z$, namely $z = k\pi$ for every $k \in \mathbb{Z}$. At such a pole, $\cos(k\pi) = (-1)^k \neq 0$ and $(\sin z)'|_{z=k\pi} = \cos(k\pi) = (-1)^k$. By the g/h form of Prop. 1.26,*

$$\operatorname*{Res}_{z=k\pi} \cot z = \frac{\cos(k\pi)}{\cos(k\pi)} = 1.$$

All residues are 1, independent of k. This uniform residue underlies the Mittag–Leffler partial-fraction expansion of cot *used later to sum series such as $\sum_{n\geq 1} 1/n^2$.*

Example 1.29 (Laurent expansions of $1/[(z-1)(z-2)]$ in three annuli). *The function $f(z) = 1/[(z-1)(z-2)]$ has singularities at $z = 1, 2$, so it is holomorphic on three disjoint concentric regions centered at the origin: the disk $|z| < 1$, the annulus $1 < |z| < 2$, and the exterior $|z| > 2$. We find its Laurent expansion in each.*

Partial fractions. *Write*

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

Clearing denominators: $1 = A(z-2) + B(z-1)$. Setting $z = 1$: $1 = A(-1)$, so $A = -1$. Setting $z = 2$: $1 = B\cdot 1$, so $B = 1$. Hence

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}. \tag{1.31}$$

(a) Disk $|z| < 1$. *Here $|z| < 1$ and $|z/2| < 1/2 < 1$; both geometric series are in non-negative powers of z. Rewrite:*

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

$$\frac{1}{z-2} = -\frac{1}{2-z} = -\frac{1}{2}\cdot\frac{1}{1-z/2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$

We have only used the geometric series $\sum_{n\geq 0} w^n = 1/(1-w)$, valid for $|w| < 1$. Adding the two expansions gives

$$f(z) = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n, \qquad |z| < 1.$$

This is the Taylor series at 0: no negative powers, consistent with f being holomorphic at the origin. Its radius of convergence is 1, matching the distance from 0 to the nearest singularity.

(b) Annulus $1 < |z| < 2$. *Now $|1/z| < 1$ but $|z/2| < 1$, so the two terms require opposite expansions.*

For $-1/(z-1)$ with $|1/z| < 1$, factor out z:

$$-\frac{1}{z-1} = -\frac{1}{z} \cdot \frac{1}{1-1/z} = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{k=1}^{\infty} z^{-k},$$

where the last equality shifts the index $k = n+1$. For $1/(z-2)$ with $|z/2| < 1$, same expansion as in (a): $1/(z-2) = -\sum_{n\geq 0} z^n/2^{n+1}$. Combining,

$$f(z) = -\sum_{k=1}^{\infty} z^{-k} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \qquad 1 < |z| < 2,$$

a genuine two-sided series.

(c) Exterior $|z| > 2$. *Now $|1/z| < 1$* and *$|2/z| < 1$; expand both in negative powers of z. For $-1/(z-1)$, same as in (b): $-\sum_{k\geq 1} z^{-k}$. For $1/(z-2)$,*

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-2/z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{z^k},$$

with $k = n+1$. Combining,

$$f(z) = \sum_{k=1}^{\infty} \frac{2^{k-1} - 1}{z^k}, \qquad |z| > 2.$$

Only negative powers, consistent with $f(z) \to 0$ as $|z| \to \infty$. The three expansions, valid on disjoint regions, are different Laurent representations of the same meromorphic function, meaning a function that is holomorphic except at isolated poles. This is a reminder that the domain of convergence is essential data attached to a series.

1.7 Applications: real integrals by contour methods

The residue theorem converts many real integrals into residue computations. The contour is chosen to make the new, non-real pieces disappear in a limit.

Example 1.30 (Rational integrand over $\mathbb{R}$). *Evaluate $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$. (The answer $I = \pi$ is known from elementary calculus via* arctan*; we redo it with contours as a sanity check.)*

Let $f(z) = 1/(1+z^2) = 1/[(z-i)(z+i)]$, which has simple poles at $z = \pm i$. Close the real segment $[-R, R]$ with a semicircular arc C_R in the upper half-plane:

$$\gamma_R = [-R, R] \cup C_R, \qquad C_R = \{Re^{i\theta} : \theta \in [0, \pi]\},$$

traversed counterclockwise. For $R > 1$, γ_R encloses only the pole at $z = i$. By Prop. 1.26 with $g(z) = 1$ and $h(z) = 1 + z^2$, $h'(i) = 2i$:

$$\operatorname*{Res}_{z=i} \frac{1}{1+z^2} = \frac{g(i)}{h'(i)} = \frac{1}{2i}.$$

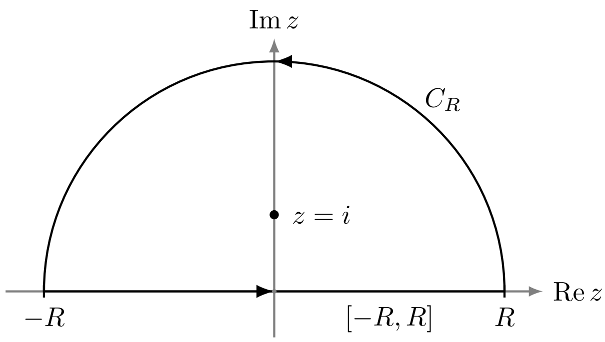

Figure 2: Semicircular contour in the upper half-plane, used to evaluate $\int_{-\infty}^{\infty} dx/(1+x^2)$ (Ex. 1.30). As $R \to \infty$, the arc contribution vanishes by the ML bound, leaving $2\pi i$ times the residue at $z = i$.

Thm. 1.25 gives

$$\oint_{\gamma_R} f(z)\,dz = 2\pi i \cdot \frac{1}{2i} = \pi.$$

Split the contour: $\oint_{\gamma_R} = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$. *On* C_R, $|z| = R$, *so by the reverse triangle inequality*

$$|1+z^2| \geq |z|^2 - 1 = R^2 - 1 \quad (R > 1),$$

hence $|f(z)| \leq 1/(R^2-1)$. *The arclength of* C_R *is* πR, *and the ML inequality (Lem. 1.9) gives*

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi R}{R^2-1} \xrightarrow{R\to\infty} 0.$$

Therefore $\int_{-R}^{R} f(x)\,dx \to \pi$, *and* $I = \pi$.

Lemma 1.31 (Jordan)**.** *For every* $R > 0$,

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta \;<\; \frac{\pi}{R}. \tag{1.32}$$

Proof. For $\theta \in [0, \pi/2]$, $\sin\theta \geq 2\theta/\pi$ (concavity of sin on $[0, \pi/2]$, which puts the graph above the chord joining $(0,0)$ to $(\pi/2, 1)$), and symmetry $\sin(\pi - \theta) = \sin\theta$ handles $[\pi/2, \pi]$. Hence

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{\pi}{R}(1 - e^{-R}) < \frac{\pi}{R}.$$

□

Example 1.32 (Oscillatory integrals via Jordan's lemma)**.** *Evaluate* $I = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$.

Replace $\cos x$ *by* $\operatorname{Re} e^{ix}$. *We integrate* $e^{iz}/(1+z^2)$, *not* $\cos z/(1+z^2)$, *because* $e^{iz} = e^{ix-y}$ *decays in the upper half-plane* $z = x + iy$ *with* $y > 0$, *while* $\cos(iy) = \cosh y$ *grows.*

Set $f(z) = e^{iz}/(1+z^2)$ *and close in the upper half-plane with* $\gamma_R = [-R, R] \cup C_R$ *as in Ex. 1.30. The only enclosed singularity for* $R > 1$ *is the simple pole at* $z = i$*; with* $g(z) = e^{iz}$ *and* $h(z) = 1 + z^2$,

$$\operatorname*{Res}_{z=i} \frac{e^{iz}}{1+z^2} = \frac{e^{i\cdot i}}{h'(i)} = \frac{e^{-1}}{2i}.$$

Hence $\oint_{\gamma_R} f(z)\,dz = 2\pi i \cdot e^{-1}/(2i) = \pi/e$.

To show the arc integral vanishes we invoke Jordan's lemma (Lem. 1.31): on C_R with $z = Re^{i\theta}$,

$$|e^{iz}| = |e^{iR(\cos\theta + i\sin\theta)}| = e^{-R\sin\theta},$$

and the lemma yields $\int_0^\pi e^{-R\sin\theta}\,d\theta < \pi/R$. Combining with $|1+z^2|^{-1} \le 1/(R^2-1)$:

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R}{R^2-1}\int_0^\pi e^{-R\sin\theta}\,d\theta \le \frac{R}{R^2-1}\cdot\frac{\pi}{R} = \frac{\pi}{R^2-1} \to 0.$$

Therefore $\int_{-R}^R f(x)\,dx \to \pi/e$. Taking real parts, $\int_{-\infty}^\infty \cos x/(1+x^2)\,dx = \pi/e$.

Example 1.33 (Keyhole contour integral). *Evaluate $I(\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx$ for $\alpha \in (0,1)$.*

For powers such as $z^{\alpha-1}$ with non-integer α, we must choose one value of the logarithm. A branch *is such a consistent single-valued choice on a region, and a* branch cut *is the curve removed from the plane so the choice does not jump while we move around. This example uses a* non-principal *branch of the logarithm, with $\arg z \in (0, 2\pi)$, so the branch cut lies on the positive real axis. The* principal *convention used later is $\arg z \in (-\pi, \pi]$, whose cut lies on the negative real axis.*

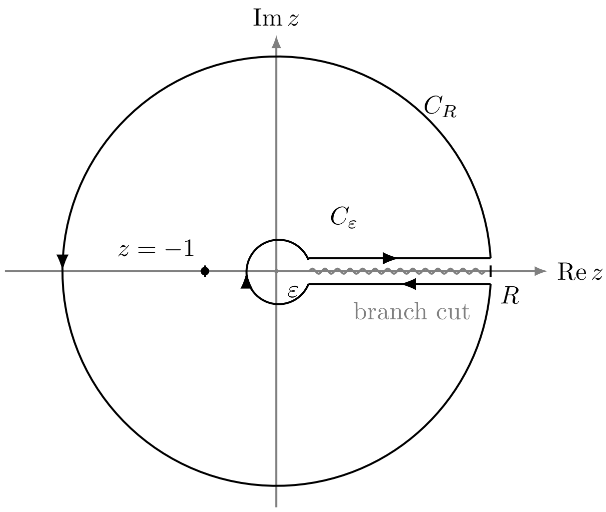

Figure 3: Keyhole contour used to evaluate $\int_0^\infty x^{\alpha-1}/(1+x)\,dx$ (Ex. 1.33). The branch cut, the chosen line where the branch jumps, runs along the positive real axis; the pole at $z = -1$ is enclosed.

Define $z^{\alpha-1} := e^{(\alpha-1)\log z}$ with this branch. Just above the cut, $\arg z \to 0^+$ and $z^{\alpha-1} = x^{\alpha-1}$; just below it, $\arg z \to 2\pi^-$ and $z^{\alpha-1} = x^{\alpha-1}e^{2\pi i(\alpha-1)}$.

Consider the keyhole contour $\gamma_{\varepsilon,R}$ consisting of: the segment from ε to R just above the cut; the large circle C_R ($|z| = R$) counterclockwise; the segment from R back to ε just below the cut; and the small circle C_ε ($|z| = \varepsilon$) clockwise. The enclosed singularity is the simple pole at $z = -1 = e^{i\pi}$, where the chosen branch has $\log(-1) = i\pi$. By Prop. 1.26 with $g(z) = z^{\alpha-1}$, $h(z) = 1+z$, $h'(-1) = 1$:

$$\operatorname*{Res}_{z=-1} \frac{z^{\alpha-1}}{1+z} = e^{(\alpha-1)\log(-1)} = e^{(\alpha-1)i\pi}.$$

Vanishing of the arc integrals. *On C_R: $|z^{\alpha-1}| = R^{\alpha-1}$, $|1+z| \ge R-1$, arclength $= 2\pi R$, so*

$$\left|\int_{C_R} \frac{z^{\alpha-1}}{1+z}\,dz\right| \le \frac{R^{\alpha-1}}{R-1}\cdot 2\pi R = \frac{2\pi R^\alpha}{R-1} \to 0 \quad (R \to \infty),$$

since $\alpha < 1$. On C_ε: $|z^{\alpha-1}| = \varepsilon^{\alpha-1}$, $|1+z| \geq 1-\varepsilon$, arclength $= 2\pi\varepsilon$, so

$$\left|\int_{C_\varepsilon} \frac{z^{\alpha-1}}{1+z}\,dz\right| \leq \frac{\varepsilon^{\alpha-1}}{1-\varepsilon}\cdot 2\pi\varepsilon = \frac{2\pi\varepsilon^{\alpha}}{1-\varepsilon} \to 0 \quad (\varepsilon \to 0),$$

since $\alpha > 0$.

Assembling. *On the upper bank of the cut, $z = x$ with x increasing from ε to R, so the straight segment tends to $I(\alpha)$. On the lower bank, use $z = xe^{2\pi i}$ with x decreasing from R to ε. Since $dz = e^{2\pi i}\,dx = dx$ and $z^{\alpha-1} = x^{\alpha-1}e^{2\pi i(\alpha-1)}$ on our branch, the lower bank contributes*

$$\int_R^{\varepsilon} \frac{e^{2\pi i(\alpha-1)}x^{\alpha-1}}{1+x}\,dx = -e^{2\pi i(\alpha-1)}I(\alpha)$$

in the limit. Therefore the two straight segments give

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx \;-\; e^{2\pi i(\alpha-1)}\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx \;=\; 2\pi i\, e^{(\alpha-1)i\pi},$$

where the minus sign on the second term comes from the reversed orientation of the lower segment. Factor:

$$\left(1 - e^{2\pi i(\alpha-1)}\right)I(\alpha) = 2\pi i\, e^{i\pi(\alpha-1)}. \tag{1.33}$$

Simplification. *Using $e^{-2\pi i} = 1$,*

$$e^{2\pi i(\alpha-1)} = e^{2\pi i\alpha}\cdot e^{-2\pi i} = e^{2\pi i\alpha}.$$

Using $e^{-i\pi} = -1$,

$$e^{i\pi(\alpha-1)} = e^{i\pi\alpha}\cdot e^{-i\pi} = -e^{i\pi\alpha}.$$

Substituting both into (1.33):

$$\left(1 - e^{2\pi i\alpha}\right)I(\alpha) = -2\pi i\, e^{i\pi\alpha}.$$

The key identity $1 - e^{2\pi i\alpha} = -2i\,e^{i\pi\alpha}\sin(\pi\alpha)$. *Factor $e^{i\pi\alpha}$ out of the left side:*

$$1 - e^{2\pi i\alpha} = e^{i\pi\alpha}\left(e^{-i\pi\alpha} - e^{i\pi\alpha}\right).$$

By Euler's formula, $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, so $e^{-i\pi\alpha} - e^{i\pi\alpha} = -2i\sin(\pi\alpha)$. Therefore

$$1 - e^{2\pi i\alpha} = -2i\,e^{i\pi\alpha}\sin(\pi\alpha). \tag{1.34}$$

Substituting (1.34):

$$-2i\,e^{i\pi\alpha}\sin(\pi\alpha)\cdot I(\alpha) = -2\pi i\,e^{i\pi\alpha},$$

and dividing both sides by $-2i\,e^{i\pi\alpha}$ (nonzero for $\alpha \in (0,1)$):

$$I(\alpha) = \frac{\pi}{\sin(\pi\alpha)}.$$

This integral will reappear in Section 3 as $B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha)$, yielding the reflection formula $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin(\pi\alpha)$.

Example 1.34 (Double pole: $\int_{-\infty}^{\infty} dx/(1+x^2)^2$). *Evaluate $J = \int_{-\infty}^{\infty} dx/(1+x^2)^2$ with the same semicircular contour as Ex. 1.30.*

Let $f(z) = 1/(1+z^2)^2 = 1/[(z-i)(z+i)]^2$. Inside γ_R (upper half-plane, $R > 1$) there is one singularity: a pole of order $m = 2$ at $z = i$. By Prop. 1.26,

$$\begin{aligned}\operatorname*{Res}_{z=i} f &= \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}\big[(z-i)^2 f(z)\big] \\ &= \lim_{z\to i}\frac{d}{dz}\frac{1}{(z+i)^2} \\ &= \lim_{z\to i}\frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}.\end{aligned}$$

Here $(z-i)^2/[(z-i)(z+i)]^2 = 1/(z+i)^2$, and $(d/dz)(z+i)^{-2} = -2(z+i)^{-3}$; substituting $z = i$ gives $(2i)^3 = -8i$.

By Thm. 1.25, $\oint_{\gamma_R} f\,dz = 2\pi i \cdot 1/(4i) = \pi/2$. On C_R, $|f| \le 1/(R^2-1)^2$ and the arclength is πR, so

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{\pi R}{(R^2-1)^2} \to 0.$$

Therefore $J = \pi/2$.

Example 1.35 (Trigonometric integral by $z = e^{i\theta}$). *Evaluate $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. Set $z = e^{i\theta}$, so as θ sweeps $[0, 2\pi]$, z traverses the unit circle once counterclockwise. Then $dz = ie^{i\theta}\,d\theta = iz\,d\theta$, so $d\theta = dz/(iz)$, and $\cos\theta = (z+z^{-1})/2$. Therefore*

$$\begin{aligned}I &= \oint_{|z|=1}\frac{1}{2+(z+z^{-1})/2}\cdot\frac{dz}{iz} \\ &= \oint_{|z|=1}\frac{2z}{(4z+z^2+1)}\cdot\frac{dz}{iz} \\ &= \frac{2}{i}\oint_{|z|=1}\frac{dz}{z^2+4z+1}.\end{aligned}$$

In the second line we multiply numerator and denominator of the first factor by $2z$; the z then cancels with the $dz/(iz)$ factor.

The roots of $z^2+4z+1 = 0$ are $z_\pm = -2 \pm \sqrt{3}$ by the quadratic formula. We have $|z_+| = |-2+\sqrt{3}| \approx 0.268 < 1$ and $|z_-| = |-2-\sqrt{3}| \approx 3.73 > 1$, so only z_+ is enclosed. The polynomial factors as $z^2+4z+1 = (z-z_+)(z-z_-)$, so

$$\operatorname*{Res}_{z=z_+}\frac{1}{z^2+4z+1} = \frac{1}{z_+-z_-} = \frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})} = \frac{1}{2\sqrt{3}}.$$

By the residue theorem,

$$I = \frac{2}{i}\cdot 2\pi i\cdot\frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

The general pattern $\int_0^{2\pi} d\theta/(a+b\cos\theta) = 2\pi/\sqrt{a^2-b^2}$ for $a > |b| > 0$ (Prob. 1.6) is obtained identically.

Example 1.36 (Essential singularity in a contour integral). *Evaluate $\oint_{|z|=1} e^{1/z}\,dz$. The only singularity inside the contour is at $z = 0$, which is essential (Ex. 1.23). The residue theorem only needs the coefficient c_{-1} of the Laurent expansion:*

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!\,z^n}.$$

Thus $c_{-1} = 1/1! = 1$, *and*

$$\oint_{|z|=1} e^{1/z}\,dz = 2\pi i \cdot 1 = 2\pi i.$$

Example 1.37 (Indented contour: $\int_{-\infty}^{\infty} \sin x/x\,dx = \pi$)**.** *The integral* $I = \int_{-\infty}^{\infty} \sin x/x\,dx$ *is a celebrated evaluation in Fourier analysis; it equals* π*. The integrand has no singularity on* $\mathbb{R}$ *(the singularity at* 0 *is removable:* $\sin x/x \to 1$*), so* I *is an ordinary improper Riemann integral. Here* improper *means it is defined as a limit over finite intervals whose endpoints tend to infinity.*

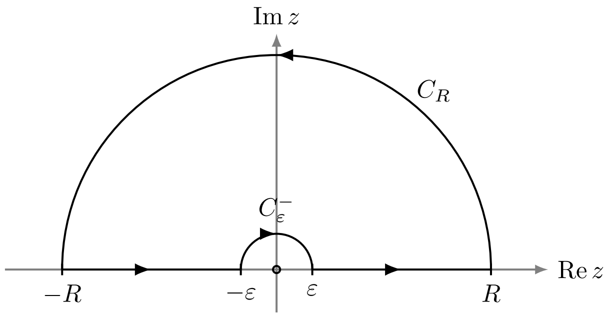

Figure 4: Indented semicircular contour used to evaluate $\int_{-\infty}^{\infty} \sin(x)/x\,dx$ (Ex. 1.37). The small semicircle C_ε^- lies in the upper half-plane and avoids the pole at $z = 0$ from above; the superscript $-$ records its clockwise orientation.

Write $\sin x/x = \mathrm{Im}(e^{ix}/x)$ *and consider* $f(z) = e^{iz}/z$*. Unlike* $\sin z/z$*, the function* e^{iz}/z does *have a pole at* $z = 0$*, so we modify the semicircular contour with a small indentation bypassing the origin. Let*

$$\gamma_{\varepsilon,R} = [-R, -\varepsilon] \cup C_\varepsilon^- \cup [\varepsilon, R] \cup C_R,$$

where C_ε^- *is the semicircle* $|z| = \varepsilon$ *in the upper half-plane traversed clockwise (from* $-\varepsilon$ *to* $+\varepsilon$*), and* C_R *is the upper semicircle* $|z| = R$ *counterclockwise. Inside* $\gamma_{\varepsilon,R}$ *there are no singularities, so by Cauchy's theorem* $\oint = 0$*, i.e.,*

$$\int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,dz + \int_{C_\varepsilon^-} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,dz + \int_{C_R} \frac{e^{iz}}{z}\,dz = 0.$$

Large semicircle. *On* C_R *parametrize* $z = Re^{i\theta}$*,* $\theta \in [0, \pi]$*,* $dz = iRe^{i\theta}d\theta$*, and* $|e^{iz}| = |e^{iR\cos\theta}| \cdot |e^{-R\sin\theta}| = e^{-R\sin\theta}$*. Hence*

$$\left| \int_{C_R} \frac{e^{iz}}{z}\,dz \right| = \left| \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}}\,iRe^{i\theta}\,d\theta \right| \le \int_0^{\pi} e^{-R\sin\theta}\,d\theta \le \frac{\pi}{R},$$

where the last inequality is Jordan's lemma (Lem. 1.31). As $R \to \infty$ *this vanishes.*

Small semicircle. *Parametrize* $z = \varepsilon e^{i\theta}$*,* θ *running from* π *to* 0 *(clockwise),* $dz = i\varepsilon e^{i\theta}\,d\theta$*:*

$$\int_{C_\varepsilon^-} \frac{e^{iz}}{z}\,dz = \int_{\pi}^{0} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} \cdot i\varepsilon e^{i\theta}\,d\theta = i\int_{\pi}^{0} e^{i\varepsilon e^{i\theta}}\,d\theta.$$

As $\varepsilon \to 0$*,* $e^{i\varepsilon e^{i\theta}} \to 1$ *uniformly in* θ*, so the integral tends to* $i(0 - \pi) = -i\pi$*.*

Straight segments. *The two real segments combine to give*

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx \longrightarrow PV\!\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx \qquad (\varepsilon \to 0,\ R \to \infty),$$

where $PV\int_{-\infty}^{\infty}$ *denotes the* Cauchy principal value*: the symmetric limit* $\lim_{R\to\infty}\int_{-R}^{R}$*, used whenever the integrand has a non-integrable singularity (here at* $x=0$*) or slowly decaying tails that would make the two-sided improper integral ambiguous.*

Taking $\varepsilon\to 0$*,* $R\to\infty$ *in the identity* $\int_{-R}^{-\varepsilon}+\int_{C_\varepsilon^-}+\int_{\varepsilon}^{R}+\int_{C_R}=0$*, we get*

$$PV\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx - i\pi + 0 = 0,$$

so $PV\int e^{ix}/x\,dx = i\pi$*. Since* $\sin x/x$ *extends continuously through* $x=0$*, its principal value is the ordinary improper integral. Taking imaginary parts:*

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \pi.$$

(The real part $PV\int\cos x/x\,dx = 0$ *by oddness, consistent with* $\mathrm{Re}(i\pi)=0$*.)*

1.8 Argument principle and Rouché's theorem

A subtler application of the residue theorem counts zeros and poles. This is useful when we want the number of solutions of $f(z)=0$ inside a region, not the solutions themselves. The mechanism is the *logarithmic derivative* f'/f: it has a simple pole at each zero or pole of f, and its residue records the multiplicity with a sign.

Theorem 1.38 (Argument principle)**.** *Let* Ω *be simply connected,* f *meromorphic on* Ω *(i.e. holomorphic except at isolated poles), and* $\gamma\subset\Omega$ *a positively oriented simple closed contour on which* f *has neither zeros nor poles. Let* Z *be the number of zeros and* P *the number of poles of* f *inside* γ*, each counted with multiplicity. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz \;=\; Z-P. \tag{1.35}$$

Proof. We compute the residue of f'/f at an arbitrary zero or pole a of f.

Case 1: a is a zero of order $m\geq 1$. Then $f(z)=(z-a)^m g(z)$ with g holomorphic near a and $g(a)\neq 0$. Differentiating,

$$f'(z) = m(z-a)^{m-1}g(z) + (z-a)^m g'(z).$$

Therefore

$$\frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1}g(z)+(z-a)^m g'(z)}{(z-a)^m g(z)} = \frac{m}{z-a}+\frac{g'(z)}{g(z)}.$$

The second term is holomorphic at a (since $g(a)\neq 0$), so f'/f has a simple pole at a with residue m.

Case 2: a is a pole of order $m\geq 1$. Then $f(z)=(z-a)^{-m}g(z)$ with g holomorphic, $g(a)\neq 0$. Differentiating by the product rule,

$$f'(z) = -m(z-a)^{-m-1}g(z) + (z-a)^{-m}g'(z),$$

so dividing by $f(z)=(z-a)^{-m}g(z)$:

$$\frac{f'(z)}{f(z)} = \frac{-m(z-a)^{-m-1}g(z)+(z-a)^{-m}g'(z)}{(z-a)^{-m}g(z)} = \frac{-m}{z-a}+\frac{g'(z)}{g(z)}.$$

The second term is holomorphic at a, so the residue at a is $-m$.

Summing over all zeros inside γ, each counted with positive multiplicity, and over all poles inside γ, each counted with negative multiplicity, and then applying the residue theorem (Thm. 1.25):

$$\oint_\gamma \frac{f'}{f}\,dz = 2\pi i(Z-P).$$

Dividing by $2\pi i$ gives (1.35). □

Remark 1.39 (Why "argument" principle). *Along γ, the values $f(z)$ trace a closed curve $f(\gamma)$ in $\mathbb{C}\setminus\{0\}$. With $w=f(z)$, the integral $\oint_\gamma f'/f\,dz$ becomes $\oint_{f(\gamma)} dw/w$. This equals $2\pi i$ times the* winding number *of $f(\gamma)$ around 0, namely the net number of turns the curve makes about the origin. Thus* (1.35) *says that the net change in* $\arg f(z)$ *along γ is $2\pi(Z-P)$.*

Corollary 1.40 (Rouché's theorem). *Let f,g be holomorphic inside and on a simple closed contour γ. If $|g(z)|<|f(z)|$ on γ, then f and $f+g$ have the same number of zeros (counted with multiplicity) inside γ.*

Proof. For $t\in[0,1]$, set $F_t=f+tg$. On γ we have

$$|F_t(z)| \ge |f(z)| - t|g(z)| \ge |f(z)| - |g(z)| > 0,$$

so every F_t is nonvanishing on γ. The argument principle therefore applies:

$$Z(t) := \frac{1}{2\pi i}\oint_\gamma \frac{F_t'(z)}{F_t(z)}\,dz$$

is the number of zeros of F_t inside γ, counted with multiplicity.

The map $(t,z)\mapsto F_t'(z)/F_t(z)$ is continuous on the compact set $[0,1]\times\gamma$, because the denominator never vanishes there. Hence it depends continuously on t, uniformly in $z\in\gamma$, and so $Z(t)$ is a continuous function of t. Since $Z(t)$ is integer-valued, it must be constant on the connected interval $[0,1]$: if it took two different integer values, the intermediate value theorem would force it to take non-integer values between them. In particular,

$$Z(0)=Z(1).$$

That is, f and $f+g$ have the same number of zeros inside γ. □

Example 1.41 (Counting roots of z^3-z-1 inside $|z|<2$). *How many roots does $p(z)=z^3-z-1$ have in the disk $|z|<2$?*

Apply Cor. 1.40 on the circle $\gamma=\{|z|=2\}$ with $f(z)=z^3$ and $g(z)=-z-1$. On γ,

$$|f(z)|=|z|^3=8, \qquad |g(z)|\le |z|+1=3.$$

Since $3<8$, $|g|<|f|$ on γ. By Rouché, $p=f+g$ has the same number of zeros inside γ as $f(z)=z^3$, which has a zero of order 3 at the origin. Therefore p has 3 roots (counted with multiplicity) in $|z|<2$. Since $\deg p = 3$, all *roots of p lie in this disk.*

The argument principle (Thm. 1.38) gives the same count: on $|z|=2$, p has no poles and no zeros (as shown above), and tracking $\arg p(z)$ *around the circle gives a net change of 6π, so $Z=3$.*

Exercises

Problem 1.1. *Verify that* $f(z) = e^z = e^x(\cos y + i\sin y)$ *satisfies the Cauchy–Riemann equations on* $\mathbb{C}$ *and find* $f'(z)$*.*

Problem 1.2. *Let* $u(x, y) = x^3 - 3xy^2$*. Show* u *is harmonic, and find the harmonic conjugate* v *such that* $f = u + iv$ *is entire. Identify* f*.*

Problem 1.3. *Compute* $\oint_{|z|=2} \frac{dz}{z^2-1}$ *two ways: (a) partial fractions + Example 1.8; (b) residue theorem.*

Problem 1.4. *Find the Laurent series of* $f(z) = \dfrac{1}{(z-1)(z-2)}$ *in (a)* $|z| < 1$*, (b)* $1 < |z| < 2$*, (c)* $|z| > 2$*.*

Problem 1.5. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ *by a semicircular contour, paying attention to the order of the pole at* $z = i$*.*

Problem 1.6. *Evaluate* $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$ *for* $a > |b| > 0$ *by substituting* $z = e^{i\theta}$ *and using the residue theorem.*

Problem 1.7. *Prove that if* f *is entire and* $|f(z)| \le C(1 + |z|^n)$ *for some constant* C *and some* $n \in \mathbb{N}$*, then* f *is a polynomial of degree* $\le n$*. (Generalize Liouville's argument using Cor. 1.14.)*

Problem 1.8. *Use Rouché's theorem (Cor. 1.40) to prove that all five roots of* $p(z) = z^5 + 3z^2 + 1$ *lie in* $|z| < 2$*. Then show that exactly two of the five roots lie in* $|z| < 1$ *(hint: on* $|z| = 1$ *compare with the dominant term* $3z^2$*).*

Problem 1.9. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$ *by an indented semicircular contour. (Write* $\sin x = \operatorname{Im} e^{ix}$*; handle the pole at* 0 *with a small upper-half-plane semicircle; see Ex. 1.37 for a worked example.)*

Problem 1.10. *Classify the singularity of* $f(z) = (1 - \cos z)/z^4$ *at* $z = 0$ *(removable, pole of order* m*, or essential?), and compute* $\operatorname{Res}_{z=0} f$*. Hint: substitute the Taylor series for* $\cos z$*.*

Problem 1.11. *(Maximum modulus principle.) Prove that if* f *is holomorphic on a domain* Ω *and* $|f|$ *attains an interior maximum at* $z_0 \in \Omega$*, then* f *is constant on the connected component containing* z_0*. Hint: apply Cauchy's integral formula on a small circle centered at* z_0 *and use that a mean of values* $\le M$ *can equal* M *only if all values equal* M*.*

Problem 1.12. *Evaluate* $\displaystyle\int_0^{\infty} \frac{\log x}{1+x^2}\,dx$ *using the principal branch of* $\log$*. Answer:* 0*. Hint: integrate* $\log z/(1+z^2)$ *on an upper-half-plane contour that follows the negative real axis from just above the branch cut, detours around* 0*, then follows the positive real axis; compare the boundary values* $\log x$ *and* $\log x + i\pi$*.*

Problem 1.13. *Show that* $f(z) = |z|^2 = x^2 + y^2$ *has ordinary partial derivatives everywhere as a two-variable real function, but is complex-differentiable only at* $z = 0$*. Compute* $f'(0)$ *directly from Def. 1.1.*

Problem 1.14. *Let* f *be an entire function with* $\operatorname{Re} f(z) \ge 0$ *for every* $z \in \mathbb{C}$*. Prove* f *is constant. Hint: apply Liouville (Cor. 1.19) to* e^{-f}*, noting* $|e^{-f}| = e^{-\operatorname{Re} f} \le 1$*.*

Problem 1.15. *Verify the Cauchy–Riemann equations for* $f(z) = \log z$ *on the slit plane* $\mathbb{C}\backslash(-\infty, 0]$ *using polar coordinates (*$u = \ln r$*,* $v = \arg z$*), and deduce* $(\log z)' = 1/z$*. Then explain why the function* $(1+i)^i := e^{i\log(1+i)}$ *is well-defined on the principal branch, and compute its value.*

2 Complex Analysis II — Asymptotics and Steepest Descent

Prerequisites. This section builds on Section 1: Cauchy's theorem (Thm. 1.10), the Cauchy integral formula (Thm. 1.13), Taylor series (Cor. 1.16), Laurent expansions (Thm. 1.21), and the residue theorem (Thm. 1.25). Contour deformations use Cauchy's theorem; crossed singularities contribute residues.

Section 1 gave us holomorphic functions, Laurent expansions, and residues. This section uses them in two ways. First, we extend holomorphic formulas beyond their original region of convergence; Definition 2.1 gives the precise name. Second, we estimate integrals with a large parameter. A large-parameter method finds the small part of the domain that contributes most: Laplace's method handles exponential concentration, stationary phase handles oscillatory cancellation, and steepest descent is the complex-contour version.

2.1 Analytic continuation and the identity theorem

Holomorphy is rigid: local data determines global behavior. This is the reason the gamma function, the Riemann zeta function ζ, and hypergeometric functions can be extended far beyond the regions where their first formulas converge.

Definition 2.1 (Analytic continuation). *Let f be holomorphic on a connected open set Ω_0. An* analytic continuation *of f to a larger connected open set Ω is a holomorphic function F on Ω such that $F = f$ on the overlap where the old formula was defined, typically on $\Omega_0 \subset \Omega$ or on a connected open subset of $\Omega_0 \cap \Omega$.*

The phrase "same function by analytic continuation" means this: the old and new formulas agree on a region where both make sense, so the identity theorem below forces them to agree everywhere they can both be compared. Thus analytic continuation is not arbitrary extension; it is extension with no freedom once the overlap is fixed.

Theorem 2.2 (Identity theorem). *Let f, g be holomorphic on a connected open set $\Omega \subset \mathbb{C}$. Suppose there is a point $a \in \Omega$ and a sequence of distinct points $z_n \in \Omega$, with $z_n \to a$, such that*

$$f(z_n) = g(z_n) \qquad \text{for every } n.$$

Equivalently, the agreement set $\{z \in \Omega : f(z) = g(z)\}$ has a limit point *inside Ω. Then $f = g$ on all of Ω.*

Proof. Let $h = f - g$ (holomorphic on Ω), and let $S = \{z \in \Omega : h(z) = 0\}$. Write S' for the set of points of Ω that are limit points of S; by hypothesis $S' \neq \emptyset$. The strategy is to show S' is both open and closed in Ω. In a connected set, the only nonempty subset that is both open and closed is the whole set, so this will force $S' = \Omega$.

Step 1 (S' is open). Fix $a \in S'$. By Cor. 1.16, h has a convergent Taylor expansion on some disk $D(a, r) \subset \Omega$:

$$h(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad c_n = \frac{h^{(n)}(a)}{n!}.$$

We claim every $c_n = 0$. Suppose not, and let k be the smallest index with $c_k \neq 0$. Factor:

$$h(z) = (z-a)^k \big[c_k + c_{k+1}(z-a) + \cdots\big] = (z-a)^k\, \tilde{h}(z),$$

with $\tilde{h}$ holomorphic and $\tilde{h}(a) = c_k \neq 0$. By continuity of $\tilde{h}$, there is a punctured neighborhood of a (a small disk around a with a itself removed) on which $\tilde{h} \neq 0$, hence on which $h \neq 0$. But then zeros of h cannot accumulate at a, contradicting $a \in S'$. So $c_n = 0$ for all n, and $h \equiv 0$ on $D(a,r)$. Every point of $D(a,r)$ is then in S', proving S' is open.

Step 2 (S' is closed in Ω). Suppose $a_n \in S'$ with $a_n \to a \in \Omega$. If $a_n = a$ for infinitely many n, then $a \in S'$ directly, because those repeated terms are themselves points of S'. Otherwise pass to a subsequence, still denoted a_n, with $a_n \neq a$ for all n. Since each a_n is a limit point of S, we can choose

$$z_n \in S, \qquad 0 < |z_n - a_n| < \min\left\{\frac{1}{n}, \frac{|a_n - a|}{2}\right\}.$$

Then $z_n \to a$. Also $z_n \neq a$, because $z_n = a$ would imply

$$|a_n - a| = |a_n - z_n| < \frac{|a_n - a|}{2},$$

which is impossible. Hence points of S distinct from a approach a, so $a \in S'$.

Step 3 (conclusion). S' is nonempty, open, and closed in the connected set Ω, so $S' = \Omega$. Thus $h \equiv 0$ on Ω. □

Remark 2.3 (Why this matters). *A holomorphic function is determined by its values on any set with a limit point: a convergent sequence, an arc, or a small disk. Hence any holomorphic continuation, if it exists, is unique. This is the basis for continuing Γ, the Riemann zeta function ζ, and the other special functions later in the notes.*

Example 2.4 (Continuing the geometric series). *The series $\sum_{n=0}^{\infty} z^n$ converges on $|z| < 1$ to $1/(1-z)$, by the geometric-series identity*

$$(1-z)\sum_{n=0}^{N} z^n = 1 - z^{N+1} \xrightarrow{N\to\infty} 1 \quad (|z| < 1).$$

The function $F(z) = 1/(1-z)$ is holomorphic on $\mathbb{C} \setminus \{1\}$, and it agrees with the series on $|z| < 1$ (a set with plenty of limit points). By Thm. 2.2, any holomorphic function on a connected open set $\Omega \supset \{|z| < 1\}$ (with $1 \notin \Omega$) that agrees with the series on $|z| < 1$ must equal F there. The series itself diverges for $|z| \geq 1$, $z \neq 1$, yet the function F persists.

2.2 Multi-valued functions and branch cuts

Some functions resist single-valued extension. *Single-valued* means that each input has exactly one output; *multi-valued* means that continuation around different paths can produce different outputs. For $\log z$, a holomorphic branch is a holomorphic function L with $e^{L(z)} = z$; necessarily $L'(z) = 1/z$. If two paths from 1 to z wind around the origin different numbers of times, integrating $1/z$ along them gives values that differ by $2\pi i k$, $k \in \mathbb{Z}$, by the residue theorem (Thm. 1.25). Thus "$\log z$" is naturally multi-valued. To make it single-valued, we delete a curve, called a *branch cut*, that prevents winding around the origin.

Definition 2.5 (Principal logarithm). *On $\Omega_0 = \mathbb{C} \setminus (-\infty, 0]$ define the* principal logarithm

$$\log z = \ln|z| + i \arg z, \qquad \arg z \in (-\pi, \pi).$$

This is holomorphic on Ω_0 with $(\log z)' = 1/z$. The ray $(-\infty, 0]$ is the principal branch cut. *Across the cut, $\log z$ jumps by $2\pi i$ (the argument jumps from $+\pi$ just above the cut to $-\pi$ just below).*

For $\alpha \in \mathbb{C}$, define $z^\alpha = e^{\alpha \log z}$, principal branch. This is the convention throughout these notes unless explicitly altered (cf. Ex. 1.33, which used $\arg z \in (0, 2\pi)$ to keep the positive real axis as the cut).

Example 2.6 ($\sqrt{z}$)**.** *Principal square root:* $\sqrt{z} = z^{1/2} = e^{\frac{1}{2}\log z}$ *on* Ω_0. *For* $z = r\,e^{i\theta}$ *with* $r > 0$ *and* $\theta \in (-\pi, \pi)$,

$$\sqrt{z} = e^{\frac{1}{2}(\ln r + i\theta)} = \sqrt{r}\,e^{i\theta/2}.$$

Approaching the cut from above ($\theta \to \pi$) *gives* $\sqrt{z} \to i\sqrt{r}$*; from below* ($\theta \to -\pi$) *gives* $-i\sqrt{r}$. *The two limiting values differ by a factor of* -1. *A different cut, e.g.* $[0, \infty)$ *with* $\arg z \in (0, 2\pi)$, *defines a different branch of the same multi-valued function.*

Remark 2.7 (Multi-sheet picture, briefly)**.** *One way to picture all values of* $\log z$ *at once is to stack many copies, or* sheets, *of the punctured plane* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. *One loop around the origin moves to the next sheet and adds* $2\pi i$. *We will not use that multi-sheet theory here. For contour integrals, the practical rule is enough: choose a branch cut away from the contour and track the branch consistently.*

2.3 Asymptotic series

Before attacking integrals, we need a precise meaning for "approximation valid at infinity". For ratios of factorials and parameter-dependent integrals, convergent power series are often the wrong tool. An *asymptotic* series may diverge, but each fixed truncation still has a controlled error at large argument.

Definition 2.8 (Asymptotic expansion)**.** *Fix an open* sector $S \subset \mathbb{C}$ *(a set of the form* $\{z : \alpha < \arg z < \beta,\ |z| > R\}$ *for some angles* $\alpha < \beta$ *and radius* $R \geq 0$*; e.g.* $|\arg z| < \pi/2$ *is the right half-plane sector). A formal series* $\sum_{n=0}^{\infty} a_n z^{-n}$ *is an* asymptotic expansion *of* $f(z)$ *as* $z \to \infty$ *in* S*, written*

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \quad (z \to \infty,\ z \in S),$$

if for every $N \geq 0$,

$$f(z) - \sum_{n=0}^{N} \frac{a_n}{z^n} = O\!\left(\frac{1}{z^{N+1}}\right) \quad (z \to \infty,\ z \in S), \tag{2.1}$$

where $g(z) = O(h(z))$ *as* $z \to \infty$ *in* S *means* $|g(z)| \leq C|h(z)|$ *for all sufficiently large* $|z|$ *in* S*, with some constant* C *independent of* z.

The series need *not* converge. The useful guarantee is only for fixed N: after truncating, the error is one order smaller than the last retained power. In practice one stops near the smallest term.

Example 2.9 (Stieltjes function, with explicit remainder)**.** *Define*

$$f(z) = \int_0^\infty \frac{e^{-t}}{1 + t/z}\,dt, \qquad \operatorname{Re} z > 0. \tag{2.2}$$

The finite geometric-series identity

$$\frac{1}{1+x} = \sum_{n=0}^{N} (-1)^n x^n + (-1)^{N+1} \frac{x^{N+1}}{1+x}$$

(check: multiply both sides by $1+x$) with $x = t/z$ gives

$$\frac{1}{1+t/z} = \sum_{n=0}^{N} (-1)^n \frac{t^n}{z^n} + (-1)^{N+1} \frac{t^{N+1}/z^{N+1}}{1+t/z}.$$

Multiply by e^{-t} and integrate t from 0 to ∞. The first piece uses

$$\int_0^\infty t^n e^{-t}\, dt = n!,$$

which follows for integer $n \ge 0$ by repeated integration by parts; Section 3 later packages this as the Euler integral $\Gamma(n+1) = n!$.

$$f(z) = \sum_{n=0}^{N} \frac{(-1)^n n!}{z^n} + R_N(z), \qquad R_N(z) = \frac{(-1)^{N+1}}{z^{N+1}} \int_0^\infty \frac{t^{N+1} e^{-t}}{1+t/z}\, dt.$$

Remainder estimate. *Fix $\varepsilon \in (0, \pi/2)$ and restrict to the closed subsector $|\arg z| \le \pi/2 - \varepsilon$. For $t \ge 0$,*

$$\left|1 + \frac{t}{z}\right| = \frac{|z+t|}{|z|} \ge \frac{\mathrm{Re}(z+t)}{|z|} \ge \frac{\mathrm{Re}\, z}{|z|} \ge \sin\varepsilon,$$

where the first $\ge$ uses $|w| \ge \mathrm{Re}\, w$ for any complex w, the second uses $t \ge 0$ (so $\mathrm{Re}(z+t) \ge \mathrm{Re}\, z$), and the third uses $\mathrm{Re}\, z/|z| = \cos(\arg z) \ge \cos(\pi/2 - \varepsilon) = \sin\varepsilon$. Therefore

$$|R_N(z)| \;\le\; \frac{1}{(\sin\varepsilon)\, |z|^{N+1}} \int_0^\infty t^{N+1} e^{-t}\, dt = \frac{(N+1)!}{(\sin\varepsilon)\, |z|^{N+1}}.$$

Thus $f(z) \sim \sum_{n\ge0} (-1)^n n!/z^n$ as $z \to \infty$ in every closed subsector of $\mathrm{Re}\, z > 0$, exactly as in Def. 2.8. On the positive real axis, $|1 + t/z| \ge 1$, so $|R_N(z)| \le (N+1)!/z^{N+1}$. There the bound is minimized near $N \approx z$, and Stirling's formula turns that minimum into an error of order $\sqrt{2\pi z}\, e^{-z}$ (exponentially small, up to an algebraic prefactor): a concrete illustration of "truncate at the smallest term".

2.4 Laplace's method

Many special-function integrals have the form

$$I(\lambda) = \int_a^b e^{\lambda\phi(t)}\, \psi(t)\, dt, \qquad \lambda \to +\infty, \tag{2.3}$$

with ϕ real-valued on $[a, b]$. The exponential $e^{\lambda\phi}$ concentrates near the maximum of ϕ; away from that point it is exponentially smaller. Thus only a small neighborhood of the maximum matters, and there ϕ is well approximated by a quadratic. The remaining integral is Gaussian.

Before stating the theorem we record the Gaussian integral we will use repeatedly.

Lemma 2.10 (Gaussian integral). *For $\alpha > 0$,*

$$\int_{-\infty}^{\infty} e^{-\alpha u^2/2}\, du = \sqrt{\frac{2\pi}{\alpha}}. \tag{2.4}$$

Proof. Denote the integral by I. Then

$$I^2 = \int_{-\infty}^{\infty} e^{-\alpha u^2/2} du \int_{-\infty}^{\infty} e^{-\alpha v^2/2} dv = \iint_{\mathbb{R}^2} e^{-\alpha(u^2+v^2)/2}\, du\, dv.$$

Switch to polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, with Jacobian $du\,dv = r\,dr\,d\theta$:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\alpha r^2/2}\, r\,dr\,d\theta = 2\pi\int_0^{\infty} r\,e^{-\alpha r^2/2}\,dr.$$

The remaining integral evaluates via $w = \alpha r^2/2$, $dw = \alpha r\,dr$:

$$\int_0^{\infty} r\,e^{-\alpha r^2/2}\,dr = \frac{1}{\alpha}\int_0^{\infty} e^{-w}\,dw = \frac{1}{\alpha}.$$

Hence $I^2 = 2\pi/\alpha$, i.e. $I = \sqrt{2\pi/\alpha}$. □

Theorem 2.11 (Laplace). *Let $\phi \in C^3[a,b]$ (thrice continuously differentiable) be real-valued, $\psi \in C[a,b]$ (continuous), and suppose ϕ attains its maximum on $[a,b]$ at a unique interior point $t_0 \in (a,b)$, with $\phi''(t_0) < 0$ and $\psi(t_0) \neq 0$. Then, as $\lambda \to +\infty$,*

$$I(\lambda) \sim \psi(t_0)\,e^{\lambda\phi(t_0)}\sqrt{\frac{2\pi}{-\lambda\,\phi''(t_0)}}. \tag{2.5}$$

Proof. Abbreviate $\phi_0 = \phi(t_0)$, $\phi_2 = \phi''(t_0) < 0$. The strategy: (i) localize to a shrinking neighborhood of t_0, (ii) Taylor-expand to quadratic order with controlled cubic error, (iii) change variable to standardize the Gaussian, (iv) extend the limits to $\pm\infty$.

Step 1 (localization). Choose $\delta(\lambda) = \lambda^{-2/5}$ (any power between $-1/2$ and $-1/3$ works; the choice is not unique). We split

$$I(\lambda) = \underbrace{\int_{|t-t_0|<\delta} e^{\lambda\phi(t)}\psi(t)\,dt}_{I_1} + \underbrace{\int_{|t-t_0|\ge\delta,\,t\in[a,b]} e^{\lambda\phi(t)}\psi(t)\,dt}_{I_2}.$$

Outside the shrinking window, $\phi(t) \le \phi_0 - c\delta^2$ for some $c > 0$ (because t_0 is a strict interior maximum with $\phi''(t_0) < 0$, and $\phi \in C^3$): indeed the quadratic Taylor bound gives $\phi(t) \le \phi_0 - \frac{1}{4}|\phi_2|(t-t_0)^2$ on a fixed neighborhood, and ϕ stays bounded away from ϕ_0 on the complement by continuity. Hence

$$|I_2| \le \|\psi\|_\infty\,(b-a)\,e^{\lambda\phi_0}\,e^{-c\lambda\delta^2} = e^{\lambda\phi_0}\cdot O\big(e^{-c\lambda^{1/5}}\big),$$

using $\lambda\delta^2 = \lambda^{1/5} \to \infty$. So I_2 is exponentially smaller than the target (2.5) (which is polynomial in λ^{-1}) and may be discarded.

Step 2 (Taylor expansion on the window). On $|t-t_0| < \delta$, Taylor's theorem with remainder gives

$$\phi(t) = \phi_0 + \tfrac{1}{2}\phi_2(t-t_0)^2 + r(t), \qquad |r(t)| \le \tfrac{1}{6}\|\phi'''\|_\infty|t-t_0|^3. \tag{2.6}$$

For $|t-t_0| < \delta = \lambda^{-2/5}$, $\lambda|r(t)| \le C\lambda\delta^3 = C\lambda^{-1/5} \to 0$. Thus $e^{\lambda r(t)} = 1 + O(\lambda^{-1/5})$ uniformly on the window. Since ψ is continuous on the compact interval $[a,b]$, its modulus of continuity

$$\omega_\psi(\rho) := \sup\{|\psi(t)-\psi(s)| : |t-s| \le \rho,\ t,s \in [a,b]\}$$

measures the largest possible change in ψ between two points no farther than ρ apart. Here sup means the supremum, or least upper bound. Continuity on the compact interval implies $\omega_\psi(\rho) \to 0$ as $\rho \to 0$. Hence $\psi(t) = \psi(t_0) + O(\omega_\psi(\delta))$ uniformly on $|t-t_0| < \delta$. Therefore

$$I_1 = e^{\lambda\phi_0}\big[\psi(t_0) + O(\omega_\psi(\delta))\big]\big[1 + O(\lambda^{-1/5})\big]\int_{|t-t_0|<\delta} e^{\frac{\lambda\phi_2}{2}(t-t_0)^2}\,dt.$$

Step 3 (rescale and extend). Substitute $u = (t-t_0)\sqrt{-\lambda\phi_2}$ (recall $\phi_2 < 0$, so the square root is real positive), $du = \sqrt{-\lambda\phi_2}\,dt$. Then $\frac{\lambda\phi_2}{2}(t-t_0)^2 = -u^2/2$, and the window $|t-t_0| < \delta$ becomes $|u| < \delta\sqrt{-\lambda\phi_2} = \sqrt{-\phi_2}\,\lambda^{1/10} \to \infty$. The Gaussian tail outside this window is $O(e^{-c\lambda^{1/5}})$, again negligible. Hence

$$\int_{|t-t_0|<\delta} e^{\frac{\lambda\phi_2}{2}(t-t_0)^2}\,dt = \frac{1}{\sqrt{-\lambda\phi_2}} \int_{-\infty}^{\infty} e^{-u^2/2} du + O(e^{-c\lambda^{1/5}}).$$

By Lem. 2.10 with $\alpha = 1$, the u-integral equals $\sqrt{2\pi}$.

Step 4 (assemble). Collecting,

$$I(\lambda) = I_1 + I_2 = \psi(t_0)\,e^{\lambda\phi_0}\sqrt{\frac{2\pi}{-\lambda\phi_2}}\big[1+o(1)\big],$$

which is (2.5). If ψ has additional smoothness, the $o(1)$ may be sharpened to a power of λ^{-1}; carrying the Taylor expansion further produces the full asymptotic series $I(\lambda) \sim e^{\lambda\phi_0}\sum_{n\geq 0} c_n\lambda^{-n-1/2}$. □

Remark 2.12 (Role of the shrinking window). *The choice $\delta = \lambda^{-2/5}$ balances two requirements: $\lambda\delta^3 \to 0$ controls the cubic Taylor error, while $\sqrt{\lambda}\delta \to \infty$ lets the Gaussian window expand to the real line.*

Example 2.13 (Stirling, real $x \to \infty$). *For this example use Euler's factorial integral*

$$\Gamma(x+1) := \int_0^\infty t^x e^{-t}\,dt, \qquad x > 0.$$

Section 3 studies this integral systematically as the Gamma function; here we only need the integral representation. Substitute $t = xs$, $dt = x\,ds$:

$$\Gamma(x+1) = \int_0^\infty (xs)^x e^{-xs}\,x\,ds = x^{x+1}\int_0^\infty e^{x(\ln s - s)}\,ds.$$

Set $\phi(s) = \ln s - s$, $\psi \equiv 1$, $\lambda = x$. Then $\phi'(s) = 1/s - 1 = 0$ at $s_0 = 1$; $\phi(1) = -1$; $\phi''(s) = -1/s^2$, so $\phi''(1) = -1$. Laplace (Thm. 2.11) gives

$$\int_0^\infty e^{x(\ln s - s)}\,ds \sim e^{-x}\sqrt{\frac{2\pi}{x}},$$

and therefore

$$\Gamma(x+1) \sim x^{x+1}\,e^{-x}\sqrt{\frac{2\pi}{x}} = \sqrt{2\pi x}\,x^x\,e^{-x}.$$

Why it is legitimate to use the finite-interval theorem on $(0,\infty)$. *The theorem was stated on a closed interval $[a,b]$. Here the same localization proof applies after splitting $(0,\infty)$ into a fixed small neighborhood of $s = 1$ and its complement. On the complement, there is an $\eta > 0$ such that $\ln s - s \leq -1-\eta$, and as $s \to 0^+$ or $s \to \infty$ the quantity $\ln s - s$ tends to $-\infty$. Those two facts make the tails exponentially smaller than the Gaussian contribution from $s = 1$.*

2.5 Stationary phase

When the exponent is *purely imaginary*, the magnitude does not concentrate. Instead, fast oscillation causes cancellation. The main contributions come from *stationary* points of the phase, where the derivative vanishes and nearby oscillations stay coherent long enough to matter.

Theorem 2.14 (Stationary phase). *Let $\phi \in C^3[a,b]$ be real, $\psi \in C^1[a,b]$ with compact support in (a,b) (so ψ is zero outside some closed subinterval of (a,b)), and suppose ϕ has a unique interior stationary point $t_0 \in (a,b)$ with $\phi'(t_0)=0$, $\phi''(t_0)\neq 0$, and $\psi(t_0)\neq 0$. Then*

$$\int_a^b e^{i\lambda\phi(t)}\psi(t)\,dt \;\sim\; \psi(t_0)\,e^{i\lambda\phi(t_0)}\sqrt{\frac{2\pi}{\lambda\,|\phi''(t_0)|}}\,e^{i\pi\sigma/4}, \quad \lambda\to+\infty, \tag{2.7}$$

where $\sigma = \operatorname{sgn}\phi''(t_0)\in\{+1,-1\}$ is the sign function ($\operatorname{sgn}(x)=+1$ if $x>0$, -1 if $x<0$).

If $\psi(t_0)=0$, the displayed leading coefficient vanishes and the symbol $\sim$ is no longer correct as written. In that degenerate case one expands ψ further; the first nonzero derivative of ψ at t_0 determines the new leading power of λ.

Proof. We mirror the four-step structure of Thm. 2.11.

Step 1 (non-stationary-phase decay away from t_0). Let $\chi_0 \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function, where $C_c^\infty(\mathbb{R})$ denotes the space of infinitely differentiable functions with compact support. Choose χ_0 so that $\chi_0\equiv 1$ on $[-1/2,1/2]$ and $\chi_0\equiv 0$ outside $[-1,1]$, with $0\le\chi_0\le 1$ everywhere. Define the rescaled cutoff $\chi_\delta(t) := \chi_0((t-t_0)/\delta)$ with $\delta=\delta(\lambda)=\lambda^{-2/5}$; this localizes the integral to a window of width 2δ around t_0 that shrinks as $\lambda\to\infty$. Split

$$I(\lambda) := \int_a^b e^{i\lambda\phi(t)}\psi(t)\,dt = \int e^{i\lambda\phi}\psi\chi_\delta\,dt + \int e^{i\lambda\phi}\psi(1-\chi_\delta)\,dt.$$

No explicit formula for χ_0 is needed: one may start with a piecewise polynomial plateau and smooth the two corners. The only properties used below are compact support, bounded derivatives, and the two values 1 near 0 and 0 away from 0. On $\operatorname{supp}(1-\chi_\delta)$, $|t-t_0|\ge\delta/2$. Since $\phi'(t_0)=0$ and $\phi''(t_0)\neq 0$, the mean value theorem gives $|\phi'(t)|\ge c\delta$ on the annulus $\delta/2\le|t-t_0|\le\eta$ for some fixed $\eta>0$; on the rest of $\operatorname{supp}\psi$, continuity and the uniqueness of the stationary point give $|\phi'(t)|\ge c_0>0$. Hence $|\phi'(t)|\ge c\delta$ everywhere on $\operatorname{supp}(1-\chi_\delta)$. Where $\phi'\neq 0$, the chain rule gives $\frac{d}{dt}e^{i\lambda\phi(t)} = i\lambda\phi'(t)\,e^{i\lambda\phi(t)}$, so $e^{i\lambda\phi} = \frac{1}{i\lambda\phi'(t)}\frac{d}{dt}e^{i\lambda\phi}$ — the standard "oscillatory-integral trick" that exchanges one factor of the large parameter λ for one derivative of the slowly varying part. Integrate by parts once with $u=\psi(1-\chi_\delta)/\phi'$ and $dv=\frac{d}{dt}e^{i\lambda\phi}\,dt/(i\lambda)$, using compact support of $\psi(1-\chi_\delta)$ so that boundary terms vanish:

$$\int e^{i\lambda\phi}\psi(1-\chi_\delta)\,dt = -\frac{1}{i\lambda}\int e^{i\lambda\phi}\,a_\lambda'(t)\,dt, \qquad a_\lambda(t) := \frac{\psi(t)(1-\chi_\delta(t))}{\phi'(t)}.$$

Because $\chi_\delta' = O(\delta^{-1})$, $|\phi'|\ge c\delta$, and ϕ'',ψ,ψ' are bounded on $[a,b]$, we have $\int|a_\lambda'(t)|dt = O(\delta^{-1})$. Therefore

$$\int e^{i\lambda\phi}\psi(1-\chi_\delta)\,dt = O\!\left(\frac{1}{\lambda\delta}\right) = O(\lambda^{-3/5}) = o(\lambda^{-1/2}),$$

which is already smaller than the leading stationary-phase term. The same estimate also controls the transition annulus $\delta/2\le|t-t_0|\le\delta$, where the smooth cutoff changes from 1 to

0. Thus replacing the cutoff-localized integral by the sharp local window $|t-t_0|<\delta$ changes only the lower-order error.

Step 2 (Taylor expansion on the window). On $|t-t_0|<\delta$, Taylor:

$$\phi(t)=\phi_0+\tfrac{1}{2}\phi_2(t-t_0)^2+\tfrac{1}{6}\phi'''(\xi(t))(t-t_0)^3, \qquad \phi_0=\phi(t_0),\ \phi_2=\phi''(t_0).$$

Since $\delta=\lambda^{-2/5}$, $\lambda\delta^3=\lambda^{-1/5}\to 0$; hence

$$e^{i\lambda\phi(t)}=e^{i\lambda\phi_0}\,e^{i\lambda\phi_2(t-t_0)^2/2}\,[1+O(\lambda^{-1/5})]$$

uniformly on $|t-t_0|<\delta$. Also $\psi(t)=\psi(t_0)+O(\delta)$ because $\psi\in C^1[a,b]$. Substitute $u=(t-t_0)\sqrt{\lambda|\phi_2|}$, $du=\sqrt{\lambda|\phi_2|}\,dt$, and set $L:=\delta\sqrt{\lambda|\phi_2|}=\sqrt{|\phi_2|}\,\lambda^{1/10}$. Then

$$\int_{|t-t_0|<\delta}e^{i\lambda\phi}\psi\,dt=\frac{\psi(t_0)\,e^{i\lambda\phi_0}}{\sqrt{\lambda|\phi_2|}}\int_{|u|<L}e^{i\sigma u^2/2}\,du\left[1+O(\lambda^{-1/5})\right],$$

where $\sigma=\operatorname{sgn}\phi_2$.

Step 3 (remove the cutoff in u). The Fresnel integral is only conditionally convergent, meaning it converges because of oscillatory cancellation rather than absolute integrability. Therefore the tail estimate should be understood as an improper oscillatory estimate, i.e. a limit of finite-interval integrals. For $N>M\geq L$, integration by parts on the finite interval $[M,N]$ gives

$$\int_M^N e^{i\sigma u^2/2}\,du=\left[\frac{e^{i\sigma u^2/2}}{i\sigma u}\right]_{u=M}^{u=N}+\int_M^N\frac{e^{i\sigma u^2/2}}{i\sigma u^2}\,du.$$

The right side is $O(M^{-1})$, uniformly in N. Thus the tails are Cauchy, the improper tail from L to infinity exists, and its size is $O(L^{-1})$. The same bound holds on $(-\infty,-L]$. Hence

$$\int_{|u|<L}e^{i\sigma u^2/2}\,du=\int_{-\infty}^{\infty}e^{i\sigma u^2/2}\,du+O(L^{-1})=\int_{-\infty}^{\infty}e^{i\sigma u^2/2}\,du+O(\lambda^{-1/10}).$$

Step 4 (Fresnel integral). The identity we need is

$$\int_{-\infty}^{\infty}e^{i\sigma u^2/2}\,du=\sqrt{2\pi}\,e^{i\sigma\pi/4}, \qquad \sigma\in\{+1,-1\}. \tag{2.8}$$

We prove this for $\sigma=+1$; the $\sigma=-1$ case follows by conjugation.

Proof of Fresnel. Consider the contour integral $\oint_{\Gamma_R}e^{iz^2/2}\,dz=0$, where Γ_R is a pie-slice contour: (i) real segment $z=u\in[0,R]$, (ii) arc $z=Re^{i\theta}$, $\theta\in[0,\pi/4]$, (iii) return along the ray $z=re^{i\pi/4}$, r from R to 0. The integrand is entire, so the integral vanishes by Cauchy's theorem (Thm. 1.10).

Arc vanishes. On the arc, $z^2=R^2e^{2i\theta}$, so $iz^2/2=(iR^2/2)(\cos 2\theta+i\sin 2\theta)$, $\operatorname{Re}(iz^2/2)=-(R^2/2)\sin 2\theta$. For $\theta\in[0,\pi/4]$, the angle 2θ lies in $[0,\pi/2]$. Concavity of sin on this interval puts the graph above the chord from $(0,0)$ to $(\pi/2,1)$, hence

$$\sin(2\theta)\geq\frac{2}{\pi}(2\theta)=\frac{4\theta}{\pi}.$$

Therefore $|e^{iz^2/2}|\leq e^{-(2R^2/\pi)\theta}$. Jordan-type estimate:

$$\left|\int_0^{\pi/4}e^{iz^2/2}\cdot iRe^{i\theta}\,d\theta\right|\leq R\int_0^{\pi/4}e^{-(2R^2/\pi)\theta}\,d\theta=\frac{\pi}{2R}(1-e^{-R^2/2})\to 0,\ R\to\infty.$$

Ray contribution. On $z = re^{i\pi/4}$, $z^2 = r^2 e^{i\pi/2} = ir^2$, so $iz^2/2 = -r^2/2$ (real negative!). $dz = e^{i\pi/4} dr$. Hence

$$-\int_0^\infty e^{-r^2/2} \cdot e^{i\pi/4}\, dr = -e^{i\pi/4}\sqrt{\pi/2},$$

where the sign is $-$ because the ray is traversed from R back to 0, i.e. in the *opposite* sense of increasing r; and $\int_0^\infty e^{-r^2/2} dr = \sqrt{\pi/2}$ by Lem. 2.10.

Summing the three pieces of Γ_R and sending $R \to \infty$:

$$\int_0^\infty e^{iu^2/2}\, du + 0 - e^{i\pi/4}\sqrt{\pi/2} = 0,$$

so $\int_0^\infty e^{iu^2/2} du = e^{i\pi/4}\sqrt{\pi/2}$. Doubling by symmetry ($u \to -u$ leaves the integrand fixed): $\int_{-\infty}^\infty e^{iu^2/2} du = \sqrt{2\pi}\, e^{i\pi/4}$, which is (2.8) for $\sigma = +1$. For $\sigma = -1$, rotate in the opposite sector $\theta \in [-\pi/4, 0]$ to get $e^{-i\pi/4}\sqrt{2\pi}$, or take the complex conjugate of the $\sigma = +1$ case.

Combining Steps 1–3 with (2.8) yields (2.7). □

Remark 2.15 (Sign interpretation). *The factor $e^{i\sigma\pi/4}$ is a phase: a minimum gives $e^{+i\pi/4}$, and a maximum gives $e^{-i\pi/4}$.*

2.6 Steepest descent

Laplace's method handles real exponents; stationary phase handles imaginary exponents. *Steepest descent* combines both ideas. For a complex exponent $\lambda f(z)$, deform the contour so the phase of λf is constant and $\operatorname{Re} f$ decreases as fast as possible away from the relevant saddle. On that contour the integral behaves like a Laplace integral.

2.6.1 Setup and saddle geometry

Consider

$$I(\lambda) = \int_\gamma e^{\lambda f(z)} g(z)\, dz, \tag{2.9}$$

with f, g holomorphic on a region containing γ and $\lambda \to +\infty$ real. Write $f(z) = u(x, y) + iv(x, y)$. The integrand has magnitude $|g|\, e^{\lambda u}$ and phase $\lambda v + \arg g$.

A *saddle point* of f is a point z_0 with $f'(z_0) = 0$. It is *simple* if $f''(z_0) \neq 0$, so the quadratic term is the first nonzero term in the Taylor expansion. The word "saddle" is literal for the surface $u = \operatorname{Re} f$: near a simple saddle, some directions make u increase and other directions make u decrease. For example, if $f(z) = z^2$ and $z = x + iy$, then $u = x^2 - y^2$, which rises in the x-direction and falls in the y-direction.

By Cauchy's theorem (Thm. 1.10), we are free to deform γ within the domain of holomorphy (keeping endpoints fixed, or the contour closed). We seek a deformation γ^* on which

1. v is *constant* along γ^* (the integrand does not oscillate), and
2. u decreases as fast as possible away from its maximum along γ^*.

Proposition 2.16 (Steepest descent paths). *Let z_0 be a simple saddle of f: $f'(z_0) = 0$ and $f''(z_0) \neq 0$. Then through z_0 there pass two curves of constant $v = \operatorname{Im} f$, meeting at right angles. On one, $u = \operatorname{Re} f$ attains a local* maximum *at z_0 (the* steepest-descent *path γ_{sd}); on the other, a local* minimum *(steepest-ascent).*

Proof. Since f is holomorphic, u and v are harmonic conjugates (Cor. 1.4); writing out Cauchy–Riemann (Thm. 1.3), $\nabla u = (u_x, u_y) = (v_y, -v_x)$, so $\nabla u \cdot \nabla v = v_y v_x + (-v_x)v_y = 0$ and $|\nabla u|^2 = v_y^2 + v_x^2 = |\nabla v|^2$ everywhere $f' \neq 0$. A curve of constant v has tangent perpendicular to ∇v (level curves are perpendicular to the gradient). Combined with $\nabla u \perp \nabla v$, the tangent is parallel to ∇u. Along the curve, u therefore changes at the fastest possible rate — this is what "steepest" means.

Near z_0 expand

$$f(z) = f(z_0) + \tfrac{1}{2} f''(z_0)(z - z_0)^2 + O\big((z - z_0)^3\big). \tag{2.10}$$

Let $\beta = \arg f''(z_0)$, so $f''(z_0) = |f''(z_0)|\, e^{i\beta}$. Set $z - z_0 = r\, e^{i\theta}$. Then

$$f(z) - f(z_0) = \tfrac{1}{2}\, |f''(z_0)|\, r^2\, e^{i(\beta + 2\theta)} + O(r^3). \tag{2.11}$$

$\mathrm{Im}(f - f(z_0))$ vanishes to leading order when $\sin(\beta + 2\theta) = 0$, i.e. when

$$\beta + 2\theta \equiv 0 \quad \text{or} \quad \pi \quad (\mathrm{mod}\ 2\pi).$$

These are the two constant-v directions, separated by the angle $\pi/2$. Along them,

$$\mathrm{Re}(f - f(z_0)) = \pm \tfrac{1}{2} |f''(z_0)| r^2,$$

with the plus sign for $\beta + 2\theta \equiv 0$ (ascent) and the minus sign for $\beta + 2\theta \equiv \pi$ (descent). □

Define the steepest-descent angle θ_0 by

$$2\theta_0 + \beta \equiv \pi \quad (\mathrm{mod}\ 2\pi), \qquad \text{i.e.} \qquad \theta_0 = \frac{\pi - \beta}{2} \quad (\mathrm{mod}\ \pi). \tag{2.12}$$

(Two choices θ_0 and $\theta_0 + \pi$ give opposite rays along the same line; the contour orientation selects one.)

The 2D saddle picture. Near a simple saddle z_0, (2.11) gives four distinguished directions:

- Two rays where $\mathrm{Im}(f - f(z_0)) = 0$ and $\mathrm{Re}(f - f(z_0)) < 0$ (steepest descent): angles θ_0 and $\theta_0 + \pi$.
- Two rays where $\mathrm{Im}(f - f(z_0)) = 0$ and $\mathrm{Re}(f - f(z_0)) > 0$ (steepest ascent): angles $\theta_0 + \pi/2$ and $\theta_0 + 3\pi/2$.
- These four rays divide the neighborhood into four sectors. Two opposite sectors (centered on the descent rays) have $u < u(z_0)$ and are the local *valleys*. The other two (centered on the ascent rays) have $u > u(z_0)$ and are the local *hills*.

The relevant saddle is determined globally: the contour must be deformable through that saddle without crossing singularities, and its endpoints must lie in descent valleys connected to it. If two saddles merge, the quadratic model is replaced by a cubic one; this is where the Airy function enters.

2.6.2 The leading-order formula

Theorem 2.17 (Steepest descent, leading order). *Suppose γ may be deformed (without crossing singularities of f or g) to a contour $\gamma_{\rm sd}$ that passes through a single simple saddle z_0, is tangent there to the steepest-descent direction θ_0 of (2.12), and satisfies* $\mathrm{Re}(f(z)-f(z_0)) \le -c < 0$ *on $\gamma_{\rm sd}$ outside some neighborhood of z_0. Then as $\lambda \to +\infty$,*

$$I(\lambda) \sim g(z_0)\,e^{\lambda f(z_0)}\sqrt{\frac{2\pi}{-\lambda f''(z_0)}}, \tag{2.13}$$

where the square root is defined as $\sqrt{2\pi/(-\lambda f''(z_0))} = e^{i\theta_0}\sqrt{2\pi/(\lambda|f''(z_0)|)}$. Equivalently,

$$I(\lambda) \sim g(z_0)\,e^{\lambda f(z_0)}\,e^{i\theta_0}\sqrt{\frac{2\pi}{\lambda\,|f''(z_0)|}}. \tag{2.14}$$

Proof. Choose a smooth parameter s on $\gamma_{\rm sd}$ with $s=0$ at z_0 and positive tangent $e^{i\theta_0}$ there. Then

$$z(s) = z_0 + s\,e^{i\theta_0} + O(s^2), \qquad dz = e^{i\theta_0}\big[1+O(s)\big]\,ds.$$

Insert this into the local expansion (2.11). By the defining relation (2.12), $\beta + 2\theta_0 \equiv \pi \pmod{2\pi}$, so

$$f(z) - f(z_0) = -\tfrac{1}{2}|f''(z_0)|\,s^2 + O(s^3). \tag{2.15}$$

The exponent is now real and negative along $\gamma_{\rm sd}$ — this is the whole point of the construction. Choose $\delta = \lambda^{-2/5}$ and split the contour into the local window $|s| < \delta$ and its complement. On the window, (2.15) gives

$$e^{\lambda[f(z(s))-f(z_0)]}\,g(z(s))\,dz = e^{-\frac{\lambda}{2}|f''(z_0)|s^2}\big[g(z_0)+O(s)\big]\big[1+O(\lambda s^3)\big]e^{i\theta_0}\,ds,$$

where the $O(\lambda s^3)$ term comes from exponentiating the cubic remainder in (2.15). This error is uniform on $|s|<\delta$ because $\lambda\delta^3 = \lambda^{-1/5} \to 0$. Therefore

$$I(\lambda) = e^{\lambda f(z_0)}\,e^{i\theta_0}\int_{|s|<\delta} e^{-\frac{\lambda}{2}|f''(z_0)|s^2}\big[g(z_0)+O(s)\big]\big[1+O(\lambda s^3)\big]\,ds + e^{\lambda f(z_0)}\,O(e^{-c\lambda\delta^2}).$$

The second term is exponentially small because $\mathrm{Re}(f(z)-f(z_0)) \le -c < 0$ off the local neighborhood, and on the local steepest-descent arc one also has $\mathrm{Re}(f(z)-f(z_0)) \le -c'|s|^2 \le -c'\delta^2$ once $|s| = \delta$. Extending the Gaussian integral from $|s|<\delta$ to $\mathbb{R}$ therefore costs only another $O(e^{-c\lambda^{1/5}})$ term, exactly as in the Laplace proof. Applying Lem. 2.10 with $\alpha = \lambda|f''(z_0)|$ gives $\sqrt{2\pi/(\lambda|f''(z_0)|)}$, so

$$I(\lambda) \sim g(z_0)\,e^{\lambda f(z_0)}\,e^{i\theta_0}\sqrt{\frac{2\pi}{\lambda|f''(z_0)|}},$$

which is (2.14).

Branch of the square root. The compact formula (2.13) uses the branch

$$\sqrt{-f''(z_0)} \;:=\; \sqrt{|f''(z_0)|}\,e^{-i\theta_0}, \tag{2.16}$$

so

$$\frac{1}{\sqrt{-f''(z_0)}} = \frac{e^{i\theta_0}}{\sqrt{|f''(z_0)|}}.$$

This is a chosen branch dictated by the steepest-descent direction θ_0; in general it is not the principal square root (which uses $\arg w \in (-\pi,\pi]$ on $w = -f''(z_0)$ and can disagree by a sign). The polar form (2.14) is usually the safer one to compute from. □

Remark 2.18 (What the formula says). *The leading term is the saddle value $e^{\lambda f(z_0)}$, times the Gaussian width from $f''(z_0)$, times the orientation phase $e^{i\theta_0}$. If $\theta_0 = 0$, the formula reduces to ordinary Laplace behavior.*

2.6.3 Recipe

1. **Locate saddles.** Solve $f'(z) = 0$.
2. **Connectivity.** Sketch level curves of $u = \operatorname{Re} f$ near each saddle; identify valleys ($u \to -\infty$) and hills ($u \to +\infty$). Determine which saddles are relevant: only saddles connected by constant-v descent paths to the endpoints of γ contribute.
3. **Deform.** Move γ onto the union of steepest-descent paths through the dominant saddles, by Cauchy's theorem (Thm. 1.10). Legality: no singularities of f or g crossed (otherwise pick up residues via Thm. 1.25).
4. **Local Gaussian.** Apply (2.13) at each contributing saddle; sum contributions.
5. **Subleading.** Further terms come from higher Taylor coefficients of f and g at z_0, yielding an asymptotic series in $1/\lambda$.

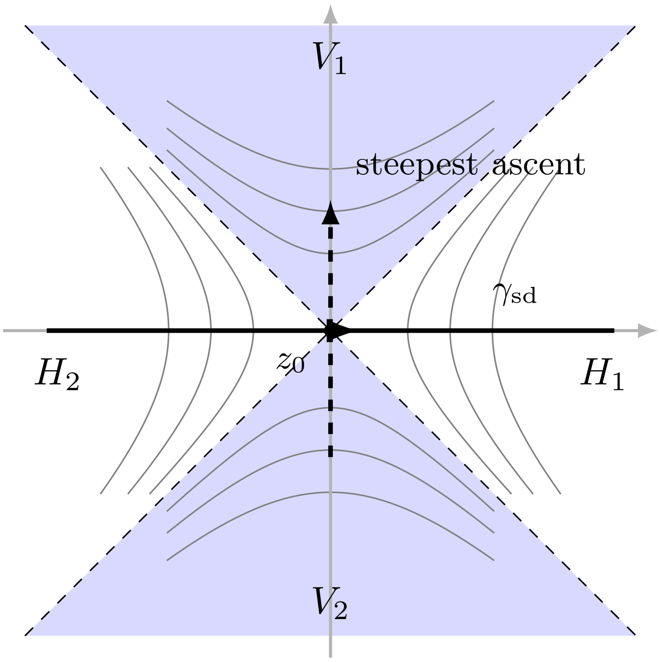

Figure 5: Level curves of $\operatorname{Re}(f - f(z_0))$ near a simple saddle of f. The steepest-descent contour (solid) passes through the saddle in the direction where $\operatorname{Re} f$ decreases fastest; the steepest-ascent direction (dashed) is orthogonal. The two *valleys* (shaded) are the regions where $\operatorname{Re} f \to -\infty$; contour deformations for Laplace-type integrals end in valleys.

2.6.4 Hankel-like contour trick

A recurring tactic: if poles or branch cuts lie on the real axis, tilt the contour into the complex plane. The new path must avoid singularities, end in descent valleys, and be equivalent by Cauchy's theorem. The Hankel contour around $(-\infty, 0]$ is the standard example; the jump across its two banks turns a contour integral into a real Laplace integral.

2.6.5 Worked example: Laplace transforms with complex poles

Before Stirling, consider a simpler example showing how poles of $g(z)$ affect coefficients. Let

$$I(\lambda) = \int_0^\infty \frac{e^{-\lambda t}}{a^2+t^2}\,dt, \qquad \lambda \to +\infty,\ a>0 \text{ fixed.}$$

The exponent $\phi(t) = -t$ is monotone on $(0,\infty)$, decreasing; the maximum is at the left endpoint $t=0$, and it is *not* a saddle in the standard sense (Thm. 2.11 does not apply). The power series of $(a^2+t^2)^{-1}$ only converges for $|t|<a$, so we do *not* integrate an infinite Taylor series out to $t=\infty$. Instead, use the finite geometric identity with $x=(t/a)^2$:

$$\frac{1}{a^2+t^2} = \frac{1}{a^2}\sum_{n=0}^{N}(-1)^n\left(\frac{t}{a}\right)^{2n} + \frac{(-1)^{N+1}t^{2N+2}}{a^{2N+2}(a^2+t^2)}.$$

Integrating term by term is now legitimate because the sum is finite, and $\int_0^\infty t^{2n}e^{-\lambda t}\,dt = (2n)!/\lambda^{2n+1}$:

$$I(\lambda) = \frac{1}{a^2\lambda}\sum_{n=0}^{N}\frac{(-1)^n(2n)!}{(a\lambda)^{2n}} + R_N(\lambda). \tag{2.17}$$

The remainder is explicit:

$$R_N(\lambda) = \frac{(-1)^{N+1}}{a^{2N+2}}\int_0^\infty \frac{t^{2N+2}e^{-\lambda t}}{a^2+t^2}\,dt,$$

so

$$|R_N(\lambda)| \le \frac{1}{a^{2N+4}}\int_0^\infty t^{2N+2}e^{-\lambda t}\,dt = \frac{(2N+2)!}{a^{2N+4}\lambda^{2N+3}}.$$

For each fixed N this is $O(\lambda^{-2N-3})$, so

$$I(\lambda) \sim \frac{1}{a^2\lambda}\left[1 - \frac{2!}{(a\lambda)^2} + \frac{4!}{(a\lambda)^4} - \cdots\right].$$

The series diverges for every λ because the Laplace moments grow factorially. The poles of $g(z) = 1/(a^2+z^2)$ at $z=\pm ia$ determine the coefficient growth. In Borel form (Problem 2.16), the corresponding transform is

$$\hat{I}(w) = \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{2n+2}}w^{2n} = \frac{1}{a^2+w^2},$$

where the “Borel form” is only a bookkeeping device: divide the factorially growing coefficient $(2n)!$ by the corresponding Laplace moment to expose the nearest singularities in the new variable w. Problem 2.16 returns to this idea systematically. The singularities at $w=\pm ia$ record the poles of the prefactor.

Off-contour poles shape the coefficients; crossed poles contribute residues (Thm. 1.25).

2.6.6 Worked example: a logarithmic saddle on the positive axis

The phase $f(s) = \log s - s$ is the simplest example in which the real contour already passes through a simple saddle, while the holomorphic continuation naturally lives on a slit plane. Consider

$$I(\lambda) = \int_0^\infty e^{\lambda(\log s - s)}\,ds, \qquad \lambda \to +\infty. \tag{2.18}$$

Promote s to a complex variable ζ and keep the principal branch of $\log\zeta$ on $\mathbb{C}\setminus(-\infty,0]$. Then

$$f(\zeta)=\log\zeta-\zeta,\qquad f'(\zeta)=\frac{1}{\zeta}-1,\qquad f''(\zeta)=-\frac{1}{\zeta^2}.$$

The unique saddle is at $\zeta_0=1$, with

$$f(1)=-1,\qquad f''(1)=-1.$$

Since $\arg f''(1)=\pi$, the steepest-descent direction is $\theta_0=(\pi-\pi)/2=0$, so the positive real axis already crosses the saddle in the correct direction. Both endpoints lie in valleys:

$$f(s)=\log s-s\to-\infty\quad\text{as } s\to 0^+ \text{ or } s\to+\infty.$$

Hence Thm. 2.17 applies directly with $g\equiv 1$ and parameter λ, giving

$$I(\lambda)\sim e^{\lambda f(1)}e^{i\theta_0}\sqrt{\frac{2\pi}{\lambda|f''(1)|}}=e^{-\lambda}\sqrt{\frac{2\pi}{\lambda}}.$$

So

$$\boxed{I(\lambda)\sim e^{-\lambda}\sqrt{\frac{2\pi}{\lambda}},\qquad \lambda\to+\infty.}$$

This is the local saddle geometry behind Stirling's formula.

2.6.7 Worked example: Airy asymptotics for $x\to+\infty$

Model geometry. The cubic $f(s)=s^3/3-s$ has saddles at $s=\pm1$ and three decay valleys. Rescaling $s\to s\sqrt{x}$ reduces the Airy problem to this same two-saddle picture.

Real-form definition. The standard Airy function is

$$\operatorname{Ai}(x)=\frac{1}{\pi}\int_0^\infty\cos\left(\frac{t^3}{3}+xt\right)dt,\qquad x\in\mathbb{R}.\tag{2.19}$$

For $x>0$, the phase $t^3/3+xt$ is strictly increasing, so there is no real stationary point and a contour deformation is needed to see the decay rate.

Passing to a contour integral. Write $\cos=\operatorname{Re}(e^{i}\cdot)$ and use the contour form with real saddles when $x>0$. Starting from

$$\cos(t^3/3+xt)=\tfrac{1}{2}\left(e^{i(t^3/3+xt)}+e^{-i(t^3/3+xt)}\right),$$

handle the two exponentials separately, keeping orientation visible. For

$$I_+:=\int_0^\infty e^{i(t^3/3+xt)}\,dt,$$

put $t=is$. Then $dt=i\,ds$, $i(t^3/3)=s^3/3$, and $ixt=-xs$. As t runs from 0 to $+\infty$ on the real axis, $s=-it$ runs from 0 to $-i\infty$, so

$$I_+=i\int_0^{-i\infty}e^{s^3/3-xs}\,ds=-i\int_{-i\infty}^0 e^{s^3/3-xs}\,ds.$$

The ray $-i\infty$ lies on the boundary of the lower decay sector $V_-=(-\pi/2,-\pi/6)$ (where $\operatorname{Re}(s^3)=0$, so the integral converges only conditionally), and it may be rotated strictly into the interior, e.g. to the standard lower ray $\infty e^{-i\pi/3}$, by Cauchy's theorem plus the quarter-arc decay estimate in Problem 2.6. Thus I_+ supplies the lower branch of the Airy contour, oriented from $\infty e^{-i\pi/3}$ to 0, with prefactor $-i$.

For

$$I_- := \int_0^\infty e^{-i(t^3/3+xt)}\,dt,$$

put $t=-is$. Then $dt=-i\,ds$, $-i(t^3/3)=s^3/3$, and $-ixt=-xs$. Now $s=it$ runs from 0 to $+i\infty$, so

$$I_- = -i\int_0^{i\infty} e^{s^3/3-xs}\,ds.$$

Rotating $i\infty$ to the standard upper ray $\infty e^{i\pi/3}$ gives the upper branch from 0 to $\infty e^{i\pi/3}$, again with prefactor $-i$. Therefore

$$\frac{1}{2\pi}(I_+ + I_-) = -\frac{i}{2\pi}\int_{\mathcal{C}} e^{s^3/3-xs}\,ds = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{s^3/3-xs}\,ds,$$

where $\mathcal{C}$ is oriented from $\infty e^{-i\pi/3}$ to $\infty e^{i\pi/3}$. The sign is worth checking once: the lower piece runs from $\infty e^{-i\pi/3}$ to 0, the upper piece runs from 0 to $\infty e^{i\pi/3}$, so they concatenate in the positive orientation of $\mathcal{C}$; the scalar identity $-i/(2\pi)=1/(2\pi i)$ gives the final prefactor. This gives

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{s^3/3-xs}\,ds, \tag{2.20}$$

where $\mathcal{C}$ runs from $\infty e^{-i\pi/3}$ to $\infty e^{i\pi/3}$ through sectors where $\mathrm{Re}(s^3)\to-\infty$. Since $\mathrm{Re}(s^3)=r^3\cos 3\phi$ for $s=re^{i\phi}$, the valley wedges are

$$V_+ := (\pi/6,\pi/2), \qquad V_- := (-\pi/2,-\pi/6), \qquad V_{\text{left}} := (5\pi/6, 7\pi/6).$$

The contour $\mathcal{C}$ enters from V_- and exits through V_+.

Saddles. Set $f(s)=s^3/3-xs$. Then

$$f'(s)=s^2-x, \qquad f''(s)=2s, \qquad f'''(s)=2.$$

For $x>0$, the saddles are $s_\pm=\pm\sqrt{x}$, with

$$f(\sqrt{x}) = \frac{x^{3/2}}{3} - x\sqrt{x} = -\frac{2x^{3/2}}{3}, \tag{2.21}$$

$$f(-\sqrt{x}) = -\frac{x^{3/2}}{3} + x\sqrt{x} = \frac{2x^{3/2}}{3}. \tag{2.22}$$

Only $+\sqrt{x}$ contributes to $\mathrm{Ai}(x)$: it connects to the endpoint wedges $V_\pm$. The saddle $-\sqrt{x}$ connects to V_{left} and corresponds to the growing companion solution.

Steepest-descent direction at $s_0=\sqrt{x}$. Here $\beta=\arg f''(\sqrt{x})=0$, so (2.12) gives $\theta_0=\pi/2$: the steepest-descent path is vertical. Writing $s=\sqrt{x}+i\sigma$,

$$(\sqrt{x}+i\sigma)^3 = x^{3/2}+3ix\sigma-3\sqrt{x}\,\sigma^2-i\sigma^3,$$

$$\tfrac{1}{3}(\sqrt{x}+i\sigma)^3 = \tfrac{x^{3/2}}{3}+ix\sigma-\sqrt{x}\,\sigma^2-i\tfrac{\sigma^3}{3}.$$

Subtracting $x(\sqrt{x}+i\sigma)$ gives

$$f(\sqrt{x}+i\sigma) = -\tfrac{2x^{3/2}}{3}-\sqrt{x}\,\sigma^2-i\tfrac{\sigma^3}{3}. \tag{2.23}$$

So along the steepest-descent contour the real part is $-2x^{3/2}/3 - \sqrt{x}\,\sigma^2$, maximized at the saddle and decreasing quadratically away from it.

Apply Thm. 2.17. Scale $s = x^{1/2}u$, so $\lambda = x^{3/2}$ and $\tilde{f}(u) = u^3/3 - u$ has saddle $u_0 = 1$ with $\tilde{f}(1) = -2/3$, $\tilde{f}''(1) = 2$. Then

$$\int_C e^{f(s)}\,ds = x^{1/2}\int_{C'} e^{\lambda \tilde{f}(u)}\,du$$
$$\sim x^{1/2}\,e^{-2\lambda/3}\,e^{i\pi/2}\sqrt{\frac{2\pi}{2\lambda}} = i\sqrt{\pi}\,x^{-1/4}e^{-2x^{3/2}/3}.$$

Dividing by $2\pi i$ from (2.20),

$$\boxed{\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}}\,x^{-1/4}\exp\!\left(-\tfrac{2}{3}x^{3/2}\right), \qquad x \to +\infty.} \tag{2.24}$$

The phase $e^{i\theta_0} = i$ cancels the i in $1/(2\pi i)$, leaving a positive real answer.

Remark 2.19 (Sign/phase check)**.** *The $e^{s^3/3 - xs}$ form is chosen only to make the contributing saddle real and the sign of $f(s_0)$ visible.*

2.6.8 Worked example: Airy asymptotics for $x \to -\infty$

We now treat the oscillatory regime. Two saddles contribute, and their phases interfere.

Start from a real representation. Writing $x = -y$ with $y > 0$,

$$\mathrm{Ai}(-y) = \frac{1}{\pi}\int_0^\infty \cos\!\left(\frac{s^3}{3} - ys\right)ds = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i(s^3/3 - ys)}\,ds, \tag{2.25}$$

because the sine part is odd and integrates to zero on the full line.

Saddles. Let $f(s) = i(s^3/3 - ys)$. Then $f'(s) = i(s^2 - y)$, so the saddles are

$$s_\pm = \pm\sqrt{y},$$

and

$$f''(s_+) = 2i\sqrt{y}, \qquad f''(s_-) = -2i\sqrt{y}, \qquad |f''(s_\pm)| = 2\sqrt{y}.$$

Their values are

$$f(s_+) = -\frac{2iy^{3/2}}{3}, \tag{2.26}$$
$$f(s_-) = \frac{2iy^{3/2}}{3}. \tag{2.27}$$

Set $\alpha = \frac{2}{3}y^{3/2}$.

Steepest-descent directions. From (2.12),

$$\theta_+ = \frac{\pi - \pi/2}{2} = \frac{\pi}{4}, \qquad \theta_- = \frac{\pi - (-\pi/2)}{2} = \frac{3\pi}{4}.$$

Pushing the real contour into the upper half-plane, the deformed contour crosses s_- along the opposite ray of its descent line and crosses s_+ along its standard descent ray:

$$\text{at } s_+:\ \text{tangent } = e^{i\pi/4}; \qquad \text{at } s_-:\ \text{tangent } = e^{-i\pi/4} = e^{i(3\pi/4 + \pi)}. \tag{2.28}$$

This orientation makes the two terms conjugate.

Saddle contributions. The Gaussian width at either saddle is

$$\sqrt{\frac{2\pi}{2\sqrt{y}}} = \sqrt{\pi}\, y^{-1/4} =: A.$$

Therefore

$$T_+ = e^{f(s_+)} e^{i\pi/4} A = A\, e^{i(\pi/4-\alpha)}, \tag{2.29}$$

$$T_- = e^{f(s_-)} e^{-i\pi/4} A = A\, e^{i(\alpha-\pi/4)} = \overline{T_+}. \tag{2.30}$$

So

$$T_+ + T_- = 2A\cos(\alpha - \pi/4).$$

Returning to (2.25),

$$\boxed{\operatorname{Ai}(-y) \sim \frac{1}{\sqrt{\pi}}\, y^{-1/4} \cos\!\left(\tfrac{2}{3} y^{3/2} - \tfrac{\pi}{4}\right), \qquad y \to +\infty.} \tag{2.31}$$

2.6.9 First correction for the logarithmic saddle

We now compute the first correction to (2.18):

$$I(\lambda) = e^{-\lambda}\sqrt{\frac{2\pi}{\lambda}}\left(1 + \frac{1}{12\lambda} + O(\lambda^{-2})\right), \qquad \lambda \to +\infty. \tag{2.32}$$

Step 1: local Taylor expansion. Write $s = 1 + w$. Since

$$\log(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + O(w^5),$$

we obtain

$$\begin{aligned} f(1+w) &= \log(1+w) - 1 - w \\ &= -1 - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + O(w^5). \end{aligned} \tag{2.33}$$

Step 2: scale to the Gaussian window. Put $u = \sqrt{\lambda}\, w$, so $w = u/\sqrt{\lambda}$ and $dw = du/\sqrt{\lambda}$. Then

$$\lambda f(1+w) = -\lambda - \frac{u^2}{2} + \frac{u^3}{3\sqrt{\lambda}} - \frac{u^4}{4\lambda} + O\big(\lambda^{-3/2}(1+|u|^5)\big),$$

uniformly on the Laplace window $|u| < \lambda^{1/10}$. Exponentiating,

$$e^{\lambda f(1+w)} = e^{-\lambda} e^{-u^2/2}\left[1 + \frac{u^3}{3\sqrt{\lambda}} + \frac{1}{\lambda}\left(-\frac{u^4}{4} + \frac{u^6}{18}\right) + O\big(\lambda^{-3/2}(1+|u|^9)\big)\right]. \tag{2.34}$$

Step 3: integrate term by term. Replacing the truncated window by $\mathbb{R}$ costs only an exponentially small error, so

$$I(\lambda) = \frac{e^{-\lambda}}{\sqrt{\lambda}} \int_{-\infty}^{\infty} e^{-u^2/2}\left[1 + \frac{u^3}{3\sqrt{\lambda}} + \frac{1}{\lambda}\left(-\frac{u^4}{4} + \frac{u^6}{18}\right)\right] du + O(e^{-\lambda}\lambda^{-2}). \tag{2.35}$$

The odd term integrates to zero. Using

$$\int_{-\infty}^{\infty} e^{-u^2/2}\,du = \sqrt{2\pi}, \qquad \int_{-\infty}^{\infty} u^4 e^{-u^2/2}\,du = 3\sqrt{2\pi}, \qquad \int_{-\infty}^{\infty} u^6 e^{-u^2/2}\,du = 15\sqrt{2\pi},$$

where the even moments follow from the recurrence

$$M_{2j} := \int_{-\infty}^{\infty} u^{2j} e^{-u^2/2}\,du = (2j-1)M_{2j-2}$$

obtained by integrating $u^{2j-1}(-e^{-u^2/2})'$ by parts. Thus $M_4 = 3M_2 = 3\sqrt{2\pi}$ and $M_6 = 5M_4 = 15\sqrt{2\pi}$. The coefficient of λ^{-1} is

$$-\frac{1}{4}\cdot 3 + \frac{1}{18}\cdot 15 = -\frac{3}{4} + \frac{5}{6} = \frac{1}{12}.$$

Substituting back gives (2.32).

Remark 2.20 (Higher orders)**.** *Continuing the expansion produces*

$$I(\lambda) \sim e^{-\lambda}\sqrt{\frac{2\pi}{\lambda}}\left(1 + \frac{1}{12\lambda} + \frac{1}{288\lambda^2} - \frac{139}{51840\,\lambda^3} + \cdots\right).$$

After reduction to a Gaussian window, later coefficients are Gaussian moments of explicit polynomials.

2.6.10 Worked example: opening the half-line by a logarithm

Consider

$$I(\lambda) = \int_0^{\infty} \exp\!\left(-\lambda\left(t + \tfrac{1}{t}\right)\right)\frac{dt}{t}, \qquad \lambda \to +\infty.$$

Although the integrand is real and positive on $(0,\infty)$, the substitution $t = e^w$ opens the multiplicative half-line into the full real line:

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-2\lambda\cosh w}\,dw.$$

Now $\phi(w) = -2\cosh w$ has a unique interior maximum at $w = 0$, with $\phi(0) = -2$ and $\phi''(0) = -2$. Laplace's method gives

$$I(\lambda) \sim e^{-2\lambda}\sqrt{\frac{2\pi}{2\lambda}} = \sqrt{\frac{\pi}{\lambda}}\,e^{-2\lambda}.$$

The change of variables turns a multiplicative saddle in t into an ordinary Gaussian saddle in $w = \log t$.

2.6.11 Worked example: a cubic turning-point model

When two simple saddles merge, the quadratic approximation vanishes and the local normal form becomes cubic. A clean model is

$$C(\lambda) = \int_0^{\infty} e^{-i\lambda\theta^3/6}\,d\theta, \qquad \lambda \to +\infty. \tag{2.36}$$

The natural rescaling is $\theta = O(\lambda^{-1/3})$: set

$$u = \left(\frac{\lambda}{2}\right)^{1/3} \theta, \qquad d\theta = \left(\frac{2}{\lambda}\right)^{1/3} du.$$

Then

$$C(\lambda) = \left(\frac{2}{\lambda}\right)^{1/3} \int_0^\infty e^{-iu^3/3}\, du.$$

Rotate the ray to $u = re^{-i\pi/6}$. This contour deformation is legal because the integrand $e^{-iu^3/3}$ is entire and the connecting arc has vanishing contribution by the same Jordan-type estimate used in the Fresnel proof. Since $u^3 = -ir^3$ there, we get

$$C(\lambda) = e^{-i\pi/6} \left(\frac{2}{\lambda}\right)^{1/3} \int_0^\infty e^{-r^3/3}\, dr. \tag{2.37}$$

Two features matter.

- The width of the contributing neighborhood is $\lambda^{-1/3}$, not $\lambda^{-1/2}$.
- The leading constant comes from a fixed cubic model integral, independent of the original problem once the local phase has been normalized.

Taking real parts in (2.37) gives

$$\int_0^\infty \cos(\lambda\theta^3/6)\, d\theta \sim \cos(\pi/6) \left(\frac{2}{\lambda}\right)^{1/3} \int_0^\infty e^{-r^3/3}\, dr.$$

More generally, if the local phase has the form $\Phi(\theta) = \theta^3/6 + O(\theta^5)$, then the next term contributes $\lambda\theta^5 = O(\lambda^{-2/3})$ on the cubic scale $\theta = O(\lambda^{-1/3})$, which is why cubic turning-point expansions advance in powers of $\lambda^{-2/3}$.

2.6.12 Worked example: two stationary points and interference

When the phase has two distinct nondegenerate stationary points on the contour (meaning $\phi' = 0$ and $\phi'' \neq 0$ at each point), each contributes a Gaussian packet and the leading term is their sum. The simplest model is

$$K(\lambda) = \int_{-\infty}^\infty e^{i\lambda(t^3/3 - t)}\, dt, \qquad \lambda \to +\infty. \tag{2.38}$$

Because the interval is unbounded, read this as an oscillatory improper integral: one may first insert a smooth cutoff and then let the cutoff expand, or insert a small damping factor and remove it at the end. Away from neighborhoods of $t = \pm 1$, the derivative $\phi'(t) = t^2 - 1$ does not vanish, so integration by parts makes the non-stationary pieces smaller than the leading $\lambda^{-1/2}$ terms. Set

$$\phi(t) = \frac{t^3}{3} - t, \qquad \phi'(t) = t^2 - 1, \qquad \phi''(t) = 2t.$$

The stationary points are

$$t_\pm = \pm 1, \qquad \phi(t_+) = -\frac{2}{3}, \qquad \phi(t_-) = \frac{2}{3}, \qquad \phi''(t_+) = 2, \qquad \phi''(t_-) = -2.$$

Applying stationary phase at the two saddles gives

$$T_+ = e^{-2i\lambda/3}\sqrt{\frac{\pi}{\lambda}}\,e^{i\pi/4},$$
$$T_- = e^{2i\lambda/3}\sqrt{\frac{\pi}{\lambda}}\,e^{-i\pi/4}.$$

These are complex conjugates, so

$$K(\lambda) \sim T_+ + T_- = 2\sqrt{\frac{\pi}{\lambda}}\cos\left(\frac{2\lambda}{3} - \frac{\pi}{4}\right).$$

Paired saddles often combine into a real cosine, with the same $\pi/4$ phase shift from Thm. 2.14.

2.6.13 Worked example: a Hankel contour model

To illustrate the contour trick from Section 2.6.4, fix $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha < 1$ and consider

$$\mathcal{H}(\alpha) = \frac{1}{2\pi i}\int_H e^t t^{-\alpha}\,dt, \tag{2.39}$$

where H is the standard Hankel contour around the cut $(-\infty, 0]$, and $t^{-\alpha} = e^{-\alpha \log t}$ is taken on the principal branch.

Evaluate piece by piece. On the lower bank $t = -r - i0$, so $t^{-\alpha} = r^{-\alpha}e^{i\pi\alpha}$; on the upper bank $t = -r + i0$, so $t^{-\alpha} = r^{-\alpha}e^{-i\pi\alpha}$. The small circle about 0 vanishes as $\rho \to 0^+$ because $\operatorname{Re}\alpha < 1$. Hence

$$\begin{aligned}\int_H e^t t^{-\alpha}\,dt &= e^{i\pi\alpha}\int_0^\infty e^{-r} r^{-\alpha}\,dr - e^{-i\pi\alpha}\int_0^\infty e^{-r} r^{-\alpha}\,dr \\ &= 2i\sin(\pi\alpha)\int_0^\infty e^{-r} r^{-\alpha}\,dr.\end{aligned}$$

The signs come from orientation. On the lower bank the contour travels from $-\infty$ toward 0, so r decreases from ∞ to ρ and $dt = -dr$, turning the integral into $+\int_\rho^\infty$. On the upper bank the contour returns from 0 to $-\infty$, so r increases from ρ to ∞ while again $dt = -dr$, giving the minus sign. Therefore

$$\mathcal{H}(\alpha) = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty e^{-r} r^{-\alpha}\,dr, \qquad \operatorname{Re}\alpha < 1.$$

The jump of $t^{-\alpha}$ across the cut has turned the contour integral into a real Laplace integral. In Section 3 this becomes the reciprocal-Gamma formula.

2.6.14 Relation to the WKB approximation (brief)

The WKB (Wentzel–Kramers–Brillouin) approximation for the Schrödinger equation

$$-\hbar^2\psi''(x) + V(x)\psi(x) = E\psi(x)$$

has oscillatory solutions when $E > V(x)$ and exponential solutions when $E < V(x)$. At a turning point $E = V$, the Gaussian approximation fails and the local model is Airy. This is the same saddle-coalescence mechanism as Section 2.6.11; the matching between (2.24) and (2.31) is the WKB connection formula in asymptotic form.

2.7 Summary of asymptotic toolkit

Given $\int e^{\lambda f(z)} g(z)\,dz$ with $\lambda \to +\infty$:

- f real on a real interval, interior maximum: Laplace (Thm. 2.11). Leading term:
$$g(t_0)e^{\lambda f(t_0)}\sqrt{\frac{2\pi}{-\lambda f''(t_0)}}.$$
- $f = i\phi$ with ϕ real: stationary phase (Thm. 2.14). Same form with phase $e^{i\pi\sigma/4}$.
- f complex, contour deformable through a simple saddle: steepest descent (Thm. 2.17). Locate, deform, evaluate.
- Two saddles coalesce: cubic local model, Airy integral, width $\lambda^{-1/3}$ (Section 2.6.11).
- Higher coalescence $f^{(k)} \neq 0$, $f^{(j)} = 0$ for $j < k$: width $\lambda^{-1/k}$, with a leading constant supplied by a universal model integral (Problem 2.15).
- Endpoint maximum with nonzero slope: expand the prefactor or integrate by parts. Leading size $O(1/\lambda)$ (see Section 2.6.5).
- Endpoint maximum with quadratic tangency: half of a Gaussian window survives, so the leading size is $O(\lambda^{-1/2})$ (Problem 2.9).

Principle: *scale near the contributing point, reduce to a Gaussian or Airy model, and integrate the model*. Subleading corrections are Gaussian moments against polynomials, as in Section 2.6.9.

Exercises

Problem 2.1. *Let f, g be entire with $f(1/n) = g(1/n)$ for all $n \in \mathbb{N}_{\geq 1}$. Prove $f = g$ on $\mathbb{C}$.*

Problem 2.2. *Show that the series $\sum_{n=0}^{\infty} z^{2^n}$ converges on $|z| < 1$ but cannot be analytically continued to any open set containing a point of the unit circle. The name* Hadamard gap theorem *refers to the general theorem behind this example: power series with rapidly growing exponent gaps can have a natural boundary, meaning a boundary across which no analytic continuation is possible.* Hint: *use the functional equation $f(z) = z + f(z^2)$ and induct along dyadic roots of unity.*

Problem 2.3. *Derive the first two terms of the asymptotic expansion of $F(\lambda) = \displaystyle\int_0^{\infty} e^{-\lambda t^2}\frac{dt}{1+t}$ as $\lambda \to +\infty$. (The maximum of $-t^2$ on $[0,\infty)$ is at the boundary $t = 0$; Taylor-expand $1/(1+t)$ rather than the exponent.)*

Problem 2.4. *Use stationary phase (Thm. 2.14) to derive the leading asymptotics of $J_0(\lambda) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(\lambda \sin\theta)\,d\theta$ as $\lambda \to +\infty$. Identify the stationary point and sum the two complex-conjugate contributions coming from the cosine. The endpoints are non-stationary, so their contributions are lower order after integration by parts. Recover $J_0(\lambda) \sim \sqrt{2/(\pi\lambda)}\cos(\lambda - \pi/4)$.*

Problem 2.5. *Apply steepest descent to $I(\lambda) = \displaystyle\int_{-\infty}^{\infty} e^{-\lambda(t^2-1)^2}\,dt$ as $\lambda \to +\infty$. The phase has stationary points at $t = -1, 0, 1$; show that $t = \pm 1$ are the two dominant maxima and that the contribution from $t = 0$ is exponentially smaller. Compute the steepest-descent direction at each dominant point and show the two leading contributions add to $I(\lambda) \sim \sqrt{\pi/\lambda}$.*

Problem 2.6. Airy contour, $t \to$ *is* rotation. *Start from the real form* (2.19), $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(t^3/3 + xt)\,dt$*. Using* $\cos = \mathrm{Re}\, e^{i\cdot}$ *and a contour rotation* $t \to$ *is in one of the two exponential pieces, derive the contour form* (2.20)*. Verify decay of the integrand on the quarter-arc at infinity, justifying the rotation by a Jordan-type arc estimate like the one used for the Fresnel integral.*

Problem 2.7. Fresnel integral (used in stationary phase). *Prove* $\displaystyle\int_{-\infty}^{\infty} e^{iu^2/2}\,du = \sqrt{2\pi}\,e^{i\pi/4}$ *by rotating the contour to* $u = re^{i\pi/4}$ *and using Lem. 2.10. State precisely where analyticity and decay on arcs enter the argument.*

Problem 2.8. Second-order logarithmic saddle. *Extend the calculation of Section 2.6.9 by one more order for* $I(x) = \int_0^\infty e^{x(\ln s - s)}\,ds$*: expand* $\ln s - s$ *to sixth order in* $w = s - 1$*, rescale, and compute the relevant Gaussian moments (you will need* $\int u^{10} e^{-u^2/2} du = 945\sqrt{2\pi}$*). Show that the coefficient of* x^{-2} *is* $1/288$.

Problem 2.9. Endpoint maximum. *Evaluate the leading-order behavior of* $\displaystyle\int_0^1 e^{\lambda\cos\theta}\,d\theta$ *as* $\lambda \to +\infty$*. The maximum of* $\cos\theta$ *on* $[0,1]$ *is at* $\theta = 0$*, a boundary point;* $\cos\theta = 1 - \theta^2/2 + O(\theta^4)$*. Show the answer is* $\sqrt{\pi/(2\lambda)}\,e^{\lambda}$ *— half of the full-range Laplace value, because only the* $\theta > 0$ *side of the Gaussian window lies in the integration range.*

Problem 2.10. Airy for $x \to -\infty$ (oscillatory regime). *Rework the oscillatory Airy calculation starting from* (2.25)*. Show that the saddles of the phase* $i(s^3/3 - xs)$ *lie at* $s = \pm\sqrt{x}$*, determine the correct tangent directions selected by the contour deformation, and prove that the two contributions are complex conjugates. Summing gives*

$$\mathrm{Ai}(-x) \sim \frac{1}{\sqrt{\pi}}\, x^{-1/4} \cos\!\left(\tfrac{2}{3}x^{3/2} - \tfrac{\pi}{4}\right), \qquad x \to +\infty.$$

Problem 2.11. Cubic turning point, subleading term. *Let* $\Phi(\theta) = \theta^3/6 - \theta^5/120 + O(\theta^7)$ *and consider* $I(n) = \int_0^\infty e^{-in\Phi(\theta)}\,d\theta$*. (a) Verify the quoted Taylor expansion. (b) Show that including the* θ^5 *term gives a correction of* relative order $n^{-2/3}$*, so*

$$I(n) = n^{-1/3}\left(C_0 + C_1 n^{-2/3} + \cdots\right)$$

for constants C_0, C_1 *independent of* n*. The exponent* $-2/3$ *in the correction is the main point.*

Problem 2.12. Complex Gaussian. *For* $\alpha \in \mathbb{C}$ *with* $\mathrm{Re}\,\alpha > 0$*, prove* $\int_{-\infty}^{\infty} e^{-\alpha u^2/2}\,du = \sqrt{2\pi/\alpha}$ *(principal branch).* Hint*: show both sides are holomorphic in* α *on the right half-plane and agree on the positive real axis (Lem. 2.10); apply Thm. 2.2.*

Problem 2.13. Laplace via a log substitution. *Evaluate* $\displaystyle\int_0^\infty t^{-1/2} e^{-\lambda(t+1/t)}\,dt$ *as* $\lambda \to +\infty$. Hint*: let* $t = e^w$ *to convert into a Laplace integral on* $\mathbb{R}$*. Compute carefully and show the answer is* $\sqrt{\pi/\lambda}\,e^{-2\lambda}$*. Explain why the principal-branch cut of* $t^{-1/2}$ *on* $(-\infty, 0]$ *causes no difficulty.*

Problem 2.14. Oscillatory integral with two stationary points. *Use stationary phase to evaluate* $\displaystyle\int_{-\infty}^{\infty} e^{i\lambda(t^3/3 - t)}\,dt$ *as* $\lambda \to +\infty$*, interpreted as an oscillatory improper integral. Show the stationary points are at* $t = \pm 1$*, each of second derivative* ± 2*, justify that the tails are lower order by integration by parts, and conclude that the sum of contributions gives* $2\sqrt{\pi/\lambda}\cos(2\lambda/3 - \pi/4)$*. Relate this to* $\mathrm{Ai}(-x)$ *in the oscillatory regime (Problem 2.10).*

Problem 2.15. Saddle of order k. *Let f be holomorphic with $f^{(j)}(z_0) = 0$ for $1 \le j < k$ and $f^{(k)}(z_0) \ne 0$ (so $k = 2$ is an ordinary simple saddle; $k = 3$ is a double coalescence, etc.). By rescaling $z - z_0 = (\lambda)^{-1/k} u$, show that*

$$\int_\gamma e^{\lambda f(z)} g(z)\, dz \;\sim\; g(z_0)\, e^{\lambda f(z_0)}\, \lambda^{-1/k} \cdot (\text{constant determined by a universal model integral and } f^{(k)}(z_0)),$$

for a contour deformable through the saddle along the appropriate descent direction. The prefactor $\lambda^{-1/2}$ of Thm. 2.17 is the $k = 2$ case. In Section 3 this model-integral constant will be rewritten in terms of Gamma values.

Problem 2.16. Borel resummation. *The series $g(z) = \sum_{n=0}^{\infty} (-1)^n n!/z^{n+1}$ diverges. Define its* Borel transform *$\hat{g}(w) = \sum_n (-1)^n w^n = 1/(1+w)$ (dividing each term by $n!$). Show that the Laplace transform $\tilde{g}(z) = \int_0^\infty e^{-zw} \hat{g}(w)\, dw = \int_0^\infty e^{-zw}/(1+w)\, dw$ converges for* $\operatorname{Re} z > 0$*, has $g(z)$ as its asymptotic expansion (in the sense of Def. 2.8), and agrees up to substitution with the Stieltjes function of Ex. 2.9. This is the prototype of* Borel resummation*: a divergent asymptotic series packaged into a well-defined analytic function.*

3 The Gamma Function

Prerequisites. This section uses elementary integration together with the residue theorem and keyhole contour of Section 1 (especially Ex. 1.33). It also uses the identity theorem from Section 2 (Thm. 2.2). A *functional equation* is an equation relating values of the same function at different inputs, such as $\Gamma(z+1) = z\Gamma(z)$ below. A *Beta–Gamma relation* is an identity connecting the Beta integral $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, introduced formally in Definition 3.8, with the Gamma function. The final Stirling subsection locally uses the real Stirling example and saddle-point ideas from Section 2 (Ex. 2.13 and Sections 2.6.6–2.6.9).

The factorial starts as a discrete object: $0! = 1$, and $n! = 1 \cdot 2 \cdots n$ for $n \geq 1$. Euler's Gamma function gives its natural complex interpolation and supplies the coefficients for many special functions used later.

3.1 Definition and analytic continuation

Definition 3.1 (Euler's integral of the second kind)**.** *For $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$,*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt. \tag{3.1}$$

Here $t > 0$ is real, so t^{z-1} means $e^{(z-1)\ln t}$ with the ordinary real logarithm $\ln t$.

Near $t = 0$, convergence is that of $\int_0^1 t^{\operatorname{Re} z - 1}\,dt$, hence requires $\operatorname{Re} z > 0$. At ∞, the factor e^{-t} dominates every power.

Here is the uniform estimate behind the holomorphy statement. If K is a compact subset of the half-plane $\operatorname{Re} z > 0$, choose numbers $\sigma > 0$ and $M > 0$ such that

$$\sigma \leq \operatorname{Re} z \leq M \qquad (z \in K).$$

For $0 < t \leq 1$,

$$|t^{z-1}| = t^{\operatorname{Re} z - 1} \leq t^{\sigma - 1},$$

which is integrable because $\sigma > 0$. For $t \geq 1$,

$$|t^{z-1}e^{-t}| \leq t^{M-1}e^{-t},$$

which is integrable because the exponential decays faster than any power. The same argument with extra factors $(\log t)^m$ controls the m-th z-derivative of t^{z-1}. Thus differentiation under the integral is justified uniformly on compact subsets, and Γ is holomorphic on $\operatorname{Re} z > 0$.

Proposition 3.2 (Basic values and the functional equation)**.** *For $\operatorname{Re} z > 0$,*

$$\Gamma(z+1) = z\,\Gamma(z), \qquad \Gamma(1) = 1, \qquad \Gamma(n+1) = n! \;\; (n = 0, 1, 2, \ldots). \tag{3.2}$$

Proof. The value at 1 is $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$. Integration by parts gives

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty t^z e^{-t}\,dt = \left[-t^z e^{-t}\right]_0^\infty + z\int_0^\infty t^{z-1}e^{-t}\,dt \\ &= z\,\Gamma(z),\end{aligned}$$

since $t^z e^{-t} \to 0$ as $t \to \infty$ and $t^z \to 0$ as $t \to 0^+$ when $\operatorname{Re} z > 0$. Iteration gives $\Gamma(n+1) = n!$. ☐

Theorem 3.3 (Continuation to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$). *$\Gamma$ extends to a meromorphic function on $\mathbb{C}$, meaning holomorphic except at isolated poles, with simple poles at $z = -n$ ($n \in \mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$) and residues*

$$\operatorname*{Res}_{z=-n} \Gamma = \frac{(-1)^n}{n!}. \tag{3.3}$$

Continuation. Read (3.2) *backward*:

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}. \tag{3.4}$$

The numerator is holomorphic on $\{\operatorname{Re} z > -n-1\}$ (by the integral definition applied at $z+n+1$), and the denominator has simple zeros at $z = 0, -1, \dots, -n$. Hence the right side provides a meromorphic continuation of Γ to that half-plane with the stated simple poles. Letting $n \to \infty$ continues Γ to all of $\mathbb{C}$ minus the non-positive integers.

For the residue, use (3.4) with the same n:

$$\begin{aligned}
\lim_{z\to -n} (z+n)\Gamma(z) &= \lim_{z\to -n} \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)} \\
&= \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} = \frac{(-1)^n}{n!}.
\end{aligned}$$

□

Example 3.4 (Half-integer values). *First compute*

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\, dt = 2\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi},$$

using $t = x^2$ and the Gaussian integral (Lem. 2.10, with $\alpha = 2$). Then (3.2) *generates every half-integer value.*

Positive half-integers.

$$\begin{aligned}
\Gamma(3/2) &= \Gamma(1/2+1) = \tfrac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}. \\
\Gamma(5/2) &= \Gamma(3/2+1) = \tfrac{3}{2}\Gamma(3/2) = \tfrac{3}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}. \\
\Gamma(7/2) &= \tfrac{5}{2}\Gamma(5/2) = \tfrac{5}{2}\cdot\frac{3\sqrt{\pi}}{4} = \frac{15\sqrt{\pi}}{8}.
\end{aligned}$$

Thus $\Gamma(n+1/2) = (2n-1)!!\sqrt{\pi}/2^n$, *where* $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$ *and* $(-1)!! := 1$.

Negative half-integers.

$$\begin{aligned}
\Gamma(-1/2) &= \frac{\Gamma(1/2)}{-1/2} = \frac{\sqrt{\pi}}{-1/2} = -2\sqrt{\pi}. \\
\Gamma(-3/2) &= \frac{\Gamma(-1/2)}{-3/2} = \frac{-2\sqrt{\pi}}{-3/2} = \frac{4\sqrt{\pi}}{3}. \\
\Gamma(-5/2) &= \frac{\Gamma(-3/2)}{-5/2} = \frac{4\sqrt{\pi}/3}{-5/2} = -\frac{8\sqrt{\pi}}{15}.
\end{aligned}$$

In general, $\Gamma(-n+1/2) = (-1)^n\, 2^n \sqrt{\pi}/(2n-1)!!$ *for* $n \geq 1$.

Example 3.5 (Gaussian moments $\int_0^\infty x^n e^{-ax^2}\,dx$)**.** *For $a>0$ and $n \ge 0$ (integer* or *half-integer),*

$$I_n(a) \equiv \int_0^\infty x^n\, e^{-ax^2}\, dx = \frac{1}{2}\, a^{-(n+1)/2}\, \Gamma\!\left(\frac{n+1}{2}\right). \tag{3.5}$$

Derivation. *Substitute $u = ax^2$:*

$$\begin{aligned} I_n(a) &= \int_0^\infty (u/a)^{n/2}\, e^{-u}\, \tfrac{1}{2} a^{-1/2} u^{-1/2}\, du \\ &= \tfrac{1}{2} a^{-n/2} a^{-1/2} \int_0^\infty u^{(n-1)/2} e^{-u}\, du \\ &= \tfrac{1}{2} a^{-(n+1)/2}\, \Gamma\!\left(\tfrac{n+1}{2}\right), \end{aligned}$$

by Def. 3.1.

Integer n. *Using Ex. 3.4 and $\Gamma(k+1) = k!$:*

$$\begin{aligned} n=0: &\quad I_0(a) = \tfrac{1}{2} a^{-1/2} \Gamma(1/2) = \tfrac{1}{2}\sqrt{\pi/a}. \\ n=1: &\quad I_1(a) = \tfrac{1}{2} a^{-1} \Gamma(1) = \tfrac{1}{2a}. \\ n=2: &\quad I_2(a) = \tfrac{1}{2} a^{-3/2} \Gamma(3/2) = \tfrac{1}{2} a^{-3/2} \cdot \tfrac{\sqrt{\pi}}{2} = \tfrac{1}{4}\sqrt{\pi/a^3}. \\ n=3: &\quad I_3(a) = \tfrac{1}{2} a^{-2} \Gamma(2) = \tfrac{1}{2a^2}. \\ n=4: &\quad I_4(a) = \tfrac{1}{2} a^{-5/2} \Gamma(5/2) = \tfrac{1}{2} a^{-5/2} \cdot \tfrac{3\sqrt{\pi}}{4} = \tfrac{3}{8}\sqrt{\pi/a^5}. \end{aligned}$$

Half-integer $n = m + 1/2$. *Then $(n+1)/2 = m/2 + 3/4$, so the answer usually remains a Gamma value; for example $I_{1/2}(1) = \frac{1}{2}\Gamma(3/4)$.*

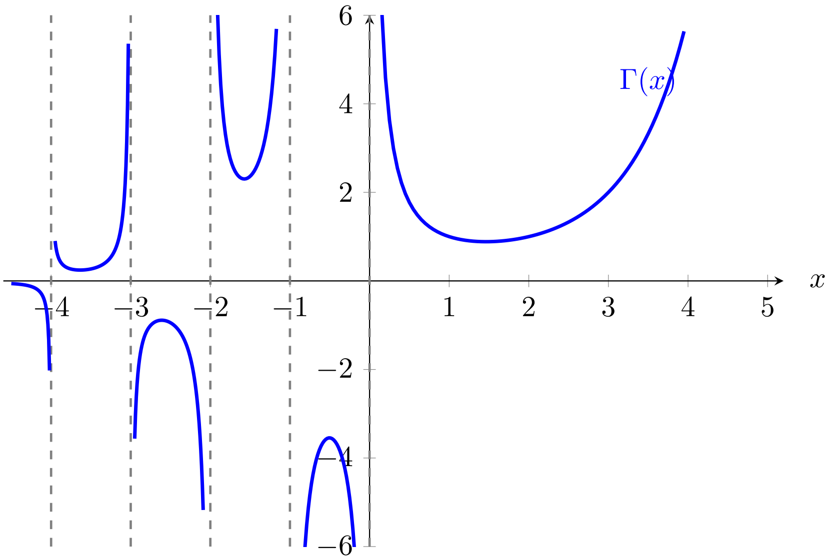

Figure 6: The Gamma function $\Gamma(x)$ on the real axis. Simple poles at $x = 0, -1, -2, \ldots$ with alternating signs. No zeros.

3.2 Weierstrass product and Euler's limit

The poles also suggest a product formula.

Theorem 3.6 (Euler's limit and Weierstrass product)**.** *For* $z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$*,*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}, \tag{3.6}$$

and equivalently

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}, \tag{3.7}$$

where

$$\gamma = \lim_{n\to\infty} \big(H_n - \ln n\big), \qquad H_n = \sum_{k=1}^{n} \frac{1}{k}, \tag{3.8}$$

is the Euler–Mascheroni constant*. The infinite product means the limit of the partial products obtained by multiplying the factors for* $n = 1,\dots,N$ *and then sending* $N \to \infty$*.*

Proof. Euler's limit for $\operatorname{Re} z > 0$*.* Define

$$\Gamma_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt. \tag{3.9}$$

Substitute $t = nu$, $dt = n\, du$:

$$\Gamma_n(z) = \int_0^1 (1-u)^n (nu)^{z-1}\, n\, du = n^z \int_0^1 (1-u)^n u^{z-1}\, du.$$

Set $A_m = \int_0^1 (1-u)^m u^{z+n-m-1}\, du$. Integration by parts, differentiating $(1-u)^m$ and integrating the power of u, gives

$$A_m = \frac{m}{z+n-m} A_{m-1}.$$

Indeed, take $U = (1-u)^m$ and $dV = u^{z+n-m-1}\, du$. Then

$$V = \frac{u^{z+n-m}}{z+n-m}, \qquad dU = -m(1-u)^{m-1}\, du.$$

The boundary term $UV|_0^1$ vanishes: at $u = 1$ the factor $(1-u)^m$ is zero, and at $u = 0$ the factor u^{z+n-m} tends to 0 because $\operatorname{Re} z > 0$. Therefore

$$A_m = -\int_0^1 V\, dU = \frac{m}{z+n-m} \int_0^1 (1-u)^{m-1} u^{z+n-m}\, du = \frac{m}{z+n-m} A_{m-1}.$$

Starting with $m = n$ and repeating down to $m = 0$,

$$\begin{aligned} \int_0^1 (1-u)^n u^{z-1}\, du &= \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)} \int_0^1 u^{z+n-1}\, du \\ &= \frac{n!}{z(z+1)\cdots(z+n)}. \end{aligned}$$

Hence

$$\Gamma_n(z) = \frac{n!\, n^z}{z(z+1)\cdots(z+n)}. \tag{3.10}$$

Since $(1-t/n)^n \mathbf{1}_{[0,n]}(t) \to e^{-t}$, where $\mathbf{1}_{[0,n]}(t)$ is the *indicator function* (equal to 1 if $t \in [0,n]$ and 0 otherwise), and for $0 \le t \le n$,

$$0 \le \left(1-\frac{t}{n}\right)^n \le e^{-t},$$

we have

$$\left|\left(1-\frac{t}{n}\right)^n \mathbf{1}_{[0,n]}(t)t^{z-1}\right| \le e^{-t}t^{\operatorname{Re}z-1}.$$

The right side is integrable when $\operatorname{Re}z > 0$, so dominated convergence gives

$$\lim_{n\to\infty} \Gamma_n(z) = \int_0^\infty e^{-t}t^{z-1}\,dt = \Gamma(z), \qquad \operatorname{Re}z > 0.$$

The point of the dominated-convergence step is that the integrands converge pointwise and are bounded by one integrable function independent of n. Comparing with (3.10) gives (3.6) on $\{\operatorname{Re}z > 0\}$. We now prove the product on this half-plane; its locally uniform convergence will then provide the continuation step.

Weierstrass product. Rewrite the right side of (3.6):

$$\frac{n!\,n^z}{z(z+1)\cdots(z+n)} = \frac{n^z}{z}\prod_{k=1}^n \frac{k}{z+k} = \frac{n^z}{z}\prod_{k=1}^n \frac{1}{1+z/k}.$$

For $\operatorname{Re}z > 0$, take reciprocals:

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} z\,n^{-z}\prod_{k=1}^n\left(1+\frac{z}{k}\right). \tag{3.11}$$

Insert $e^{z/k}e^{-z/k} = 1$ into each factor:

$$\begin{aligned} z\,n^{-z}\prod_{k=1}^n\left(1+\frac{z}{k}\right) &= z\,n^{-z}\left[\prod_{k=1}^n e^{z/k}\right]\prod_{k=1}^n\left(1+\frac{z}{k}\right)e^{-z/k} \\ &= z\,n^{-z}\,e^{zH_n}\prod_{k=1}^n\left(1+\frac{z}{k}\right)e^{-z/k}. \end{aligned}$$

Since $n^{-z}e^{zH_n} = e^{z(H_n-\ln n)}$ and $H_n - \ln n \to \gamma$, the exponential factor tends to $e^{\gamma z}$. On $|z| \le R$,

$$\left(1+\frac{z}{k}\right)e^{-z/k} = 1 - \frac{z^2}{2k^2} + O_R(k^{-3}),$$

where $O_R(k^{-3})$ means the implied constant depends on R but the bound is uniform for $|z| \le R$ (so the M-test applies on any compact set). Hence the product converges locally uniformly and defines an entire function

$$W(z) = z\,e^{\gamma z}\prod_{k=1}^\infty\left(1+\frac{z}{k}\right)e^{-z/k}.$$

Taking the limit in (3.11) gives $W(z) = 1/\Gamma(z)$ for $\operatorname{Re}z > 0$. Since Γ has already been continued meromorphically in Theorem 3.3, the meromorphic identity $W(z)\Gamma(z) = 1$ extends from the right half-plane to all of $\mathbb{C}$. Thus

$$\frac{1}{\Gamma(z)} = z\,e^{\gamma z}\prod_{k=1}^\infty\left(1+\frac{z}{k}\right)e^{-z/k},$$

which is (3.7). Finally, the finite products on the right of (3.11) converge to $1/\Gamma(z)$; at every point where Γ is finite this limit is nonzero, so taking reciprocals gives (3.6) on $\mathbb{C}\setminus\{0,-1,-2,\dots\}$. □

Remark 3.7. $1/\Gamma$ *is entire, with zeros exactly at* $z = 0, -1, -2, \dots$.

3.3 The Beta function

The Beta function packages integrals with two endpoint powers.

Definition 3.8 (Beta). *For* $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$,

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt. \tag{3.12}$$

An equivalent form is obtained by the substitution $t = u/(1+u)$, $dt = du/(1+u)^2$, $1-t = 1/(1+u)$:

$$B(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}\,du. \tag{3.13}$$

Proposition 3.9 (Beta–Gamma relation). *For* $\operatorname{Re} p, \operatorname{Re} q > 0$,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{3.14}$$

Proof. Start with $\Gamma(p)\Gamma(q)$ as a double integral over the first quadrant:

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \left(\int_0^\infty s^{p-1}e^{-s}\,ds\right)\left(\int_0^\infty t^{q-1}e^{-t}\,dt\right)\\ &= \iint_{[0,\infty)^2} s^{p-1}t^{q-1}e^{-(s+t)}\,ds\,dt,\end{aligned}$$

where the second equality is Fubini (absolutely convergent for $\operatorname{Re} p, \operatorname{Re} q > 0$).

Change variables $s = rx$, $t = r(1-x)$, with $r \in (0,\infty)$ and $x \in (0,1)$. The Jacobian determinant is

$$\frac{\partial(s,t)}{\partial(r,x)} = \begin{vmatrix} x & r \\ 1-x & -r \end{vmatrix} = -rx - r(1-x) = -r,$$

so $|\partial(s,t)/\partial(r,x)| = r$, and $s + t = r$. Therefore

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^\infty \int_0^1 (rx)^{p-1}(r(1-x))^{q-1}e^{-r}\,r\,dx\,dr\\ &= \left(\int_0^\infty r^{p+q-1}e^{-r}\,dr\right)\left(\int_0^1 x^{p-1}(1-x)^{q-1}\,dx\right)\\ &= \Gamma(p+q)B(p,q),\end{aligned}$$

which rearranges to (3.14). □

Example 3.10 (Beta at equal arguments). *Set* $q = p$ *in the Beta–Gamma identity* (3.14). *Then*

$$B(p,p) = \frac{\Gamma(p)\Gamma(p)}{\Gamma(p+p)} = \frac{\Gamma(p)^2}{\Gamma(2p)}.$$

For instance, $B(1/2,1/2) = \Gamma(1/2)^2/\Gamma(1) = \pi$.

Example 3.11 (Worked Beta evaluations). *Evaluate* $\int_0^1 t^3(1-t)^5\,dt$, $\int_0^1 t^{1/2}(1-t)^{1/2}dt$, *and* $\int_0^1 (1-t^4)^{1/2}dt$ *in closed form.*

(i) *Direct from* (3.12), $\int_0^1 t^3(1-t)^5\,dt = B(4,6) = \Gamma(4)\Gamma(6)/\Gamma(10) = (3!)(5!)/9! = (6)(120)/362880 = 720/362880 = 1/504$.

(ii) $\int_0^1 t^{1/2}(1-t)^{1/2}\,dt = B(3/2,3/2) = \Gamma(3/2)^2/\Gamma(3) = (\sqrt{\pi}/2)^2/2 = \pi/8.$
(iii) *Substitute* $u = t^4$, $du = 4t^3dt$, $dt = du/(4u^{3/4})$*:*

$$\int_0^1 (1-t^4)^{1/2}dt = \int_0^1 (1-u)^{1/2}\frac{du}{4u^{3/4}} = \tfrac{1}{4}\int_0^1 u^{-3/4}(1-u)^{1/2}\,du$$
$$= \tfrac{1}{4}B(1/4,3/2) = \tfrac{1}{4}\cdot\frac{\Gamma(1/4)\Gamma(3/2)}{\Gamma(7/4)}.$$

Using $\Gamma(3/2) = \sqrt{\pi}/2$ *and* $\Gamma(7/4) = \frac{3}{4}\Gamma(3/4)$*, this becomes* $\sqrt{\pi}\,\Gamma(1/4)/[6\Gamma(3/4)]$.

3.4 Reflection formula

Theorem 3.12 (Euler reflection)**.** *For* $z \in \mathbb{C}\setminus\mathbb{Z}$*,*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{3.15}$$

Proof. First take real $x \in (0,1)$. By Prop. 3.9,

$$\Gamma(x)\Gamma(1-x) = \Gamma(1)B(x,1-x) = B(x,1-x), \tag{3.16}$$

using $\Gamma(p+q) = \Gamma(1) = 1$. Insert the form (3.13) with $p = x$, $q = 1-x$:

$$B(x,1-x) = \int_0^\infty \frac{u^{x-1}}{1+u}\,du.$$

Ex. 1.33 evaluates this integral as

$$\int_0^\infty \frac{u^{x-1}}{1+u}\,du = \frac{\pi}{\sin(\pi x)}.$$

Thus (3.15) holds for real $x \in (0,1)$. Inside the vertical strip $0 < \operatorname{Re} z < 1$, both sides are holomorphic functions of z: $\Gamma(z)\Gamma(1-z)$ is holomorphic there, and $\sin(\pi z)$ has no zeros in the strip except on the real endpoints, which are not included. Since the two holomorphic functions agree on the real interval $(0,1)$, which has limit points in the strip, the identity theorem extends the formula to the whole strip.

To continue beyond the strip without worrying about poles, invert the identity. The functions

$$\frac{1}{\Gamma(z)\Gamma(1-z)} \quad\text{and}\quad \frac{\sin(\pi z)}{\pi},$$

are entire: the possible poles of $\Gamma(z)$ and $\Gamma(1-z)$ become zeros of the reciprocal, and $\sin(\pi z)$ is entire. They agree on the strip, so the identity theorem makes them agree on all of $\mathbb{C}$. Taking reciprocals away from the zeros of $\sin(\pi z)$ gives (3.15); at the zeros, the reciprocal statement records the corresponding poles of the gamma factors. □

Corollary 3.13 (Γ is zero-free)**.** *If* $z_0 \notin \mathbb{Z}$ *and* $\Gamma(z_0) = 0$*, then* (3.15) *gives a contradiction, since* $\pi/\sin(\pi z_0)$ *is finite and nonzero. If* $z_0 \in \mathbb{Z}_{\geq 1}$*, then* $\Gamma(z_0) = (z_0-1)! \neq 0$*; if* $z_0 \in \mathbb{Z}_{\leq 0}$*, then* z_0 *is a pole.*

Corollary 3.14 ($\Gamma(1/2)$)**.** $\Gamma(1/2) = \sqrt{\pi}$.

Proof. Set $z = 1/2$ in (3.15): $\Gamma(1/2)^2 = \pi/\sin(\pi/2) = \pi$. The integrand $t^{-1/2}e^{-t}$ in (3.1) is positive, so $\Gamma(1/2) > 0$; take the positive square root. □

3.5 Duplication formula

Theorem 3.15 (Legendre duplication). *For* $z \in \mathbb{C} \setminus \{0, -1/2, -1, -3/2, \dots\}$,

$$\Gamma(z)\,\Gamma\big(z + \tfrac{1}{2}\big) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{3.17}$$

Proof. Begin with $\operatorname{Re} z > 0$, so Prop. 3.9 applies at $p = q = z$:

$$B(z,z) = \frac{\Gamma(z)^2}{\Gamma(2z)} = \int_0^1 t^{z-1}(1-t)^{z-1}\,dt. \tag{3.18}$$

Substitute $t = (1+u)/2$, so $dt = du/2$, $1-t = (1-u)/2$, and $t(1-t) = (1-u^2)/4$. The integrand becomes

$$t^{z-1}(1-t)^{z-1} = \big(t(1-t)\big)^{z-1} = \left(\frac{1-u^2}{4}\right)^{z-1} = 4^{1-z}(1-u^2)^{z-1},$$

and the interval $t \in (0,1)$ maps to $u \in (-1,1)$:

$$B(z,z) = 4^{1-z}\int_{-1}^{1}(1-u^2)^{z-1}\,\frac{du}{2} = 2^{1-2z}\int_{-1}^{1}(1-u^2)^{z-1}\,du.$$

Since the integrand is even in u,

$$\int_{-1}^{1}(1-u^2)^{z-1}\,du = 2\int_0^1(1-u^2)^{z-1}\,du.$$

In the last integral put $v = u^2$, $dv = 2u\,du$, so $du = dv/(2\sqrt{v})$:

$$2\int_0^1(1-u^2)^{z-1}\,du = \int_0^1(1-v)^{z-1}\,v^{-1/2}\,dv = B(\tfrac{1}{2},z) = \frac{\Gamma(1/2)\,\Gamma(z)}{\Gamma(z+1/2)},$$

where the last equality is Prop. 3.9. Assembling,

$$\frac{\Gamma(z)^2}{\Gamma(2z)} = 2^{1-2z}\cdot\frac{\sqrt{\pi}\,\Gamma(z)}{\Gamma(z+1/2)},$$

using $\Gamma(1/2) = \sqrt{\pi}$ (Cor. 3.14). Since Γ has no zeros (Cor. 3.13), cancel one $\Gamma(z)$ and rearrange to obtain (3.17) on the half-plane $\operatorname{Re} z > 0$.

To extend beyond that half-plane, observe that both sides of (3.17) are meromorphic and zero-free on $\mathbb{C} \setminus \{0, -1/2, -1, -3/2, \dots\}$. Their reciprocals,

$$\frac{1}{\Gamma(z)\Gamma(z+1/2)} \qquad \text{and} \qquad \frac{2^{2z-1}}{\sqrt{\pi}\,\Gamma(2z)},$$

are entire and agree for $\operatorname{Re} z > 0$. The identity theorem therefore extends (3.17) to every z where both sides are defined. □

3.6 Hankel contour representation

Hankel's contour integral represents $1/\Gamma$ directly as an entire function.

Theorem 3.16 (Hankel)**.** *For all $z \in \mathbb{C}$,*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_H e^t\, t^{-z}\, dt, \tag{3.19}$$

where H is a Hankel contour*: it comes in from $-\infty$ just below the negative real axis, loops counter-clockwise around the origin along a small circle of some fixed radius $\rho > 0$, and returns to $-\infty$ just above the axis. The branch of $t^{-z} = e^{-z \log t}$ is the principal branch off the cut, with limiting boundary values* $\arg t = -\pi$ *on the lower edge and* $\arg t = \pi$ *on the upper edge. The value of the integral is independent of the chosen ρ.*

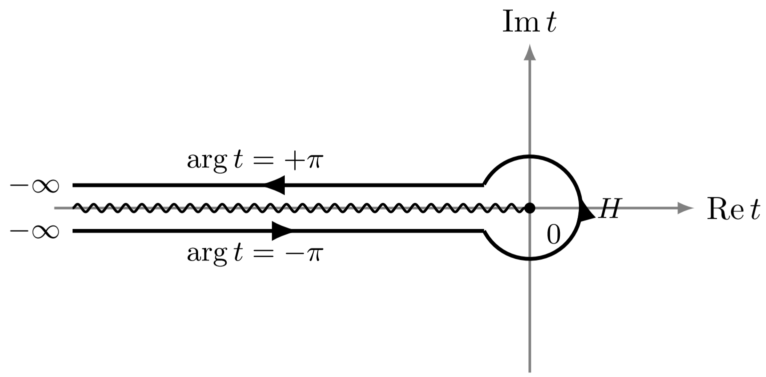

Figure 7: The Hankel contour H of Thm. 3.16. The branch cut runs along $(-\infty, 0]$; the contour comes in from $-\infty$ below the cut, circles the origin counterclockwise, and returns to $-\infty$ above the cut.

Proof. Intuition. The idea is to collapse the Hankel contour onto the two sides of the negative real axis. The integrand $e^t t^{-z}$ decays as $t \to -\infty$ because $e^t = e^{-s}$ on the negative real axis. We first assume $\operatorname{Re} z < 1$ so that the small circle around the origin contributes negligibly; once we obtain an identity on that half-plane, analytic continuation extends it to all z.

Assume $\operatorname{Re} z < 1$ throughout; the result then extends by analytic continuation (see end).

Collapse to the cut. On the small circle $|t| = \varepsilon$, the integrand is $O(\varepsilon^{-\operatorname{Re} z})$ and the length is $O(\varepsilon)$, so the contribution is $O(\varepsilon^{1-\operatorname{Re} z}) \to 0$.

On the upper edge write $t = s e^{i\pi}$, and on the lower edge write $t = s e^{-i\pi}$. Then $t = -s$, $e^t = e^{-s}$, and

$$\begin{aligned}
\int_{\text{lower}} e^t t^{-z}\, dt &= \int_{s=\infty}^{s=\varepsilon} e^{-s} (s\, e^{-i\pi})^{-z} (-ds) \\
&= \int_{\varepsilon}^{\infty} e^{-s} s^{-z} e^{i\pi z}\, ds, \\
\int_{\text{upper}} e^t t^{-z}\, dt &= \int_{s=\varepsilon}^{s=\infty} e^{-s} (s\, e^{i\pi})^{-z} (-ds) \\
&= -\int_{\varepsilon}^{\infty} e^{-s} s^{-z} e^{-i\pi z}\, ds,
\end{aligned}$$

Summing and taking $\varepsilon \to 0^+$,

$$\begin{aligned}
\int_H e^t t^{-z}\, dt &= \left(e^{i\pi z} - e^{-i\pi z}\right) \int_0^{\infty} e^{-s} s^{-z}\, ds \\
&= 2i \sin(\pi z)\, \Gamma(1-z),
\end{aligned}$$

where the last step recognizes the Euler integral (3.1) with argument $1-z$ (legitimate because $\mathrm{Re}(1-z)>0$).

Divide by $2\pi i$:

$$\frac{1}{2\pi i}\int_H e^t t^{-z}\,dt = \frac{\sin(\pi z)}{\pi}\Gamma(1-z). \tag{3.20}$$

Identify the result. By the reflection formula,

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)} \qquad \Longleftrightarrow \qquad \frac{\sin(\pi z)\,\Gamma(1-z)}{\pi}=\frac{1}{\Gamma(z)},$$

so (3.20) becomes (3.19).

Radius and continuation. If $0<\rho'<\rho$, the difference between the two Hankel integrals is the integral around the boundary of the slit annulus (a ring-shaped region cut open along the negative real axis)

$$\{\,t:\rho'\le |t|\le \rho,\ \arg t\in(-\pi,\pi)\,\},$$

which vanishes by Cauchy's theorem. For fixed ρ, differentiating in z only inserts powers of $-\log t$, still integrable on the rays because of e^{-s}. Thus the integral is entire in z. Since it agrees with $1/\Gamma(z)$ on $\mathrm{Re}\,z<1$, the identity theorem gives (3.19) for all z. □

3.7 Stirling's formula for complex z

Stirling's expansion follows from Euler's product limit and Euler–Maclaurin summation.

Bernoulli numbers and polynomials.

The *Bernoulli numbers* B_m are the Taylor coefficients of the generating function

$$\frac{w}{e^w-1}=\sum_{m=0}^{\infty}B_m\frac{w^m}{m!} \qquad (|w|<2\pi). \tag{3.21}$$

The left side is well-defined and holomorphic at $w=0$ even though e^w-1 vanishes there: $e^w-1=w+w^2/2+w^3/6+\cdots$ has a simple zero at $w=0$ that cancels the explicit w in the numerator, leaving the value 1 at the origin. The next nearest singularities are the zeros $w=\pm 2\pi i$ of e^w-1, fixing the radius of convergence at 2π. The first values are

$$B_0=1,\ \ B_1=-\tfrac{1}{2},\ \ B_2=\tfrac{1}{6},\ \ B_3=0,\ \ B_4=-\tfrac{1}{30},\ \ B_5=0,\ \ B_6=\tfrac{1}{42},\ldots \tag{3.22}$$

Also $B_{2k+1}=0$ for $k\ge 1$.

The *Bernoulli polynomials* $B_m(x)$ are defined by the two-variable generating function

$$\frac{w\,e^{xw}}{e^w-1}=\sum_{m=0}^{\infty}B_m(x)\frac{w^m}{m!}, \qquad B_m:=B_m(0). \tag{3.23}$$

Then $B_m=B_m(0)$, and differentiation in x gives

$$B'_{m+1}(x)=(m+1)B_m(x). \tag{3.24}$$

Also $B_m(1)=B_m(0)$ for $m\ne 1$, while $B_1(1)=1/2$.

The first few polynomials are $B_0(x)=1$, $B_1(x)=x-\frac{1}{2}$, $B_2(x)=x^2-x+\frac{1}{6}$, $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$.

Periodic Bernoulli functions.

Let $\{t\} = t - \lfloor t \rfloor \in [0,1)$ denote the fractional part. The *periodic Bernoulli function* of order m is

$$\widetilde{B}_m(t) := B_m(\{t\}), \tag{3.25}$$

the 1-periodic extension of $B_m|_{[0,1)}$. For $m \geq 2$ it is continuous and satisfies $\widetilde{B}'_{m+1} = (m+1)\widetilde{B}_m$ away from integers. Also $\widetilde{B}_1(t) = t - \lfloor t \rfloor - \frac{1}{2}$ off the integers.

Proposition 3.17 (Euler–Maclaurin). *Let $f \in C^{2K}[0,n]$. Then*

$$\begin{aligned} \sum_{k=1}^{n} f(k) = \int_0^n f(t)\,dt + \frac{f(n)-f(0)}{2} \\ + \sum_{m=1}^{K-1} \frac{B_{2m}}{(2m)!}\big(f^{(2m-1)}(n) - f^{(2m-1)}(0)\big) \\ - \int_0^n \frac{\widetilde{B}_{2K}(t)}{(2K)!} f^{(2K)}(t)\,dt. \end{aligned} \tag{3.26}$$

Proof. For each integer $k = 0, \ldots, n-1$, integrate by parts with $u = t - k$ ($du = dt$) and $dv = f'(t)\,dt$ ($v = f(t)$):

$$\int_k^{k+1} (t-k)f'(t)\,dt = \big[(t-k)f(t)\big]_k^{k+1} - \int_k^{k+1} f(t)\,dt = f(k+1) - \int_k^{k+1} f(t)\,dt.$$

Equivalently,

$$f(k+1) - \int_k^{k+1} f(t)\,dt = \int_k^{k+1} (t-k)f'(t)\,dt.$$

On $(k, k+1)$ we have $\widetilde{B}_1(t) = t - k - \frac{1}{2}$, so summing over k gives

$$\sum_{k=1}^{n} f(k) = \int_0^n f(t)\,dt + \frac{f(n)-f(0)}{2} + \int_0^n \widetilde{B}_1(t) f'(t)\,dt. \tag{3.27}$$

For $m \geq 1$, the periodic Bernoulli functions satisfy $\widetilde{B}'_{m+1}(t) = (m+1)\widetilde{B}_m(t)$ away from integers, and $\widetilde{B}_{m+1}(0) = \widetilde{B}_{m+1}(n) = B_{m+1}$ at the endpoints. Hence integration by parts yields

$$\int_0^n \widetilde{B}_m(t) f^{(m)}(t)\,dt = \frac{B_{m+1}}{m+1}\big(f^{(m)}(n) - f^{(m)}(0)\big) - \int_0^n \frac{\widetilde{B}_{m+1}(t)}{m+1} f^{(m+1)}(t)\,dt.$$

Starting from $m = 1$ in (3.27) and repeating this step $2K-1$ times gives the full formula. Since $B_{2m+1} = 0$ for every $m \geq 1$, only even Bernoulli numbers remain. □

Theorem 3.18 (Stirling). *As $|z| \to \infty$ in any sector $|\arg z| \leq \pi - \delta$ with $\delta > 0$, and with $\log z$ denoting the principal branch,*

$$\log\Gamma(z+1) = \big(z + \tfrac{1}{2}\big)\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{m=1}^{K-1} \frac{B_{2m}}{2m(2m-1)z^{2m-1}} + R_K(z), \tag{3.28}$$

where

$$R_K(z) = -\frac{1}{2K}\int_0^\infty \frac{\widetilde{B}_{2K}(t)}{(z+t)^{2K}}\,dt, \tag{3.29}$$

and $|R_K(z)| \leq C_{K,\delta}|z|^{1-2K}$. Equivalently,

$$\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \cdots\right). \tag{3.30}$$

Proof. Strategy. Take the log of Euler's limit formula (3.6), apply the Euler–Maclaurin summation formula to the resulting sum of logs, let $n \to \infty$, and isolate the constant and correction terms. The known real Stirling result (Ex. 2.13) gives the leading constant $\sqrt{2\pi}$. The error estimate for the remainder term bounds the subleading corrections.

Fix $K \geq 1$ and use the principal branch of log. By Euler's limit,

$$\log\Gamma(z+1) = \lim_{n\to\infty}\left[\log(n!) + z\log n - \sum_{k=1}^{n}\log(z+k)\right]. \tag{3.31}$$

The sector condition keeps the ray $z+[0,\infty)$ off the principal branch cut $(-\infty,0]$. Thus the principal logarithm is single-valued along this ray, and the derivatives of $f_z(t) := \log(z+t)$ are the ordinary derivatives of one fixed branch. Apply Proposition 3.17 to f_z on $[0,n]$. Since

$$f_z^{(m)}(t) = (-1)^{m-1}\frac{(m-1)!}{(z+t)^m},$$

we obtain

$$\sum_{k=1}^{n}\log(z+k) = \int_0^n \log(z+t)\,dt + \frac{\log(z+n)-\log z}{2} + \sum_{m=1}^{K-1}\frac{B_{2m}}{2m(2m-1)}\left((z+n)^{1-2m} - z^{1-2m}\right) + E_{K,n}(z),$$

where

$$E_{K,n}(z) = \frac{1}{2K}\int_0^n \frac{\widetilde{B}_{2K}(t)}{(z+t)^{2K}}\,dt.$$

Also

$$\int_0^n \log(z+t)\,dt = \left[(z+t)\log(z+t) - (z+t)\right]_{t=0}^{t=n} = (z+n)\log(z+n) - z\log z - n.$$

Therefore

$$\sum_{k=1}^{n}\log(z+k) = \left(z+n+\tfrac{1}{2}\right)\log(z+n) - \left(z+\tfrac{1}{2}\right)\log z - n + \sum_{m=1}^{K-1}\frac{B_{2m}}{2m(2m-1)}\left((z+n)^{1-2m} - z^{1-2m}\right) + E_{K,n}(z).$$

Substituting this into (3.31) gives

$$\log\Gamma(z+1) = \left(z+\tfrac{1}{2}\right)\log z + \sum_{m=1}^{K-1}\frac{B_{2m}}{2m(2m-1)}z^{1-2m} + \lim_{n\to\infty}A_n(z) - \lim_{n\to\infty}E_{K,n}(z),$$

where

$$A_n(z) := \log(n!) + z\log n - \left(z+n+\tfrac{1}{2}\right)\log(z+n) + n - \sum_{m=1}^{K-1}\frac{B_{2m}}{2m(2m-1)}(z+n)^{1-2m}.$$

Use the real Stirling asymptotic $\log(n!) = (n+\frac{1}{2})\log n - n + \frac{1}{2}\log(2\pi) + o(1)$ (Ex. 2.13). Since

$$\left(z+n+\tfrac{1}{2}\right)\log(z+n) = \left(z+n+\tfrac{1}{2}\right)\left(\log n + \log\left(1+\frac{z}{n}\right)\right),$$

the $\log n$ terms cancel, $(z+n+\frac{1}{2})\log(1+z/n) \to z$, and $(z+n)^{1-2m} \to 0$. Hence

$$\lim_{n\to\infty} A_n(z) = -z + \tfrac{1}{2}\log(2\pi).$$

Also $E_{K,n}(z) \to -R_K(z)$, because R_K was defined with the opposite sign and the upper limit extended to ∞. Hence (3.28) follows.

To bound the remainder uniformly in the sector $|\arg z| \le \pi-\delta$, let $r=|z|$. For $t \ge 0$,

$$|z+t|^2 = r^2+t^2+2rt\cos(\arg z) \ge (r+t)^2\sin^2(\delta/2),$$

so $|z+t| \ge (r+t)\sin(\delta/2)$. If $M_{2K} := \sup_{x\in[0,1]}|B_{2K}(x)|$, then

$$|R_K(z)| \le \frac{M_{2K}}{2K\,\sin^{2K}(\delta/2)}\int_0^\infty \frac{dt}{(r+t)^{2K}} = \frac{M_{2K}}{2K(2K-1)\sin^{2K}(\delta/2)}\,r^{1-2K},$$

which is the stated estimate.

Finally, exponentiating (3.28) gives (3.30); the first terms are obtained from

$$\exp\!\left(\frac{1}{12z}-\frac{1}{360z^3}+\cdots\right) = 1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840\,z^3}+\cdots.$$

□

Example 3.19 (Numerical check of Stirling). *Compute* $\ln\Gamma(10) = \ln(9!) = \ln 362880 \approx 12.80183$. *In* (3.28) *we use* $z=9$, *because* $\Gamma(10)=\Gamma(z+1)$ *with* $z=9$. *The leading approximation is*

$$\left(z+\tfrac{1}{2}\right)\ln z - z + \tfrac{1}{2}\ln(2\pi)\Big|_{z=9} = 9.5\ln 9 - 9 + \tfrac{1}{2}\ln(2\pi) \approx 12.79257.$$

The first correction is $1/(12z) = 1/108 \approx 0.009259$, *giving* 12.80183 *to five digits. The next correction is*

$$\frac{B_4}{4\cdot 3z^3} = -\frac{1}{360\cdot 9^3} \approx -3.81\times 10^{-6},$$

so the two-correction approximation is unchanged at the displayed precision. The point of the example is that the first correction already explains the visible error in the leading Stirling approximation.

3.8 The digamma function

Definition 3.20 (Digamma). *For* $z \notin \{0,-1,-2,\dots\}$, *define*

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\log\Gamma(z).$$

The function ψ *is meromorphic; it has simple poles at the poles of* Γ.

Taking the logarithmic derivative of the Weierstrass product (3.7),

$$-\log\Gamma(z) = \log z + \gamma z + \sum_{n=1}^{\infty}\left[\log\!\left(1+\frac{z}{n}\right)-\frac{z}{n}\right],$$

$$-\psi(z) = \frac{1}{z}+\gamma+\sum_{n=1}^{\infty}\left[\frac{1}{n+z}-\frac{1}{n}\right],$$

Combine the $-\gamma$ and $-1/z$ terms with the telescoping series, then re-index $n \mapsto n+1$ to absorb $-1/z$ as the $n=0$ term:

$$\psi(z) = -\gamma-\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) = -\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right). \tag{3.32}$$

Proposition 3.21 (Special values of ψ).

$$\psi(1) = -\gamma, \qquad \psi(1/2) = -\gamma - 2\ln 2. \tag{3.33}$$

The recurrence is $\psi(z+1) = \psi(z) + 1/z$; the reflection identity is $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$.

Proof. Value at 1. Set $z = 1$ in (3.32):

$$\psi(1) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+1}\right) = -\gamma.$$

Value at $1/2$. Set $z = 1/2$ in (3.32):

$$\psi(1/2) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+1/2}\right) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{2}{2n+1}\right).$$

The partial sum up to N:

$$S_N = \sum_{n=0}^{N}\frac{1}{n+1} - 2\sum_{n=0}^{N}\frac{1}{2n+1} = \sum_{k=1}^{N+1}\frac{1}{k} - 2\sum_{k=1,\,k\text{ odd}}^{2N+1}\frac{1}{k}.$$

Write the second sum as (sum of reciprocals $1,\dots,2N+1$) − (sum of even reciprocals):

$$\sum_{k=1,\,k\text{ odd}}^{2N+1}\frac{1}{k} = \sum_{k=1}^{2N+1}\frac{1}{k} - \sum_{k=1}^{N}\frac{1}{2k} = H_{2N+1} - \tfrac{1}{2}H_N.$$

Hence

$$S_N = H_{N+1} - 2\big(H_{2N+1} - \tfrac{1}{2}H_N\big) = H_{N+1} + H_N - 2H_{2N+1}.$$

Use $H_M = \ln M + \gamma + o(1)$:

$$\begin{aligned} S_N &= \big[\ln(N+1) + \gamma\big] + \big[\ln N + \gamma\big] - 2\big[\ln(2N+1) + \gamma\big] + o(1) \\ &= \ln\frac{N(N+1)}{(2N+1)^2} + o(1) \to \ln\frac{1}{4} = -2\ln 2. \end{aligned}$$

Therefore $\psi(1/2) = -\gamma + (-2\ln 2) = -\gamma - 2\ln 2$.

Recurrence and reflection. Differentiating $\log\Gamma(z+1) = \log z + \log\Gamma(z)$ gives $\psi(z+1) = 1/z + \psi(z)$. Differentiating $\log\Gamma(z) + \log\Gamma(1-z) = \log\pi - \log\sin(\pi z)$ (log of Thm. 3.12) gives $\psi(z) - \psi(1-z) = -\pi\cot(\pi z)$, i.e. $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$. □

3.9 The Pochhammer symbol

Iterating $\Gamma(a+n) = (a+n-1)\cdots(a+1)a\,\Gamma(a)$ gives a rising product.

Definition 3.22 (Pochhammer symbol). $(a)_0 = 1$, *and for* $n \in \mathbb{Z}_{\geq 1}$

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

Since each factor shifts the argument of Γ up by one, the functional equation (3.2) gives

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{3.34}$$

for all a such that the two Gamma values in the quotient are finite. If a is a non-positive integer, one should keep the finite product as the primary definition: the Gamma quotient may contain poles in both numerator and denominator even though the product $a(a+1)\cdots(a+n-1)$ is perfectly well defined.

Example 3.23 ($(1/2)_n$ and $(-n)_k$). The half-integer case. *By the shift and Cor. 3.14,*

$$(1/2)_n = \frac{\Gamma(n+1/2)}{\Gamma(1/2)} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}}.$$

Iterating the functional equation,

$$\Gamma(n+1/2) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})\cdots\tfrac{1}{2}\cdot\Gamma(1/2) = \frac{(2n-1)(2n-3)\cdots 1}{2^n}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi},$$

so

$$(1/2)_n = \frac{(2n-1)!!}{2^n} = \frac{(2n)!}{4^n\, n!}, \tag{3.35}$$

using $(2n)! = (2n)!!\,(2n-1)!! = 2^n n!\,(2n-1)!!$.

Negative-integer first argument. *For* $n \in \mathbb{Z}_{\geq 0}$ *and* $k \in \mathbb{Z}_{\geq 0}$,

$$(-n)_k = (-n)(-n+1)\cdots(-n+k-1).$$

If $k \leq n$, *every factor is nonzero and equals* $(-1)^k$ *times* $n(n-1)\cdots(n-k+1) = n!/(n-k)!$:

$$(-n)_k = (-1)^k \frac{n!}{(n-k)!} = (-1)^k\, k! \binom{n}{k}, \qquad 0 \leq k \leq n. \tag{3.36}$$

If $k > n$, *one factor is* 0, *so* $(-n)_k = 0$.

3.10 Physical applications

Three common uses:

Example 3.24 (Volume of the unit d-ball). *Let* $V_d = \mathrm{vol}\{x \in \mathbb{R}^d : |x| \leq 1\}$. *Set*

$$I_d = \int_{\mathbb{R}^d} e^{-|x|^2}\, d^d x. \tag{3.37}$$

By Fubini and the one-dimensional Gaussian integral,

$$I_d = \prod_{j=1}^{d} \int_{-\infty}^{\infty} e^{-x_j^2}\, dx_j = \pi^{d/2}. \tag{3.38}$$

In polar coordinates, the volume of a ball of radius r *is* $V_d r^d$. *Therefore the radial density of volume is the derivative*

$$\frac{d}{dr}\big(V_d r^d\big) = d V_d r^{d-1}.$$

Equivalently, a thin shell of thickness dr *has volume* $dV_d r^{d-1}\, dr$ *to first order. Hence*

$$I_d = d\, V_d \int_0^\infty e^{-r^2}\, r^{d-1}\, dr. \tag{3.39}$$

Substitute $u = r^2$, $du = 2r\, dr$, $r^{d-1}\, dr = \frac{1}{2} u^{d/2-1}\, du$:

$$\int_0^\infty e^{-r^2} r^{d-1}\, dr = \frac{1}{2}\int_0^\infty e^{-u} u^{d/2-1}\, du = \frac{1}{2}\Gamma(d/2),$$

by Def. 3.1. Combining with (3.38) *and* (3.39),

$$\pi^{d/2} = d\, V_d \cdot \tfrac{1}{2}\Gamma(d/2) = V_d \cdot \tfrac{d}{2}\Gamma(d/2) = V_d\, \Gamma(d/2+1),$$

using $\frac{d}{2}\Gamma(d/2) = \Gamma(d/2+1)$ *(Prop. 3.2). Therefore*

$$V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}. \tag{3.40}$$

For example, $V_1 = 2$*,* $V_2 = \pi$*,* $V_3 = 4\pi/3$*, and* $V_4 = \pi^2/2$*.*

Surface area. *Since* $S_{d-1} = dV_d$*,*

$$S_{d-1} = \frac{d\,\pi^{d/2}}{\Gamma(d/2+1)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{3.41}$$

using $\Gamma(d/2+1) = (d/2)\Gamma(d/2)$*. Checks:* $S_0 = 2$*,* $S_1 = 2\pi$*,* $S_2 = 4\pi$*,* $S_3 = 2\pi^2$*.*

Example 3.25 (Trig form of Beta and classical integrals). *Substitute* $t = \sin^2\theta$ *in* (3.12)*:* $dt = 2\sin\theta\cos\theta\,d\theta$*,* $t^{p-1} = \sin^{2p-2}\theta$*,* $(1-t)^{q-1} = \cos^{2q-2}\theta$*, and* $t \in (0,1) \longleftrightarrow \theta \in (0,\pi/2)$*. Then*

$$\begin{aligned} B(p,q) &= \int_0^{\pi/2} \sin^{2p-2}\theta\cos^{2q-2}\theta \cdot 2\sin\theta\cos\theta\,d\theta \\ &= 2\int_0^{\pi/2} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta, \end{aligned}$$

so

$$\int_0^{\pi/2} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta = \frac{1}{2}B(p,q) = \frac{\Gamma(p)\Gamma(q)}{2\Gamma(p+q)}. \tag{3.42}$$

At $p = q = 1/2$*, both sides equal* $\pi/2$*. Also,*

$$\begin{aligned} \int_0^{\pi/2} \sin^{2n}\theta\,d\theta &= \tfrac{1}{2}B(n+1/2,1/2) = \frac{\Gamma(n+1/2)\sqrt{\pi}}{2\,\Gamma(n+1)} = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}, \\ \int_0^{\pi/2} \sin^{2n+1}\theta\,d\theta &= \tfrac{1}{2}B(n+1,1/2) = \frac{n!\,\sqrt{\pi}}{2\,\Gamma(n+3/2)} = \frac{(2n)!!}{(2n+1)!!}, \end{aligned}$$

where the last forms use $\Gamma(n+1) = n!$*,* $\Gamma(n+1/2) = (2n-1)!!\sqrt{\pi}/2^n$*, and* $(2n)!! = 2^n n!$*. These formulas reduce even and odd powers of sine to Gamma values.*

- **Planck spectrum.** The Stefan–Boltzmann law reduces to $\int_0^\infty x^3/(e^x-1)\,dx = \Gamma(4)\,\zeta(4) = 6\cdot\pi^4/90 = \pi^4/15$, where $\zeta(s) = \sum_{n\ge1} n^{-s}$ for $\mathrm{Re}\,s > 1$ and $\zeta(4) = \pi^4/90$. The integral identity follows by expanding $1/(e^x-1) = \sum_{n\ge1} e^{-nx}$ for $x > 0$. Since the terms x^3e^{-nx} are nonnegative, Tonelli's theorem—the theorem that permits interchanging a sum and an integral for nonnegative terms—justifies exchanging the infinite sum and the integral:

$$\int_0^\infty x^3e^{-nx}\,dx = \frac{1}{n^4}\int_0^\infty u^3e^{-u}\,du = \frac{\Gamma(4)}{n^4},$$

  where $u = nx$. Summing over n gives $\Gamma(4)\sum_{n\ge1} n^{-4} = \Gamma(4)\zeta(4)$.

- **Quantum harmonic oscillator normalization.** Hermite polynomial norms involve $\Gamma(n+1/2) = (2n-1)!!\,\sqrt{\pi}/2^n$ from (3.35).

Exercises

Problem 3.1. *Show* $\Gamma(z+1/2) = (2z-1)!!\,\sqrt{\pi}/2^z$ *for* $z \in \mathbb{N}$ *(with the convention* $(-1)!! := 1$ *for* $z = 0$*), and give the analogous formula for* $z \in \mathbb{N} + 1/2$*.*

Problem 3.2. *Evaluate* $\int_0^{\pi/2} \sin^{2p-1}\theta\, \cos^{2q-1}\theta\, d\theta$ *in closed form. (Substitute* $t = \sin^2\theta$*; recognise the Beta integral.)*

Problem 3.3. *Prove* $\Gamma(n+1) < n^{n+1/2}e^{-n}\sqrt{2\pi}\, e^{1/(12n)}$ *for* $n \geq 1$ *using the explicit error bound of Stirling's series.*

Problem 3.4. *Derive the digamma reflection* $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$ *by taking the logarithmic derivative of Thm. 3.12.*

Problem 3.5. *Let* F *be holomorphic near* $z = -n$*. Prove that* $\operatorname{Res}_{z=-n}[\Gamma(z)F(z)] = (-1)^n F(-n)/n!$*. Apply this to* $F(z) = e^z$ *and to* $F(z) = 1/(z-1)$*.*

Problem 3.6. *Verify the duplication formula* (3.17) *at* $z = 1/2$*: both sides should equal* $\sqrt{\pi}$*.*

Problem 3.7. *Use the Hankel representation* (3.19) *to show that* $1/\Gamma(z)$ *has a simple zero at each* $z = -n$ *with derivative* $(1/\Gamma)'(-n) = (-1)^n n!$*, and deduce* $\operatorname{Res}_{z=-n}\Gamma = (-1)^n/n!$*. Hint: write* $z = -n + \varepsilon$*, expand* $t^{-z} = t^n e^{-\varepsilon \log t}$*, and deform* H *onto a small circle around* $t = 0$ *to isolate the term linear in* ε*.*

Problem 3.8 (Gaussian moments)**.** *Using Ex. 3.5, evaluate*

$$\int_0^\infty x^{2k}e^{-x^2}\,dx \quad \text{and} \quad \int_{-\infty}^\infty x^{2k}e^{-x^2}\,dx \qquad (k \in \mathbb{N})$$

in closed form. Check your answer against the operator identity $\int_{-\infty}^\infty x^{2k}e^{-ax^2}dx = (-\partial_a)^k\sqrt{\pi/a}$ *evaluated at* $a = 1$*.*

Problem 3.9 (Four-dimensional ball)**.** *From* (3.40)*, compute* V_4 *and the surface area* S_3 *of the unit* 3*-sphere in* $\mathbb{R}^4$ *(use* $S_{d-1} = d\,V_d$*). Verify* $V_4 = \int_{-1}^1 V_3\,(1-x^2)^{3/2}\,dx$ *by carrying out the one-dimensional integral (substitute* $x = \sin\theta$*; reduce to a Beta integral).*

Problem 3.10 (Beta at half-integer)**.** *Evaluate* $B(1/2, n+1/2)$ *in closed form for* $n \in \mathbb{N}$*, and interpret the result as* $\int_0^{\pi/2}\cos^{2n}\theta\,d\theta$ *(use the trig form from Problem 3.2).*

Problem 3.11 (Wallis's product via duplication + Stirling)**.** *Evaluate Legendre duplication* (3.17) *at* $z = n$ *(integer* $n \geq 1$*). Take the ratio of the duplication identity at* $n+1$ *and at* n*, use Stirling* (3.30)*, and deduce*

$$\prod_{k=1}^\infty \frac{4k^2}{4k^2-1} = \frac{\pi}{2}.$$

Problem 3.12 (Digamma and harmonic sums)**.** *Iterate* $\psi(z+1) = \psi(z) + 1/z$ *starting from* $\psi(1) = -\gamma$ *to obtain* $\psi(n+1) = H_n - \gamma$ *for* $n \in \mathbb{N}$*, where* $H_n = \sum_{k=1}^n 1/k$*. Then use digamma duplication* $\psi(2z) = \frac{1}{2}[\psi(z) + \psi(z+1/2)] + \log 2$ *(Problem: derive this from log-differentiating Thm. 3.15) to evaluate* $\psi(3/2)$ *and* $\psi(5/2)$ *in closed form.*

Problem 3.13 (Stirling correction by direct expansion)**.** *Starting from the local expansion*

$$f(1+\sigma) = -1 - \frac{\sigma^2}{2} + \frac{\sigma^3}{3} - \frac{\sigma^4}{4} + \frac{\sigma^5}{5} - \frac{\sigma^6}{6} + O(\sigma^7)$$

for $f(s) = \ln s - s$ *and the Gaussian-moment identity* $\int \tau^{2k}e^{-\tau^2/2}d\tau = (2k-1)!!\sqrt{2\pi}$*, reproduce the coefficient* $1/288$ *of the* z^{-2} *correction in* (3.30)*. You will need to keep* $\tau^3/\sqrt{z}, \tau^4/z, \tau^5/z^{3/2}, \tau^6/z^2$ *terms in the expansion of* $e^{zf(1+\sigma)}$*, square* τ^3*, and cross-multiply consistently.*

4 Bessel Functions

Bessel functions are the natural harmonics of cylindrical problems: drum modes, electromagnetic scattering by a wire, and the radial Schrödinger equation in a 2D central potential. We take the complex-analytic route: define J_n by a generating function, extract coefficients by Cauchy's formula, then derive the series, recurrences, integral representations, and ODE from that one object. The general-order functions J_ν, Y_ν, $H_\nu^{(1,2)}$, I_ν, and K_ν follow afterward.

Prerequisites. This section builds on:

- Laurent expansion and Laurent-coefficient formula (Theorem 1.21) and the residue theorem (Theorem 1.25) from Section 1.
- The steepest-descent method (Theorem 2.17, Proposition 2.16) and the identity theorem (Theorem 2.2) from Section 2.
- The Gamma function's functional equation (Proposition 3.2), the Beta–Gamma identity (Proposition 3.9), the Legendre duplication formula (Theorem 3.15), and the Hankel-contour representation of $1/\Gamma$ (Theorem 3.16) from Section 3.

4.1 The generating function for integer order

A *generating function* stores a whole sequence in one analytic function. The choice below is natural because, when $t = e^{i\theta}$, it becomes $e^{iz\sin\theta}$, so $J_n(z)$ is the n-th Fourier coefficient of a plane wave in cylindrical harmonics.

Definition 4.1 (Bessel generating function). *For $z \in \mathbb{C}$ and $t \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, set*

$$G(z,t) := \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right]. \tag{4.1}$$

For fixed z, the map $t \mapsto (z/2)(t - 1/t)$ is holomorphic on $\mathbb{C}^*$, and exp is entire, so $G(z,\cdot)$ is holomorphic on $\mathbb{C}^*$ with an essential singularity at $t = 0$ (and at $t = \infty$). It therefore admits a Laurent expansion convergent on every annulus, i.e. every ring-shaped region $0 < r_1 < |t| < r_2$:

Definition 4.2 (Bessel J_n, integer order). *For $n \in \mathbb{Z}$, $J_n(z)$ is the coefficient of t^n in the Laurent expansion of $G(z,t)$:*

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n. \tag{4.2}$$

By the Laurent coefficient formula (Theorem 1.21):

$$J_n(z) = \frac{1}{2\pi i} \oint_{|t|=1} \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] t^{-n-1}\, dt. \tag{4.3}$$

This is the working definition for integer order.

4.2 Series expansion

Proposition 4.3 (Power series). *For $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $z \in \mathbb{C}$,*

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{z}{2}\right)^{n+2k}. \tag{4.4}$$

Moreover, $J_{-n}(z) = (-1)^n J_n(z)$ for every $n \in \mathbb{N}_0$.

Proof. Step 1 (factorization). Since $zt/2$ and $-z/(2t)$ commute as scalars, $e^{A+B} = e^A e^B$ gives

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \exp\left(\frac{zt}{2}\right) \cdot \exp\left(-\frac{z}{2t}\right). \tag{4.5}$$

Each factor is an absolutely convergent power/Laurent series in t for $t \neq 0$, so we may multiply:

$$\exp\left(\frac{zt}{2}\right)\exp\left(-\frac{z}{2t}\right) = \left(\sum_{p=0}^{\infty} \frac{1}{p!}\left(\frac{z}{2}\right)^p t^p\right)\left(\sum_{q=0}^{\infty} \frac{(-1)^q}{q!}\left(\frac{z}{2}\right)^q t^{-q}\right). \tag{4.6}$$

The double series of absolute values converges on any annulus, so we may multiply the two series term by term and collect powers of t in any order.

Step 2 (extract the t^n coefficient). Pairs (p,q) contributing to t^n satisfy $p - q = n$. For $n \geq 0$, the constraint $p, q \geq 0$ forces $q \geq 0$, $p = n + q$. Writing $q = k$: (Here and throughout, $[t^n]F(z,t)$ denotes the coefficient of t^n in the Laurent expansion of F in t.)

$$[t^n]G(z,t) = \sum_{k=0}^{\infty} \frac{1}{(n+k)!\,k!}\left(\frac{z}{2}\right)^{n+k} \cdot (-1)^k \left(\frac{z}{2}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{z}{2}\right)^{n+2k}, \tag{4.7}$$

which is (4.4).

Step 3 (negative index: J_{-n}). Fix $m \in \mathbb{N}$, $m \geq 1$, and compute $[t^{-m}]G(z,t)$. The constraint is $p - q = -m$, i.e. $q = p + m$; since $p \geq 0$, we have $q \geq m$. Let $\ell := p$, so $q = \ell + m$ with $\ell \geq 0$:

$$[t^{-m}]G(z,t) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(\frac{z}{2}\right)^{\ell} \cdot \frac{(-1)^{\ell+m}}{(\ell+m)!}\left(\frac{z}{2}\right)^{\ell+m}. \tag{4.8}$$

Collect the $(z/2)$ powers and the signs:

$$[t^{-m}]G(z,t) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+m}}{\ell!\,(\ell+m)!}\left(\frac{z}{2}\right)^{m+2\ell}. \tag{4.9}$$

Factor $(-1)^{\ell+m} = (-1)^m \cdot (-1)^{\ell}$:

$$[t^{-m}]G(z,t) = (-1)^m \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!\,(\ell+m)!}\left(\frac{z}{2}\right)^{m+2\ell} = (-1)^m J_m(z), \tag{4.10}$$

where the last equality uses (4.4) with $n = m$, $k = \ell$. Since $J_{-m}(z) = [t^{-m}]G(z,t)$ by (4.2), we conclude $J_{-m}(z) = (-1)^m J_m(z)$. □

For non-integer order, replace factorials with Gamma functions:

Definition 4.4 (J_ν for arbitrary $\nu \in \mathbb{C}$). *For $\nu \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$, define*

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{\nu+2k}, \tag{4.11}$$

where $(z/2)^\nu = e^{\nu \log(z/2)}$ uses the principal logarithm (branch cut on $(-\infty, 0]$).

For non-integer ν, the point $z = 0$ is a branch point, meaning a point around which analytic continuation changes the chosen branch, because of the factor z^ν. For integer $\nu = n$, the powers are ordinary integer powers and the function extends through 0 to an entire function.

For $\nu = -m$ with $m \in \mathbb{N}$, the first m terms vanish because $1/\Gamma(-m+k+1) = 0$ for $0 \leq k < m$; here $1/\Gamma$ is understood as the entire reciprocal-Gamma function from Theorem 3.16. The remaining series reproduces $(-1)^m J_m$. Convergence: for $k \to \infty$ the ratio of the $(k+1)$-th term to the k-th is $-(z/2)^2/[(k+1)(\nu+k+1)] \to 0$; by the ratio test the series (4.11) therefore converges absolutely for every fixed $z \in \mathbb{C} \setminus (-\infty, 0]$. Thus J_ν is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$, and integer-order J_n is entire.

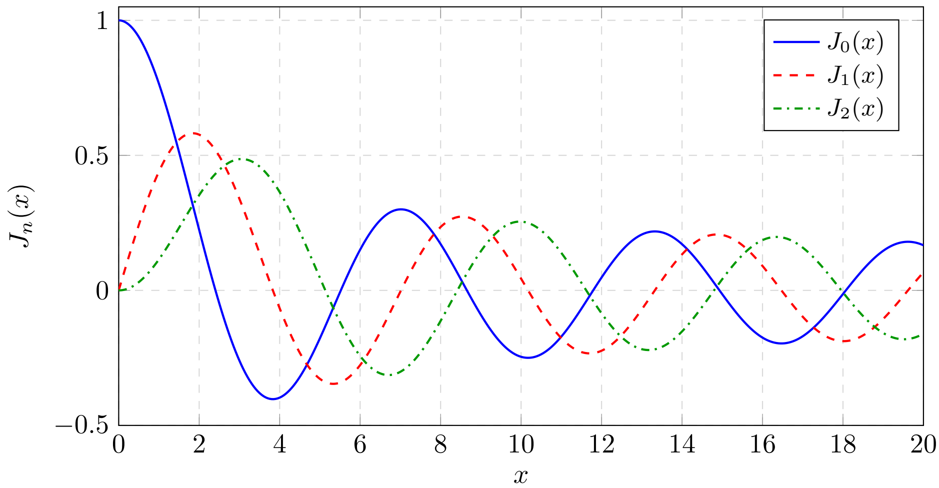

Figure 8: The first three Bessel functions of the first kind on the real axis. The series (4.11) yields $J_0(0) = 1$ and $J_n(0) = 0$ for $n \geq 1$. All three oscillate with slowly decaying amplitude as x grows.

4.3 Recurrences

Proposition 4.5 (Basic recurrences). *For $\nu \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$,*

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z), \tag{4.12}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z), \tag{4.13}$$

$$\frac{d}{dz}\big[z^\nu J_\nu(z)\big] = z^\nu J_{\nu-1}(z), \tag{4.14}$$

$$\frac{d}{dz}\big[z^{-\nu} J_\nu(z)\big] = -z^{-\nu} J_{\nu+1}(z). \tag{4.15}$$

Proof. Step 1: (4.12) *and* (4.13) *for integer $\nu = n$ via generating function.* Differentiate (4.1) in t. The inner derivative is $(z/2)(1 + 1/t^2)$, so

$$\frac{\partial G}{\partial t} = \frac{z}{2}\left(1 + \frac{1}{t^2}\right) G(z,t) = \sum_{n=-\infty}^{\infty} n J_n(z)\, t^{n-1}, \tag{4.16}$$

where on the right we differentiated (4.2) term by term (justified because the Laurent series converges uniformly on compact subsets of $\mathbb{C}^*$). Multiply by t:

$$\frac{z}{2}\left(t + \frac{1}{t}\right) \sum_{n=-\infty}^{\infty} J_n(z)\, t^n = \sum_{n=-\infty}^{\infty} n J_n(z)\, t^n. \tag{4.17}$$

Expand the left side, using $t \cdot t^n = t^{n+1}$ and $t^{-1} \cdot t^n = t^{n-1}$, and relabel ($n+1 \to n$ in the first sum, $n-1 \to n$ in the second):

$$\frac{z}{2} \sum_{n=-\infty}^{\infty} J_n(z)\, t^{n+1} + \frac{z}{2} \sum_{n=-\infty}^{\infty} J_n(z)\, t^{n-1} = \frac{z}{2} \sum_{n=-\infty}^{\infty} \big[J_{n-1}(z) + J_{n+1}(z)\big] t^n. \tag{4.18}$$

Matching coefficients of t^n between (4.17) and (4.18):

$$\frac{z}{2}\big[J_{n-1}(z) + J_{n+1}(z)\big] = n J_n(z), \tag{4.19}$$

which rearranges to (4.12) for integer $\nu = n$.

For (4.13), differentiate G in z:

$$\frac{\partial G}{\partial z} = \frac{1}{2}\left(t - \frac{1}{t}\right) G(z,t) = \sum_{n=-\infty}^{\infty} J_n'(z)\, t^n. \tag{4.20}$$

The identical index shift gives $\frac{1}{2}[J_{n-1}(z) - J_{n+1}(z)] = J_n'(z)$, which is (4.13).

Step 2: (4.14) *by direct series manipulation.* From (4.11),

$$z^{\nu} J_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \cdot \frac{z^{2\nu+2k}}{2^{\nu+2k}}. \tag{4.21}$$

Differentiate term by term (justified by absolute uniform convergence on compacts):

$$\frac{d}{dz}\big[z^{\nu} J_{\nu}(z)\big] = \sum_{k=0}^{\infty} \frac{(-1)^k (2\nu+2k)}{k!\,\Gamma(\nu+k+1)} \cdot \frac{z^{2\nu+2k-1}}{2^{\nu+2k}}. \tag{4.22}$$

Use the reciprocal-Gamma form of the functional equation, $(\nu+k)/\Gamma(\nu+k+1) = 1/\Gamma(\nu+k)$, which is valid also at the exceptional values by continuity:

$$\frac{d}{dz}\big[z^{\nu} J_{\nu}(z)\big] = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2}{k!\,\Gamma(\nu+k)} \cdot \frac{z^{2\nu+2k-1}}{2^{\nu+2k}} = z^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k)} \left(\frac{z}{2}\right)^{\nu-1+2k}, \tag{4.23}$$

where we pulled z^{ν} and a 1/2 (from $2/2^{\nu+2k} = 1/2^{\nu+2k-1}$) out. The remaining series is $J_{\nu-1}(z)$ by (4.11) with $\nu \to \nu-1$ and $\Gamma((\nu-1)+k+1) = \Gamma(\nu+k)$. Hence (4.14).

Step 3: (4.15) *similarly.* From (4.11),

$$z^{-\nu} J_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \cdot \frac{z^{2k}}{2^{\nu+2k}}. \tag{4.24}$$

Differentiate: only $k \ge 1$ contributes, and $d(z^{2k})/dz = 2kz^{2k-1}$:

$$\frac{d}{dz}\big[z^{-\nu} J_{\nu}(z)\big] = \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 2k}{k!\,\Gamma(\nu+k+1)} \cdot \frac{z^{2k-1}}{2^{\nu+2k}}. \tag{4.25}$$

Use $k/k! = 1/(k-1)!$ and shift index $k = m+1$, $m \ge 0$:

$$\frac{d}{dz}\big[z^{-\nu} J_{\nu}(z)\big] = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \cdot 2}{m!\,\Gamma(\nu+m+2)} \cdot \frac{z^{2m+1}}{2^{\nu+2m+2}}. \tag{4.26}$$

Rewrite the coefficient in (4.26) as $-1/2^{\nu+2m+1}$:

$$\frac{d}{dz}\big[z^{-\nu} J_{\nu}(z)\big] = -\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu+m+2)} \frac{z^{2m+1}}{2^{\nu+2m+1}}. \tag{4.27}$$

On the other hand, using (4.11) with $\nu \to \nu+1$,

$$z^{-\nu} J_{\nu+1}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu+m+2)} z^{-\nu} \left(\frac{z}{2}\right)^{\nu+1+2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu+m+2)} \frac{z^{2m+1}}{2^{\nu+2m+1}}.$$

Comparing with (4.27) gives

$$\frac{d}{dz}\big[z^{-\nu} J_{\nu}(z)\big] = -z^{-\nu} J_{\nu+1}(z).$$

Step 4: (4.12) and (4.13) for general ν. Expand the left side of (4.14):

$$\frac{d}{dz}[z^\nu J_\nu] = \nu z^{\nu-1} J_\nu + z^\nu J_\nu' = z^\nu J_{\nu-1} \quad \Longrightarrow \quad J_\nu' + \frac{\nu}{z} J_\nu = J_{\nu-1}. \tag{4.28}$$

Similarly from (4.15):

$$\frac{d}{dz}[z^{-\nu} J_\nu] = -\nu z^{-\nu-1} J_\nu + z^{-\nu} J_\nu' = -z^{-\nu} J_{\nu+1} \quad \Longrightarrow \quad J_\nu' - \frac{\nu}{z} J_\nu = -J_{\nu+1}. \tag{4.29}$$

Adding (4.28) and (4.29) gives $2J_\nu' = J_{\nu-1} - J_{\nu+1}$, i.e. (4.13). Subtracting gives $2\nu J_\nu/z = J_{\nu-1} + J_{\nu+1}$, i.e. (4.12). ☐

4.4 Bessel's differential equation

Theorem 4.6 (Bessel's ODE)**.** *On its domain of definition (the slit plane for non-integer ν, and all of $\mathbb{C}$ for integer order), J_ν satisfies*

$$z^2 J_\nu''(z) + z J_\nu'(z) + (z^2 - \nu^2) J_\nu(z) = 0. \tag{4.30}$$

Proof. We give two derivations: one short via the recurrences, and one by direct series verification.

Derivation A (via recurrences). From (4.28), $zJ_\nu' + \nu J_\nu = zJ_{\nu-1}$. Differentiate both sides in z:

$$J_\nu' + zJ_\nu'' + \nu J_\nu' = J_{\nu-1} + zJ_{\nu-1}'. \tag{4.31}$$

Apply (4.29) at order $\nu - 1$, i.e. with $\nu \to \nu - 1$: this gives $J_{\nu-1}' - (\nu-1)J_{\nu-1}/z = -J_\nu$, equivalently

$$zJ_{\nu-1}' = (\nu-1)J_{\nu-1} - zJ_\nu. \tag{4.32}$$

Substitute (4.32) into (4.31):

$$(1+\nu)J_\nu' + zJ_\nu'' = J_{\nu-1} + (\nu-1)J_{\nu-1} - zJ_\nu = \nu J_{\nu-1} - zJ_\nu. \tag{4.33}$$

From (4.28): $J_{\nu-1} = J_\nu' + \nu J_\nu/z$, so $\nu J_{\nu-1} = \nu J_\nu' + \nu^2 J_\nu/z$. Insert:

$$(1+\nu)J_\nu' + zJ_\nu'' = \nu J_\nu' + \frac{\nu^2}{z} J_\nu - zJ_\nu. \tag{4.34}$$

Rearrange (subtract $\nu J_\nu'$ from both sides):

$$J_\nu' + zJ_\nu'' = \frac{\nu^2}{z} J_\nu - zJ_\nu. \tag{4.35}$$

Multiply through by z:

$$z^2 J_\nu'' + zJ_\nu' = \nu^2 J_\nu - z^2 J_\nu, \tag{4.36}$$

which rearranges to (4.30).

Derivation B (direct series). Define the Bessel operator $\mathcal{L}_\nu y := z^2 y'' + zy' + (z^2 - \nu^2)y$. Apply to (4.11); write $a_k := (-1)^k/[k!\,\Gamma(\nu+k+1)\,2^{\nu+2k}]$ and $\alpha_k := \nu+2k$ so $J_\nu(z) = \sum_k a_k z^{\alpha_k}$. Term by term,

$$z^2 \cdot \frac{d^2}{dz^2}[a_k z^{\alpha_k}] = a_k\, \alpha_k(\alpha_k - 1) z^{\alpha_k}, \qquad z \cdot \frac{d}{dz}[a_k z^{\alpha_k}] = a_k\, \alpha_k\, z^{\alpha_k}. \tag{4.37}$$

Summing these two:

$$z^2(a_k z^{\alpha_k})'' + z(a_k z^{\alpha_k})' = a_k[\alpha_k(\alpha_k - 1) + \alpha_k] z^{\alpha_k} = a_k \alpha_k^2 z^{\alpha_k}. \tag{4.38}$$

Hence

$$\mathcal{L}_\nu J_\nu = \sum_{k=0}^{\infty} a_k(\alpha_k^2 - \nu^2) z^{\alpha_k} + \sum_{k=0}^{\infty} a_k z^{\alpha_k+2}. \tag{4.39}$$

Compute $\alpha_k^2 - \nu^2 = (\nu + 2k)^2 - \nu^2 = 4k\nu + 4k^2 = 4k(\nu + k)$:

$$\mathcal{L}_\nu J_\nu = \sum_{k=0}^{\infty} 4k(\nu + k) a_k z^{\alpha_k} + \sum_{k=0}^{\infty} a_k z^{\alpha_k+2}. \tag{4.40}$$

The $k = 0$ term of the first sum vanishes (factor of k). In the first sum replace $k = j$, $j \geq 1$; in the second replace $k = j - 1$, $j \geq 1$ (so $\alpha_{j-1} + 2 = \alpha_j$):

$$\mathcal{L}_\nu J_\nu = \sum_{j=1}^{\infty} \big[4j(\nu + j) a_j + a_{j-1}\big] z^{\alpha_j}. \tag{4.41}$$

Evaluate the bracket. From a_k's definition,

$$4j(\nu + j) a_j = \frac{(-1)^j \cdot 4j(\nu + j)}{j!\,\Gamma(\nu + j + 1)\, 2^{\nu+2j}}, \qquad a_{j-1} = \frac{(-1)^{j-1}}{(j-1)!\,\Gamma(\nu + j)\, 2^{\nu+2j-2}}. \tag{4.42}$$

In $4j(\nu + j) a_j$: cancel j using $j! = j \cdot (j-1)!$, and use $(\nu + j)\Gamma(\nu + j) = \Gamma(\nu + j + 1)$:

$$4j(\nu + j) a_j = \frac{(-1)^j \cdot 4}{(j-1)!\,\Gamma(\nu + j)\, 2^{\nu+2j}} = \frac{(-1)^j}{(j-1)!\,\Gamma(\nu + j)\, 2^{\nu+2j-2}}, \tag{4.43}$$

where the last step uses $4/2^{\nu+2j} = 1/2^{\nu+2j-2}$. Comparing with a_{j-1}:

$$4j(\nu + j) a_j = -a_{j-1}. \tag{4.44}$$

Therefore the bracket in (4.41) vanishes for every $j \geq 1$, and $\mathcal{L}_\nu J_\nu = 0$. □

Remark 4.7. *Bessel's ODE is second-order and thus has two linearly independent solutions. For $\nu \notin \mathbb{Z}$, $\{J_\nu, J_{-\nu}\}$ is a fundamental pair. For $\nu = n \in \mathbb{Z}$, $J_{-n} = (-1)^n J_n$ so the pair degenerates; we introduce the second-kind function Y_ν below to fill the gap.*

4.5 Integral representations

The contour formula (4.3) is already an integral representation. Deforming and parametrizing yields familiar real-variable forms.

Theorem 4.8 (Bessel's integral for integer order)**.** *For $n \in \mathbb{Z}$ and $z \in \mathbb{C}$,*

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta. \tag{4.45}$$

Proof. Parametrize $|t| = 1$ by $t = e^{i\theta}$, $\theta \in [-\pi, \pi]$, so $dt = ie^{i\theta}\, d\theta$, and

$$t - \frac{1}{t} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta, \qquad t^{-n-1} = e^{-i(n+1)\theta}. \tag{4.46}$$

Insert into (4.3):

$$J_n(z) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{(z/2)\cdot 2i\sin\theta}\, e^{-i(n+1)\theta}\cdot ie^{i\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(z\sin\theta - n\theta)}\,d\theta, \tag{4.47}$$

where $i/(2\pi i) = 1/(2\pi)$ and $e^{-i(n+1)\theta}\cdot e^{i\theta} = e^{-in\theta}$. Write $e^{i\phi} = \cos\phi + i\sin\phi$ with $\phi := z\sin\theta - n\theta$. For *real* z, $\sin\phi$ is odd in θ and integrates to zero on $[-\pi,\pi]$, while $\cos\phi$ is even:

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(z\sin\theta - n\theta)\,d\theta = \frac{1}{\pi}\int_{0}^{\pi} \cos(z\sin\theta - n\theta)\,d\theta, \tag{4.48}$$

then use $\cos(\phi) = \cos(-\phi)$ to rewrite as $\cos(n\theta - z\sin\theta)$. For complex z, both sides of (4.45) are entire functions of z and agree on $\mathbb{R}$; by the identity theorem (Theorem 2.2) they agree on $\mathbb{C}$. □

Theorem 4.9 (Schläfli integral representation). *For $\nu \in \mathbb{C}$ and $z \in \mathbb{C}\setminus(-\infty, 0]$, define the rotated Hankel contour*

$$\mathcal{C}_z := \{t \in \mathbb{C}^* : zt/2 \in H\},$$

i.e. the set of t-values that the map $u = zt/2$ sends onto the standard Hankel contour H of Theorem 3.16. Then

$$J_\nu(z) = \frac{1}{2\pi i}\int_{\mathcal{C}_z} e^{(z/2)(t-1/t)}\, t^{-\nu-1}\,dt, \tag{4.49}$$

where $t^{-\nu-1}$ is interpreted by pulling back the principal branch from $u = zt/2$:

$$t^{-\nu-1} := \left(\frac{z}{2}\right)^{\nu+1}\left(\frac{zt}{2}\right)^{-\nu-1}.$$

The power $(zt/2)^{-\nu-1}$ uses the principal branch off the cut. The point of this convention is that we choose the branch once in the u-plane, where the contour is the fixed Hankel contour, and then pull that choice back to the rotated t-contour. When $z > 0$ is real, the map $t \mapsto zt/2$ is a positive dilation, so this convention agrees on $\mathcal{C}_z$ with the Hankel-contour boundary values from Theorem 3.16: $\arg t = -\pi$ on the lower edge and $\arg t = \pi$ on the upper edge. For general $z \in \mathbb{C}\setminus(-\infty, 0]$ the identity is then obtained by analytic continuation in z. Equivalently, for each fixed z, $\mathcal{C}_z$ is a z-dependent rotated Hankel contour in the t-plane.

Proof. Motivation. Formula (4.3) extracts an integer-order coefficient by a closed contour. For complex order, $t^{-\nu-1}$ has a branch cut, so the natural replacement is a Hankel contour around that cut. The proof reduces this contour integral to Hankel's representation of $1/\Gamma$.

Strategy. Push the t-contour forward by $u = zt/2$. By definition of $\mathcal{C}_z$, this sends the integral to the standard Hankel contour H, where Theorem 3.16 applies term by term.

Step 1 (reduce to the standard Hankel contour). Theorem 3.16 states that for every $w \in \mathbb{C}$,

$$\frac{1}{\Gamma(w)} = \frac{1}{2\pi i}\int_H e^u u^{-w}\,du. \tag{4.50}$$

Denote the Schläfli integral (4.49) by $I(z,\nu)$. By the definition of $\mathcal{C}_z = \{t : zt/2 \in H\}$, the map $t \mapsto u = zt/2$ sends $\mathcal{C}_z$ bijectively onto H; its inverse $u \mapsto t = 2u/z$ shows $\mathcal{C}_z$ is the image of H under the fixed scalar dilation $u \mapsto (2/z)u$, justifying the shorthand $\mathcal{C}_z = (2/z)H$. Also

$$t = \frac{2u}{z}, \qquad dt = \frac{2}{z}\,du, \qquad \frac{1}{t} = \frac{z}{2u}, \qquad t^{-\nu-1} = \left(\frac{2u}{z}\right)^{-\nu-1},$$

with the branch of $t^{-\nu-1}$ understood via the branch of $u^{-\nu-1}$. Therefore

$$I(z,\nu)=\frac{1}{2\pi i}\int_H e^{u-z^2/(4u)}\left(\frac{2u}{z}\right)^{-\nu-1}\frac{2}{z}\,du=\left(\frac{z}{2}\right)^{\nu}\frac{1}{2\pi i}\int_H e^{u-z^2/(4u)}\,u^{-\nu-1}\,du. \tag{4.51}$$

Step 2 (expand the remaining factor and integrate term by term). Fix the standard Hankel contour H with a circular part of radius $\rho>0$. Then $|u|\geq\rho$ everywhere on H, so the Laurent series

$$e^{-z^2/(4u)}=\sum_{k=0}^{\infty}\frac{(-z^2/4)^k}{k!}\,u^{-k}$$

converges uniformly on H, because

$$\sum_{k=0}^{\infty}\frac{|z|^{2k}}{4^k k!|u|^k}\leq\sum_{k=0}^{\infty}\frac{|z|^{2k}}{4^k k!\rho^k}=\exp\!\left(\frac{|z|^2}{4\rho}\right).$$

On the circular part of H the contour has finite length, so uniform convergence is enough. On the two rays of H, write $u=-se^{\pm i0}$ with $s\geq\rho$; then $|e^u|=e^{-s}$. For fixed z and ν, the absolute value of the expanded integrand is bounded by

$$C_{\rho,z,\nu}\,e^{-s}s^{-\operatorname{Re}\nu-1},$$

which is integrable on $[\rho,\infty)$ because of e^{-s}. Hence dominated convergence allows termwise integration:

$$I(z,\nu)=\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-z^2/4)^k}{k!}\cdot\frac{1}{2\pi i}\int_H e^u\,u^{-\nu-k-1}\,du. \tag{4.52}$$

By (4.50) with $w=\nu+k+1$, the inner integral equals $1/\Gamma(\nu+k+1)$:

$$I(z,\nu)=\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k\,(z/2)^{2k}}{k!\,\Gamma(\nu+k+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{\nu+2k}=J_\nu(z), \tag{4.53}$$

matching (4.11).

Step 3 (integer ν). When $\nu=n\in\mathbb{Z}$, t^{-n-1} is single-valued and the integrand has no branch. The two rays of C_z then traverse the same path in opposite directions with identical integrand, so they cancel. The remaining small loop around 0 may be deformed to any simple closed contour enclosing the origin, for example $|t|=1$. We recover (4.3).

Sanity check at $\nu=0$, $z=1$ real. Then C_1 is the standard Hankel contour H (no rotation), $t^{-1}\,dt$ is single-valued, and the rays cancel. The small loop around $t=0$ gives

$$J_0(1)\;=\;\frac{1}{2\pi i}\oint_{|t|=\rho}e^{(1/2)(t-1/t)}\,t^{-1}\,dt\;=\;\frac{1}{2\pi}\int_0^{2\pi}e^{i\sin\theta}\,d\theta\;=\;\frac{1}{\pi}\int_0^{\pi}\cos(\sin\theta)\,d\theta,$$

which is (4.45) at $n=0$. The integral evaluates numerically to $J_0(1)\approx 0.7652$, in agreement with the series (4.4). □

Theorem 4.10 (Poisson/Sommerfeld integral). *For* $\operatorname{Re}\nu>-1/2$ *and* $z\in\mathbb{C}\setminus(-\infty,0]$,

$$J_\nu(z)=\frac{(z/2)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_{-1}^{1}(1-s^2)^{\nu-1/2}\cos(zs)\,ds. \tag{4.54}$$

Here $(z/2)^\nu$ uses the same principal branch as in (4.11). If $\nu=n\in\mathbb{N}_0$, the same formula gives the entire extension of J_n.

Proof. For $s \in (-1,1)$, the factor $(1-s^2)^{\nu-1/2}$ means

$$\exp\left(\left(\nu - \frac{1}{2}\right)\log(1-s^2)\right),$$

where the logarithm is the real logarithm of the positive number $1-s^2$. The condition $\operatorname{Re}\nu > -1/2$ makes the endpoint singularities integrable. Denote the right-hand side by $R_\nu(z)$. We show $R_\nu(z) = J_\nu(z)$ on the slit plane by expanding $\cos(zs)$, integrating term by term, and comparing with (4.11).

Step 1 (series for $\cos$*).* For $z, s \in \mathbb{C}$, $\cos(zs) = \sum_{k=0}^{\infty}(-1)^k(zs)^{2k}/(2k)!$; this Taylor series converges uniformly for $s \in [-1,1]$ and fixed z, and $(1-s^2)^{\nu-1/2} \in L^1(-1,1)$ (its absolute value is integrable) for $\operatorname{Re}\nu > -1/2$. Therefore we may integrate term by term (dominated convergence):

$$R_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \int_{-1}^{1} s^{2k}(1-s^2)^{\nu-1/2}\,ds. \tag{4.55}$$

Step 2 (Beta integral). The integrand of the inner integral is even in s, so

$$\int_{-1}^{1} s^{2k}(1-s^2)^{\nu-1/2}\,ds = 2\int_{0}^{1} s^{2k}(1-s^2)^{\nu-1/2}\,ds. \tag{4.56}$$

Substitute $u = s^2$, so $ds = du/(2\sqrt{u})$ and $s^{2k} = u^k$:

$$2\int_0^1 s^{2k}(1-s^2)^{\nu-1/2}\,ds = 2\int_0^1 u^{k-1/2}(1-u)^{\nu-1/2}\cdot\frac{du}{2} = \int_0^1 u^{k+1/2-1}(1-u)^{\nu+1/2-1}\,du. \tag{4.57}$$

By Definition 3.8 of the Beta function, this is $B(k+1/2,\, \nu+1/2)$. Apply the Beta–Gamma identity (Proposition 3.9):

$$B(k+1/2,\, \nu+1/2) = \frac{\Gamma(k+1/2)\,\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}. \tag{4.58}$$

Step 3 (insert and simplify). Substitute (4.58) into (4.55):

$$R_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \cdot \frac{\Gamma(k+1/2)}{\Gamma(\nu+k+1)}, \tag{4.59}$$

after the $\Gamma(\nu+1/2)$ cancels. We now reduce $\Gamma(k+1/2)/(2k)!$. By the Legendre duplication formula (Theorem 3.15) applied with $w = k+1/2$:

$$\Gamma(k+1/2)\,\Gamma(k+1) = 2^{1-2(k+1/2)}\sqrt{\pi}\,\Gamma(2k+1) = 2^{-2k}\sqrt{\pi}\,(2k)!, \tag{4.60}$$

where $\Gamma(2k+1) = (2k)!$. Hence

$$\frac{\Gamma(k+1/2)}{(2k)!} = \frac{\sqrt{\pi}}{2^{2k}\,\Gamma(k+1)} = \frac{\sqrt{\pi}}{2^{2k}\,k!}. \tag{4.61}$$

Insert into (4.59):

$$R_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}\sqrt{\pi}}{2^{2k}\,k!\,\Gamma(\nu+k+1)} = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}, \tag{4.62}$$

using $z^{2k}/2^{2k} = (z/2)^{2k}$. Absorbing $(z/2)^\nu$ into the summand gives

$$R_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k} = J_\nu(z), \tag{4.63}$$

by (4.11).

If $\nu = n \in \mathbb{N}_0$, then $(z/2)^n$ is entire and the same computation yields the entire power series for $J_n(z)$. So the identity extends from the slit plane to all $z \in \mathbb{C}$ in the integer-order case. □

Example 4.11 ($\nu = 1/2$). *From* (4.54) *with* $\nu = 1/2$, $\Gamma(1) = 1$*:*

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \tag{4.64}$$

using $\int_{-1}^{1} \cos(zs)\,ds = 2\sin z/z$ *and* $(z/2)^{1/2} \cdot 2/z = \sqrt{2}/\sqrt{z}$*, hence* $\sqrt{2}/\sqrt{z} \cdot 1/\sqrt{\pi} = \sqrt{2/(\pi z)}$*.*

4.6 Asymptotics

Theorem 4.12 (Large argument). *For fixed* $\nu \in \mathbb{C}$*, as* $|z| \to \infty$ *in any sector* $|\arg z| \le \pi - \delta$ *with* $\delta > 0$*,*

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[\cos\!\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\!\left(\frac{e^{|\operatorname{Im} z|}}{|z|}\right) \right]. \tag{4.65}$$

On the positive real axis, away from zeros of the leading cosine, this is often written as $J_\nu(z) \sim \sqrt{2/(\pi z)} \cos(z - \nu\pi/2 - \pi/4)$*. The displayed form is safer because the leading cosine itself has zeros.*

Proof. Write $z = re^{i\phi}$ with $r = |z|$ and $|\phi| \le \pi - \delta$. Since $\mathcal{C}_z = (2/z)H = e^{-i\phi}(2/r)H$, the Schläfli contour is the Hankel contour rotated by $-\phi$. Apply steepest descent (Theorem 2.17) to (4.49) with $\lambda := r/2$ and

$$f_\phi(t) = e^{i\phi}\left(t - \frac{1}{t}\right), \qquad g(t) = t^{-\nu-1}, \tag{4.66}$$

so the integrand is $e^{\lambda f_\phi(t)} g(t)$.

Step 1 (saddles). Since $f_\phi'(t) = e^{i\phi}(1 + 1/t^2)$, the saddles are still

$$t = \pm i.$$

Step 2 (values at the saddles). Since $i - 1/i = 2i$ and $-i - 1/(-i) = -2i$,

$$f_\phi(i) = 2ie^{i\phi}, \qquad f_\phi(-i) = -2ie^{i\phi}.$$

Therefore

$$\lambda f_\phi(i) = iz, \qquad \lambda f_\phi(-i) = -iz. \tag{4.67}$$

Step 3 (second derivatives and descent angles). We have

$$f_\phi''(t) = -\frac{2e^{i\phi}}{t^3},$$

so

$$f_\phi''(i) = -2ie^{i\phi}, \qquad f_\phi''(-i) = 2ie^{i\phi}.$$

Hence $|f''_\phi(\pm i)| = 2$, while

$$\arg f''_\phi(i) = \phi - \frac{\pi}{2}, \qquad \arg f''_\phi(-i) = \phi + \frac{\pi}{2}.$$

By Proposition 2.16, the steepest-descent directions are

$$\theta_+ = \frac{\pi - (\phi - \pi/2)}{2} = \frac{3\pi}{4} - \frac{\phi}{2}, \qquad \theta_- = \frac{\pi - (\phi + \pi/2)}{2} = \frac{\pi}{4} - \frac{\phi}{2}. \tag{4.68}$$

Step 4 (deform the contour). The rotated Hankel contour can be deformed through the two saddles along the descent directions (4.68) without crossing singularities or changing the endpoint behavior. The remaining pieces are lower order uniformly for $|\phi| \le \pi - \delta$, so the leading term is the sum of the two saddle contributions.

This is the only global steepest-descent input in the proof. Locally, the descent directions come from the quadratic calculation above. Globally, the two ends of the rotated Hankel contour lie in sectors where $\mathrm{Re}\{e^{i\phi}(t - 1/t)\} \to -\infty$, and the branch point at $t = 0$ remains encircled rather than crossed. Thus Cauchy's theorem permits the contour to slide onto the two descent arcs through $t = \pm i$, with the connecting arcs placed in regions of smaller real part. A *Stokes line* is a curve across which the relative exponential dominance of saddle contributions changes; drawing those lines gives the same conclusion. The algebra below is the local computation of the two resulting saddle contributions.

Step 5 (leading-order contributions). By Theorem 2.17, the contribution of an isolated simple saddle t_* with $f''_\phi(t_*) \ne 0$ and descent angle θ_* is

$$\mathcal{I}(t_*) \sim e^{\lambda f_\phi(t_*)} g(t_*) e^{i\theta_*} \sqrt{\frac{2\pi}{\lambda |f''_\phi(t_*)|}}.$$

The power $g(t) = t^{-\nu-1}$ uses the branch pulled back from the standard Hankel variable $u = zt/2$. To fix the saddle constants, start on the positive real z-axis: the cut in the t-plane is then the negative real axis, so the continuous saddle values are $i^{-\nu-1} = e^{-i(\nu+1)\pi/2}$ and $(-i)^{-\nu-1} = e^{i(\nu+1)\pi/2}$. For $|\arg z| \le \pi - \delta$, rotating z carries this pulled-back branch continuously, and the final expression is the corresponding analytic continuation in the sector. At $t_* = i$, use (4.67) and (4.68). Since $|f''_\phi(i)| = 2$,

$$\mathcal{I}_+ \sim \sqrt{\frac{2\pi}{r}}\, e^{iz}\, i^{-\nu-1} e^{i\theta_+} = \sqrt{\frac{2\pi}{r}}\, e^{iz}\, e^{-i(\nu+1)\pi/2} e^{i(3\pi/4 - \phi/2)}. \tag{4.69}$$

At $t_* = -i$,

$$\mathcal{I}_- \sim \sqrt{\frac{2\pi}{r}}\, e^{-iz}\, (-i)^{-\nu-1} e^{i\theta_-} = \sqrt{\frac{2\pi}{r}}\, e^{-iz}\, e^{i(\nu+1)\pi/2} e^{i(\pi/4 - \phi/2)}.$$

Step 6 (combine). By Schläfli's formula (4.49),

$$J_\nu(z) \sim \frac{1}{2\pi i}\big(\mathcal{I}_+ + \mathcal{I}_-\big).$$

Since $z^{-1/2} = r^{-1/2} e^{-i\phi/2}$ on the principal branch,

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\, e^{-i\pi/2} \left[e^{iz} e^{-i(\nu+1)\pi/2} e^{i3\pi/4} + e^{-iz} e^{i(\nu+1)\pi/2} e^{i\pi/4} \right],$$

because $\sqrt{2\pi/r}\, e^{-i\phi/2}/(2\pi) = 1/\sqrt{2\pi z}$. Collect exponent phases. First term:

$$e^{-i\pi/2} e^{-i(\nu+1)\pi/2} e^{i3\pi/4} = e^{-i(\nu\pi/2 + \pi/4)}.$$

Second term:

$$e^{-i\pi/2}e^{i(\nu+1)\pi/2}e^{i\pi/4} = e^{i(\nu\pi/2+\pi/4)}.$$

Let $\chi := z - \nu\pi/2 - \pi/4$. Then the first bracketed term is $e^{i(z-\nu\pi/2-\pi/4)} = e^{i\chi}$ (since iz plus the first phase gives $i(z - \nu\pi/2 - \pi/4)$), and the second is $e^{-i\chi}$. Hence the leading saddle sum is

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\big(e^{i\chi} + e^{-i\chi}\big) = \sqrt{\frac{2}{\pi z}}\cos\chi,$$

which is the main term in (4.65). Keeping the next terms in the local steepest-descent expansions at the two saddles gives the stated remainder. □

Remark 4.13 (Sign/phase check). *The two saddle contributions are complex conjugates of each other when $\nu \in \mathbb{R}$ and $z > 0$: $\mathcal{I}_- = \overline{\mathcal{I}_+}$. The sum is real, and the resulting* $\cos$ *is* $2\,\mathrm{Re}(e^{i\chi})/2 = \cos\chi$. *Any off-by-$\pi/2$ error in the phase would flip* $\cos \longleftrightarrow \sin$ *and is easy to catch by testing $\nu = 1/2$: there $J_{1/2}(z) = \sqrt{2/(\pi z)}\sin z$ (see* (4.64)*), and* (4.65) *predicts* $\cos(z - \pi/4 - \pi/4) = \cos(z - \pi/2) = \sin z$. *Consistent.*

Theorem 4.14 (Large order). *As $\nu \to +\infty$ with $z = \nu$:*

$$J_\nu(\nu) \sim \frac{2^{1/3}}{3^{2/3}\,\Gamma(2/3)}\,\nu^{-1/3}. \tag{4.70}$$

Proof. Picture. At $z = \nu$, the factor $t^{-\nu} = e^{-\nu\log t}$ joins the exponential phase. The resulting phase has a double saddle at $t = 1$ and cubic leading behavior, so the natural scale is $\nu^{-1/3}$.

Take (4.49) with $z = \nu > 0$ real. Since $\mathcal{C}_\nu = (2/\nu)H$, we may write

$$J_\nu(\nu) = \frac{1}{2\pi i}\int_{\mathcal{C}_\nu} e^{\nu\varphi(t)}\,\frac{dt}{t}, \tag{4.71}$$

where

$$\varphi(t) := \tfrac{1}{2}(t - 1/t) - \log t. \tag{4.72}$$

Step 1 (coalescent saddle and cubic expansion). We have

$$\varphi'(t) = \tfrac{1}{2}\left(1 + \frac{1}{t^2}\right) - \frac{1}{t} = \frac{(t-1)^2}{2t^2},$$

so $t = 1$ is a double saddle. Differentiate:

$$\varphi''(t) = -\frac{1}{t^3} + \frac{1}{t^2}, \qquad \varphi'''(t) = \frac{3}{t^4} - \frac{2}{t^3}.$$

At $t = 1$,

$$\varphi(1) = 0, \qquad \varphi'(1) = 0, \qquad \varphi''(1) = 0, \qquad \varphi'''(1) = 1.$$

The vanishing of both φ' and φ'' at $t = 1$ is what makes the saddle "double" or "cubic": the quadratic Taylor term is gone, and the leading nontrivial behavior is the cubic one. This is the saddle-coalescence scenario flagged generally in the asymptotics summary (Section 2.7). Hence, with $t = 1 + s$,

$$\varphi(1+s) = \frac{s^3}{6} + O(s^4), \qquad \frac{1}{1+s} = 1 + O(s), \tag{4.73}$$

as $s \to 0$.

Step 2 (scale the cubic saddle). The steepest-descent rays through $t = 1$ have $\arg s = \pm\pi/3$. Deform $\mathcal{C}_\nu$ through these rays; away from a small disk about $t = 1$, the contour contributes only $O(e^{-c\nu})$.

Now set

$$s = 2^{1/3}\nu^{-1/3}\sigma, \qquad ds = 2^{1/3}\nu^{-1/3}\,d\sigma. \tag{4.74}$$

Then

$$\nu\varphi(1+s) = \frac{\sigma^3}{3} + O(\nu^{-1/3}\sigma^4), \qquad \frac{1}{1+s} = 1 + O(\nu^{-1/3}\sigma), \tag{4.75}$$

uniformly on the scaled contour for bounded σ. The local contour tends to the cubic Hankel contour Γ_c (the same contour shape used in the Hankel representation of $1/\Gamma$, Thm. 3.16), and dominated convergence gives

$$J_\nu(\nu) = \frac{2^{1/3}\nu^{-1/3}}{2\pi i}\int_{\Gamma_c} e^{\sigma^3/3}\,d\sigma + O(\nu^{-2/3}). \tag{4.76}$$

Step 3 (cubic Hankel integral). Put

$$u = \frac{\sigma^3}{3}. \tag{4.77}$$

On the ray $\sigma = re^{i\pi/3}$ with $r > 0$, we have $\sigma^3 = r^3 e^{i\pi} = -r^3$, so $u = -r^3/3 < 0$, with u approached from above the real axis (since $\operatorname{Im}\sigma^3$ vanishes from the positive side as one rotates onto the ray). Similarly the ray $\sigma = re^{-i\pi/3}$ sends $u \to -r^3/3$ from below. Thus the two cubic descent rays of Γ_c map onto the two banks of the negative-real-axis cut for u, while the small connector around $\sigma = 0$ maps to a small circle around $u = 0$. In this orientation the image is the standard Hankel contour H. Since $\sigma = (3u)^{1/3} = 3^{1/3}u^{1/3}$ on the corresponding branch, differentiating gives

$$d\sigma = 3^{-2/3}u^{-2/3}\,du. \tag{4.78}$$

Therefore

$$\int_{\Gamma_c} e^{\sigma^3/3}\,d\sigma = 3^{-2/3}\int_H e^u u^{-2/3}\,du = \frac{2\pi i}{3^{2/3}\Gamma(2/3)}, \tag{4.79}$$

by Hankel's formula (3.19) with $z = 2/3$.

Step 4 (assemble the constant). Substitute (4.79) into (4.76):

$$J_\nu(\nu) \sim \frac{2^{1/3}\nu^{-1/3}}{2\pi i}\cdot\frac{2\pi i}{3^{2/3}\Gamma(2/3)} = \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)}\,\nu^{-1/3}.$$

□

Remark 4.15. (4.70) *is the leading transition value in the Debye expansion. The* $\nu^{-1/3}$ *scale comes from the cubic saddle.*

4.7 The second-kind and Hankel functions

For $\nu \notin \mathbb{Z}$, define the *Bessel function of the second kind* (Weber function)

$$Y_\nu(z) := \frac{\cos(\nu\pi)J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)}. \tag{4.80}$$

This is a linear combination of two solutions of Bessel's ODE (4.30), hence is also a solution. The next paragraph shows why it supplies a second solution rather than just another multiple of J_ν.

Why this combination is natural. For non-integer ν, the leading powers near $z = 0$ are different:

$$J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \qquad J_{-\nu}(z) \sim \frac{(z/2)^{-\nu}}{\Gamma(1-\nu)}.$$

Thus J_ν and $J_{-\nu}$ give two independent local behaviors. At integer $\nu = n$, they collapse via $J_{-n} = (-1)^n J_n$. The numerator in (4.80) is built to vanish at the same points as $\sin(\nu\pi)$, so the quotient has a finite limit and supplies the missing second solution.

For integer order $\nu = n \in \mathbb{Z}$, define by limit:

$$Y_n(z) := \lim_{\nu \to n} Y_\nu(z). \tag{4.81}$$

Both numerator $N(\nu, z) := \cos(\nu\pi) J_\nu(z) - J_{-\nu}(z)$ and denominator $D(\nu) := \sin(\nu\pi)$ vanish at $\nu = n$, and both are analytic in ν. L'Hôpital's rule applied in ν gives

$$Y_n(z) = \left.\frac{\partial_\nu N}{\partial_\nu D}\right|_{\nu=n} = \frac{1}{\pi(-1)^n}\left[-\pi \sin(\nu\pi) J_\nu + \cos(\nu\pi)\frac{\partial J_\nu}{\partial \nu} - \frac{\partial J_{-\nu}}{\partial \nu}\right]_{\nu=n}, \tag{4.82}$$

where $\partial_\nu D = \pi \cos(\nu\pi)|_{\nu=n} = \pi(-1)^n$, and the first bracketed term vanishes at $\nu = n$. Simplifying,

$$Y_n(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu(z)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(z)}{\partial \nu}\right]_{\nu=n}. \tag{4.83}$$

The ν-derivatives of (4.11) contain $\partial_\nu (z/2)^{\nu+2k} = (z/2)^{\nu+2k}\log(z/2)$. Thus Y_n has a logarithmic singularity at $z = 0$, visible in Figure 9; for example $Y_0(z) = (2/\pi)[\log(z/2) + \gamma] J_0(z) + \cdots$.

The *Hankel functions* (first and second kind) are

$$H_\nu^{(1)}(z) := J_\nu(z) + i\,Y_\nu(z), \qquad H_\nu^{(2)}(z) := J_\nu(z) - i\,Y_\nu(z). \tag{4.84}$$

They form another fundamental pair for Bessel's ODE and separate outgoing from incoming waves.

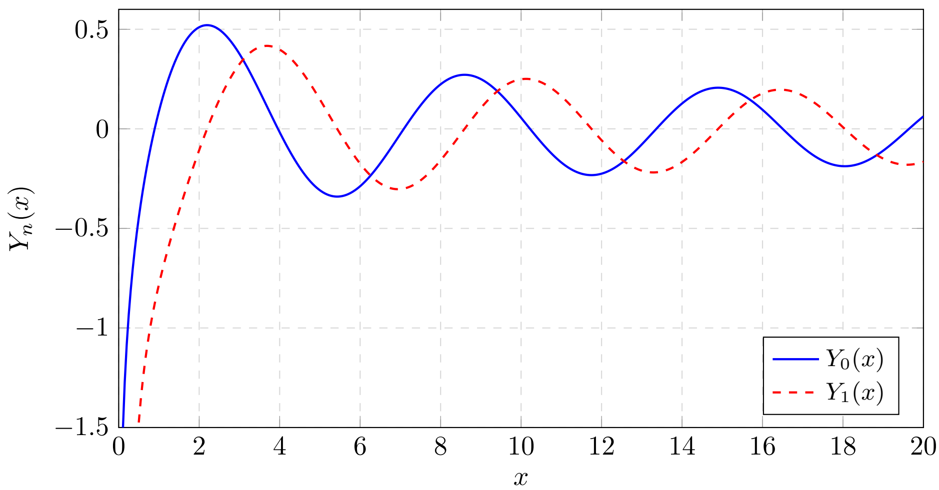

Figure 9: Bessel functions of the second kind Y_0 and Y_1 on $(0, 20]$. Both diverge as $x \to 0^+$ (logarithmically for Y_0, like $-2/(\pi x)$ for Y_1), making them inadmissible solutions on any domain containing the origin; for x bounded away from 0 they oscillate like J_ν but phase-shifted by $\pi/2$ (Corollary 4.16).

Corollary 4.16 (Hankel asymptotics). *As $|z| \to \infty$ in any sector $|\arg z| \le \pi - \delta$ with $\delta > 0$,*

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i(z-\nu\pi/2-\pi/4)}, \qquad H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i(z-\nu\pi/2-\pi/4)}. \tag{4.85}$$

Proof. Let

$$A(z) := \sqrt{\frac{2}{\pi z}}, \qquad \chi := z - \frac{\nu\pi}{2} - \frac{\pi}{4}.$$

Using the leading part of Theorem 4.12,

$$J_\nu(z) \sim A(z)\cos\chi, \tag{4.86}$$

and, with ν replaced by $-\nu$,

$$J_{-\nu}(z) \sim A(z)\cos(\chi + \nu\pi). \tag{4.87}$$

For $\nu \notin \mathbb{Z}$, insert these into (4.80):

$$\begin{aligned} Y_\nu(z) &\sim A(z)\,\frac{\cos(\nu\pi)\cos\chi - \cos(\chi+\nu\pi)}{\sin(\nu\pi)} \\ &= A(z)\sin\chi. \end{aligned} \tag{4.88}$$

The trigonometric simplification uses $\cos(\chi+\nu\pi) = \cos\chi\cos(\nu\pi) - \sin\chi\sin(\nu\pi)$. Therefore

$$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + iY_\nu(z) \sim A(z)\big(\cos\chi + i\sin\chi\big) = A(z)e^{i\chi}, \\ H_\nu^{(2)}(z) &= J_\nu(z) - iY_\nu(z) \sim A(z)\big(\cos\chi - i\sin\chi\big) = A(z)e^{-i\chi}. \end{aligned}$$

For integer ν, the same formulas follow by taking the limit $\mu \to \nu$ from non-integer μ. □

With time dependence $e^{-i\omega t}$, $H_\nu^{(1)}(kr)e^{-i\omega t} \propto e^{i(kr-\omega t)}$ is an *outgoing* cylindrical wave, while $H_\nu^{(2)}(kr)$ is incoming.

4.7.1 Wronskian identities

For differentiable functions f and g, their *Wronskian* is $W[f,g](z) := f(z)g'(z) - f'(z)g(z)$. A nonzero Wronskian proves that two solutions are linearly independent.

Theorem 4.17 (Wronskian identities). *For $\nu \in \mathbb{C}$ and $z \in \mathbb{C}\setminus(-\infty,0]$,*

$$W[J_\nu, J_{-\nu}](z) = -\frac{2\sin(\nu\pi)}{\pi z}, \tag{4.89}$$

$$W[J_\nu, Y_\nu](z) = \frac{2}{\pi z}, \tag{4.90}$$

$$W\big[H_\nu^{(1)}, H_\nu^{(2)}\big](z) = -\frac{4i}{\pi z}. \tag{4.91}$$

Proof. Step 1 (Abel's identity). Put Bessel's ODE (4.30) in normal form:

$$y'' + \frac{1}{z}y' + \left(1 - \frac{\nu^2}{z^2}\right)y = 0.$$

For any two solutions y_1, y_2, their Wronskian $W(z) := y_1y_2' - y_1'y_2$ satisfies Abel's identity

$$W'(z) = -p(z)W(z), \qquad p(z) = \frac{1}{z}, \tag{4.92}$$

so $W(z) = C/z$. Thus $zW(z)$ is constant, and we may compute it in any convenient limit.

Here is the one-line derivation. For a general equation $y'' + p(z)y' + q(z)y = 0$,

$$W' = y_1 y_2'' - y_1'' y_2 = y_1(-py_2' - qy_2) - (-py_1' - qy_1)y_2 = -p(y_1 y_2' - y_1' y_2) = -pW.$$

In the Bessel case $p(z) = 1/z$, so $W'/W = -1/z$ and therefore $W = C/z$ on any simply connected region avoiding $z = 0$.

Step 2 (compute $zW[J_\nu, J_{-\nu}]$ at $z \to \infty$). Since zW is constant, evaluate it using the large-z asymptotics. From (4.65):

$$J_\nu(z) \sim A\cos\chi_\nu, \qquad J_\nu'(z) \sim -A\sin\chi_\nu + O(z^{-3/2}),$$

with $A = \sqrt{2/(\pi z)}$ and $\chi_\nu = z - \nu\pi/2 - \pi/4$. Analogously for $J_{-\nu}$: $\chi_{-\nu} = z + \nu\pi/2 - \pi/4$, so $\chi_{-\nu} - \chi_\nu = \nu\pi$. Compute:

$$\begin{aligned} W[J_\nu, J_{-\nu}] &\sim A^2[\cos\chi_\nu \cdot (-\sin\chi_{-\nu}) - (-\sin\chi_\nu)\cos\chi_{-\nu}] \\ &= A^2[\sin\chi_\nu \cos\chi_{-\nu} - \cos\chi_\nu \sin\chi_{-\nu}] \\ &= A^2 \sin(\chi_\nu - \chi_{-\nu}) = A^2\sin(-\nu\pi) = -A^2\sin(\nu\pi) \\ &= -\frac{2}{\pi z}\sin(\nu\pi). \end{aligned} \tag{4.93}$$

This is (4.89). By Abel, $zW(z)$ is constant, so the value computed from the large-z limit holds for every $z \in \mathbb{C} \setminus (-\infty, 0]$.

Step 3 (Wronskian $W[J_\nu, Y_\nu]$). From (4.80), Y_ν is a linear combination. The Wronskian is bilinear, meaning linear in each of its two inputs:

$$W[J_\nu, Y_\nu] = \frac{1}{\sin(\nu\pi)}\Big(\cos(\nu\pi)\, W[J_\nu, J_\nu] - W[J_\nu, J_{-\nu}]\Big). \tag{4.94}$$

Since $W[J_\nu, J_\nu] = 0$ and by (4.89):

$$W[J_\nu, Y_\nu] = \frac{-W[J_\nu, J_{-\nu}]}{\sin(\nu\pi)} = \frac{2\sin(\nu\pi)/(\pi z)}{\sin(\nu\pi)} = \frac{2}{\pi z}. \tag{4.95}$$

The right side has no singularity at integer ν (the $\sin(\nu\pi)$'s cancel); by continuity, (4.90) extends to integer ν.

Step 4 (Wronskian of Hankel pair). By bilinearity and antisymmetry of the Wronskian,

$$W[J + iY, J - iY] = -2i\, W[J, Y] = -\frac{4i}{\pi z},$$

which is (4.91). □

4.8 Modified Bessel functions

Motivation. Ordinary Bessel functions oscillate on $\mathbb{R}_{>0}$, so they fit wave propagation. Diffusion and electrostatic problems in cylindrical coordinates instead need real growing or decaying radial profiles. Formally, this occurs when the separation constant changes sign, or equivalently when Bessel's equation is evaluated at iz.

Let $Y(w)$ solve Bessel's equation (4.30) in the variable w, and set $y(z) = Y(iz)$. Then $y'(z) = iY'(iz)$ and $y''(z) = -Y''(iz)$. Substituting $w = iz$ into Bessel's equation gives

$$-z^2 Y''(iz) + izY'(iz) + (-z^2 - \nu^2)Y(iz) = 0.$$

Since $Y(iz) = y(z)$, $Y'(iz) = y'(z)/i$, and $Y''(iz) = -y''(z)$, this becomes

$$z^2 y'' + z\, y' - (z^2 + \nu^2)\, y = 0, \tag{4.96}$$

the *modified Bessel equation*. The sign of the z^2 term has flipped, turning oscillatory real-axis behavior into exponential behavior.

A first solution is $J_\nu(iz)$, but it carries a phase. Remove it by defining

$$I_\nu(z) := i^{-\nu} J_\nu(iz) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}, \tag{4.97}$$

with no alternating sign: $i^{2k} = (-1)^k$ cancels the sign in the J_ν series.

Branch convention. For non-integer ν, use the principal branch for $(iz)^\nu$; the identity $I_\nu(z) = i^{-\nu} J_\nu(iz)$ then holds in the right half-plane and extends by analytic continuation.

A linearly independent solution is defined by

$$K_\nu(z) := \frac{\pi}{2} \cdot \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \tag{4.98}$$

with integer order defined by limit (via l'Hôpital, exactly as for Y_n). This combination decays as $z \to +\infty$ (Theorem 4.18), while I_ν grows.

Theorem 4.18 (Asymptotics of I_ν and K_ν)**.** *As $|z| \to \infty$ in any sector $|\arg z| \le \pi/2 - \delta$ with $\delta > 0$:*

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \tag{4.99}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}. \tag{4.100}$$

Proof. Derivation of (4.99). If $|\arg z| \le \pi/2 - \delta$, then $w = iz$ lies in a sector $|\arg w| \le \pi - \delta$, so Theorem 4.12 applies at $w = iz$. With $\eta := \nu\pi/2 + \pi/4$,

$$J_\nu(iz) \sim \sqrt{\frac{2}{\pi i z}} \cos(iz - \eta). \tag{4.101}$$

$$\cos(iz - \eta) = \frac{e^{i(iz-\eta)} + e^{-i(iz-\eta)}}{2} = \frac{e^{-z-i\eta} + e^{z+i\eta}}{2}.$$

Because $\operatorname{Re} z \ge |z| \sin\delta > 0$ in this sector, the first exponential is smaller than the second by the factor $e^{-2\operatorname{Re} z}$, so

$$\cos(iz - \eta) \sim \frac{1}{2} e^{z+i\eta}. \tag{4.102}$$

$$\sqrt{\frac{1}{iz}} = e^{-i\pi/4} z^{-1/2},$$

again on the principal branch. Therefore

$$J_\nu(iz) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i\pi/4} \cdot \frac{1}{2} e^{z+i\eta} = \frac{e^z}{\sqrt{2\pi z}}\, e^{i(\eta-\pi/4)}. \tag{4.103}$$

Multiply by $i^{-\nu} = e^{-i\nu\pi/2}$:

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(iz) \sim \frac{e^z}{\sqrt{2\pi z}}, \tag{4.104}$$

because $\eta - \pi/4 = \nu\pi/2$.

Derivation of (4.100). For $\nu \notin \mathbb{Z}$, combine (4.98), (4.97), and (4.80) at the argument iz:

$$K_\nu(z) = \frac{\pi}{2}\frac{i^\nu J_{-\nu}(iz) - i^{-\nu}J_\nu(iz)}{\sin(\nu\pi)}$$
$$= \frac{\pi}{2}\left(\frac{i^\nu\cos(\nu\pi) - i^{-\nu}}{\sin(\nu\pi)}J_\nu(iz) - i^\nu Y_\nu(iz)\right). \tag{4.105}$$

Set $a = e^{i\nu\pi/2} = i^\nu$. Then $a^2 = e^{i\nu\pi}$, $a^{-2} = e^{-i\nu\pi}$, so by Euler's formula $\cos(\nu\pi) = (a^2 + a^{-2})/2$ and $\sin(\nu\pi) = (a^2 - a^{-2})/(2i)$. Substituting,

$$\frac{i^\nu\cos(\nu\pi) - i^{-\nu}}{\sin(\nu\pi)} = \frac{a\,(a^2 + a^{-2})/2 - a^{-1}}{(a^2 - a^{-2})/(2i)}.$$

The numerator equals $(a^3 + a^{-1})/2 - a^{-1} = (a^3 - a^{-1})/2 = a(a^2 - a^{-2})/2$ (factoring a out). So

$$\frac{a(a^2 - a^{-2})/2}{(a^2 - a^{-2})/(2i)} = ia = i^{\nu+1}.$$

Hence (4.105) becomes

$$K_\nu(z) = \frac{\pi}{2}\,i^{\nu+1}\big(J_\nu(iz) + iY_\nu(iz)\big) = \frac{\pi}{2}\,i^{\nu+1}H_\nu^{(1)}(iz). \tag{4.106}$$

Now apply Corollary 4.16 at the argument iz:

$$H_\nu^{(1)}(iz) \sim \sqrt{\frac{2}{\pi iz}}\,e^{i(iz - \nu\pi/2 - \pi/4)} = \sqrt{\frac{2}{\pi z}}\,e^{-i\pi/4}e^{-z}e^{-i\nu\pi/2}e^{-i\pi/4}. \tag{4.107}$$

Multiply by $(\pi/2)\,i^{\nu+1} = (\pi/2)\,e^{i\pi(\nu+1)/2}$:

$$K_\nu(z) \sim \frac{\pi}{2}\,e^{i\pi(\nu+1)/2}\sqrt{\frac{2}{\pi z}}\,e^{-z}\,e^{-i\pi/4 - i\nu\pi/2 - i\pi/4}. \tag{4.108}$$

Collect phases: $\pi(\nu+1)/2 - \pi/4 - \nu\pi/2 - \pi/4 = \nu\pi/2 + \pi/2 - \pi/2 - \nu\pi/2 = 0$. Therefore all phase factors multiply to 1:

$$K_\nu(z) \sim \frac{\pi}{2}\sqrt{\frac{2}{\pi z}}\,e^{-z} = \sqrt{\frac{\pi^2}{4}\cdot\frac{2}{\pi z}}\,e^{-z} = \sqrt{\frac{\pi}{2z}}\,e^{-z}, \tag{4.109}$$

which is (4.100). The integer-order case follows by taking the limit $\mu \to \nu$ from non-integer μ. □

Physically, I_ν and K_ν appear in cylindrical Laplace-type problems, 2D screened Coulomb interactions, and below-cutoff waveguide modes.

4.9 Orthogonality and Fourier–Bessel series

On $[0, a]$ with weight r, the rescaled functions $J_\nu(\alpha_{\nu,m}r/a)$ behave like sine modes on an interval. Here $\alpha_{\nu,m}$ is the m-th positive zero of J_ν.

Theorem 4.19 (Bessel orthogonality). *For $\nu \geq 0$, $a > 0$, and positive integers m, m',*

$$\int_0^a J_\nu\!\left(\frac{\alpha_{\nu,m}\,r}{a}\right)J_\nu\!\left(\frac{\alpha_{\nu,m'}\,r}{a}\right)r\,dr = \frac{a^2}{2}\big[J_{\nu+1}(\alpha_{\nu,m})\big]^2\,\delta_{m,m'}. \tag{4.110}$$

Here $\delta_{m,m'}$ is the Kronecker delta: it equals 1 when $m = m'$ and 0 otherwise.

Proof. Sturm–Liouville framing. A Sturm–Liouville problem is a second-order ODE arranged so that integration by parts produces orthogonality of different eigenfunctions. Here the radial equation has weight r, meaning the natural inner product is $\int_0^a u(r)v(r)\,r\,dr$. Thus distinct eigenfunctions are orthogonal; it remains to compute the normalization.

Write $u_m(r) := J_\nu(\alpha_{\nu,m} r/a)$ and $\beta_m := \alpha_{\nu,m}/a$. From Bessel's ODE (4.30) with argument $\beta_m r$, u_m satisfies

$$(ru_m')' + \left(\beta_m^2 r - \frac{\nu^2}{r}\right)u_m = 0, \tag{4.111}$$

Set $z = \beta_m r$. Then $u_m(r) = J_\nu(z)$, $u_m'(r) = \beta_m J_\nu'(z)$, and $ru_m'(r) = zJ_\nu'(z)$, so

$$(ru_m')'(r) = \beta_m\big(zJ_\nu'(z)\big)' = \beta_m\big(J_\nu'(z) + zJ_\nu''(z)\big),$$

where the outer prime is d/dr. Dividing Bessel's equation (4.30) by z gives $zJ_\nu'' + J_\nu' = (\nu^2/z - z)J_\nu$. Substituting,

$$(ru_m')' = \beta_m\left(\frac{\nu^2}{z} - z\right)J_\nu(z) = \left(\frac{\nu^2}{r} - \beta_m^2 r\right)u_m,$$

using $z = \beta_m r$. This rearranges to (4.111). The same equation at index m' is

$$(ru_{m'}')' + \left(\beta_{m'}^2 r - \frac{\nu^2}{r}\right)u_{m'} = 0. \tag{4.112}$$

Step 1 (cross-integrate). Multiply (4.111) by $u_{m'}$, (4.112) by u_m, and subtract:

$$u_{m'}(ru_m')' - u_m(ru_{m'}')' + (\beta_m^2 - \beta_{m'}^2)\,r\,u_m u_{m'} = 0. \tag{4.113}$$

The first two terms form a derivative:

$$\frac{d}{dr}\big[r(u_m' u_{m'} - u_m u_{m'}')\big] = (\beta_{m'}^2 - \beta_m^2)\,r\,u_m u_{m'}. \tag{4.114}$$

Integrate over $[0,a]$:

$$\big[r(u_m' u_{m'} - u_m u_{m'}')\big]_{r=0}^{r=a} = (\beta_{m'}^2 - \beta_m^2)\int_0^a u_m(r)u_{m'}(r)\,r\,dr. \tag{4.115}$$

At $r = 0$, the series (4.11) gives

$$u_m(r) = \frac{(\beta_m r/2)^\nu}{\Gamma(\nu+1)} + O(r^{\nu+2}), \qquad ru_m'(r) = \frac{\nu(\beta_m/2)^\nu}{\Gamma(\nu+1)}r^\nu + O(r^{\nu+2}) \quad (\nu > 0),$$

while for $\nu = 0$,

$$u_m(r) = 1 + O(r^2), \qquad ru_m'(r) = O(r^2).$$

Hence $r(u_m' u_{m'} - u_m u_{m'}') \to 0$ as $r \to 0^+$. At $r = a$, $u_m(a) = J_\nu(\alpha_{\nu,m}) = 0$ and likewise $u_{m'}(a) = 0$, so the boundary term there also vanishes.

Therefore the left side of (4.115) is zero, and for $m \neq m'$ we have $\beta_m \neq \beta_{m'}$, giving

$$\int_0^a u_m(r)u_{m'}(r)\,r\,dr = 0 \qquad (m \neq m'). \tag{4.116}$$

Step 2 (diagonal). For $m = m'$, evaluate

$$N_m := \int_0^a J_\nu(\beta_m r)^2\,r\,dr. \tag{4.117}$$

Multiply (4.111) by $2ru_m'$ and integrate:

$$\int_0^a 2ru_m'(ru_m')'\,dr + \int_0^a 2ru_m'\left(\beta_m^2 r - \frac{\nu^2}{r}\right)u_m\,dr = 0. \tag{4.118}$$

The first integral is

$$[ru_m'(r)]^2\Big|_0^a = a^2(u_m'(a))^2,$$

because $ru_m'(r) \to 0$ as $r \to 0^+$ by the same small-r estimates used above. The second integral, via $d(u_m^2)/dr = 2u_m u_m'$, becomes $\int_0^a \beta_m^2 r^2\, d(u_m^2)/dr\,dr - \nu^2 \int_0^a d(u_m^2)/dr\,dr$. Integrate by parts the first of these:

$$\int_0^a \beta_m^2 r^2 \frac{d(u_m^2)}{dr}\,dr = \beta_m^2 r^2 u_m^2\big|_0^a - 2\beta_m^2 \int_0^a r\,u_m^2\,dr = 0 - 2\beta_m^2 N_m, \tag{4.119}$$

using $u_m(a) = 0$ and $r^2u_m^2 \to 0$ at $r = 0$. The second term evaluates to $-\nu^2\big[u_m(a)^2 - u_m(0)^2\big]$. Collecting,

$$a^2(u_m'(a))^2 - 2\beta_m^2 N_m - \nu^2\big[u_m(a)^2 - u_m(0)^2\big] = 0. \tag{4.120}$$

Since $u_m(a) = 0$, and since $u_m(0) = 0$ for $\nu > 0$ while the coefficient ν^2 already annihilates the $u_m(0)^2$ term when $\nu = 0$, the last bracket vanishes:

$$N_m = \frac{a^2}{2\beta_m^2}(u_m'(a))^2. \tag{4.121}$$

Now $u_m'(a) = \beta_m J_\nu'(\alpha_{\nu,m})$, and from (4.29) at $z = \alpha_{\nu,m}$ (where $J_\nu(\alpha_{\nu,m}) = 0$), $J_\nu'(\alpha_{\nu,m}) = -J_{\nu+1}(\alpha_{\nu,m})$. Hence $(u_m'(a))^2 = \beta_m^2 J_{\nu+1}(\alpha_{\nu,m})^2$ and

$$N_m = \frac{a^2}{2\beta_m^2}\cdot \beta_m^2 J_{\nu+1}(\alpha_{\nu,m})^2 = \frac{a^2}{2}J_{\nu+1}(\alpha_{\nu,m})^2. \tag{4.122}$$

This establishes (4.110). □

The expansion

$$f(r) = \sum_{m=1}^{\infty} c_m J_\nu\!\left(\frac{\alpha_{\nu,m} r}{a}\right), \qquad c_m = \frac{2}{a^2 J_{\nu+1}(\alpha_{\nu,m})^2}\int_0^a f(r)J_\nu\!\left(\frac{\alpha_{\nu,m} r}{a}\right) r\,dr, \tag{4.123}$$

is the *Fourier–Bessel series* of f: the radial analogue of the Fourier series. It converges in the weighted L^2 sense for suitable square-integrable f on $[0,a]$ with weight $r\,dr$. For pointwise convergence at $r = a$, the series takes the Dirichlet boundary value 0, just as a sine series does at the endpoints.

4.10 Physical examples

Example 4.20 (Circular drumhead: lowest modes explicitly). *Consider the 2D wave equation on a disk of radius a with Dirichlet boundary $\psi|_{r=a} = 0$. Separation of variables $\psi(r,\phi,t) = R(r)\Phi(\phi)T(t)$ reduces the spatial part to*

$$R'' + R'/r + (k^2 - m^2/r^2)R = 0, \tag{4.124}$$

with $\Phi(\phi) = e^{im\phi}$, $m \in \mathbb{Z}$, $T(t) = e^{-i\omega t}$, $\omega = ck$. The radial equation depends on m^2, so the radial order is $|m|$; the modes m and $-m$ have the same frequency and correspond to the two real *angular shapes* $\cos(m\phi)$ *and* $\sin(m\phi)$ *when* $m \neq 0$*. Regularity at* $r = 0$ *selects* $R(r) = J_{|m|}(kr)$ *(*$Y_{|m|}$ *diverges). The boundary condition* $J_{|m|}(ka) = 0$ *quantizes* $k = \alpha_{|m|,n}/a$*; eigenfrequencies*

$$\omega_{m,n} = c\,\frac{\alpha_{|m|,n}}{a}. \tag{4.125}$$

Numerical zeros (positive zeros of J_m*):*

	$n=1$	$n=2$	$n=3$
$m=0$	2.4048	5.5201	8.6537
$m=1$	3.8317	7.0156	10.1735
$m=2$	5.1356	8.4172	11.6198
$m=3$	6.3802	9.7610	13.0152

Frequency ratios relative to the fundamental $\omega_{0,1}$*:*

$$\omega_{1,1}/\omega_{0,1} \approx 3.8317/2.4048 \approx 1.593, \quad \omega_{2,1}/\omega_{0,1} \approx 2.135, \quad \omega_{0,2}/\omega_{0,1} \approx 2.295,$$

$$\omega_{3,1}/\omega_{0,1} \approx 2.653, \quad \omega_{1,2}/\omega_{0,1} \approx 2.917.$$

These are not small-integer ratios, so a circular drum has inharmonic overtones. A 1D string instead has integer overtones $2:3:4:\ldots$*. The mode* $(m,n) = (0,1)$ *is axisymmetric, meaning independent of the angular coordinate, with one peak at the center;* $(1,1)$ *has a nodal diameter;* $(0,2)$ *has a nodal circle at* $r = a\alpha_{0,1}/\alpha_{0,2} \approx 0.436\,a$*.*

Example 4.21 (Cylindrical waveguide: TE and TM cutoffs)**.** *A perfectly conducting hollow circular cylinder of inner radius a supports transverse-magnetic (TM) and transverse-electric (TE) guided modes. With z along the guide and propagation factor* $e^{i(k_z z - \omega t)}$*, the axial fields* E_z, H_z *satisfy the 2D Helmholtz equation*

$$\left(\nabla_\perp^2 + \gamma^2\right)\psi = 0, \qquad \gamma^2 := \frac{\omega^2}{c^2} - k_z^2. \tag{4.126}$$

In polar coordinates, $\psi(r,\phi) \propto J_m(\gamma r)\,e^{im\phi}$ *(the* Y_m *solution is discarded by regularity at* $r = 0$*). As in the drum problem, the cutoff depends only on* $|m|$*; in the formulas below* $m \geq 0$ *is the listed angular index, while the angular factors with signs m and* $-m$ *are degenerate when* $m \neq 0$*. Boundary conditions on the perfect conductor fix* γ*:*

TM modes *(axial* E_z*):* $E_z|_{r=a} = 0$*, so* $J_m(\gamma a) = 0$*, giving* $\gamma_{m,n} = \alpha_{m,n}/a$*. Cutoff frequency (below which* k_z *is imaginary and the mode is evanescent, i.e. decays instead of propagating along the guide):*

$$f_c^{\mathrm{TM}_{m,n}} = \frac{c}{2\pi}\frac{\alpha_{m,n}}{a}. \tag{4.127}$$

This comes from setting $k_z = 0$ *at cutoff in* $\gamma^2 = \omega^2/c^2 - k_z^2$*, so* $\omega_c = c\gamma$ *and* $f_c = \omega_c/(2\pi)$*.*

TE modes *(axial* H_z*):* $\partial H_z/\partial r|_{r=a} = 0$*, so* $J_m'(\gamma a) = 0$*, giving* $\gamma_{m,n} = \alpha'_{m,n}/a$*, where* $\alpha'_{m,n}$ *is the n-th positive zero of* J_m'*:*

$$f_c^{\mathrm{TE}_{m,n}} = \frac{c}{2\pi}\frac{\alpha'_{m,n}}{a}. \tag{4.128}$$

Numerical zeros of J_m'*:*

$$\alpha'_{1,1} \approx 1.8412, \quad \alpha'_{2,1} \approx 3.0542, \quad \alpha'_{0,1} \approx 3.8317, \quad \alpha'_{1,2} \approx 5.3314.$$

Note $\alpha'_{0,1} = \alpha_{1,1}$*, a consequence of* $J_0' = -J_1$*.*

Ordering *(from low to high cutoff):*

$$\begin{aligned}
TE_{1,1}:&\ \alpha'_{1,1} \approx 1.8412 \qquad \textit{(fundamental)}\\
TM_{0,1}:&\ \alpha_{0,1} \approx 2.4048\\
TE_{2,1}:&\ \alpha'_{2,1} \approx 3.0542\\
TE_{0,1} = TM_{1,1}:&\ \approx 3.8317\\
TM_{2,1}:&\ \alpha_{2,1} \approx 5.1356.
\end{aligned}$$

Numerical cutoff at $a = 1$ cm, $c = 3 \times 10^{10}$ cm/s:

$$\begin{aligned}
f_c(TE_{1,1}) &= \frac{3 \times 10^{10} \cdot 1.8412}{2\pi \cdot 1} \approx 8.79 \text{ GHz},\\
f_c(TM_{0,1}) &\approx 11.49 \text{ GHz},\\
f_c(TE_{2,1}) &\approx 14.58 \text{ GHz}.
\end{aligned}$$

The single-mode bandwidth *is the frequency interval between $TE_{1,1}$ and the next cutoff, $TM_{0,1}$: here ≈ 8.79–11.49 GHz. In this window only $TE_{1,1}$ propagates. Below it all modes are evanescent; above it multiple modes propagate and dispersion, the frequency-dependence of propagation speed, becomes harder to control.*

Example 4.22 (2D Helmholtz Green's function)**.** *A Green's function is the response to an ideal point source. The outgoing Green's function for $(-\nabla^2 - k^2)G = \delta^{(2)}(\mathbf{r})$ in the plane is*

$$G(\mathbf{r}) = \frac{i}{4} H_0^{(1)}(kr). \tag{4.129}$$

Here $\delta^{(2)}(\mathbf{r})$ is the two-dimensional Dirac delta: it is zero away from the origin and integrates to 1 over any region containing the origin. The three checks are as follows. Away from the origin, $H_0^{(1)}(kr)$ solves Bessel's equation at $\nu = 0$. Near $r = 0$,

$$H_0^{(1)}(kr) = 1 + \frac{2i}{\pi}\left[\log\left(\frac{kr}{2}\right) + \gamma\right] + O(r^2|\log r|),$$

so

$$G(\mathbf{r}) = -\frac{1}{2\pi}\log r + O(1),$$

and $\Delta \log r = 2\pi\,\delta^{(2)}(\mathbf{r})$ gives $-\Delta G = \delta^{(2)}(\mathbf{r})$ at the singular point. The remaining term $-k^2 G$ is locally integrable near $r = 0$, so it contributes no delta mass, meaning no additional point-source contribution. Finally, $G \sim (i/4)\sqrt{2/(\pi k r)}\,e^{i(kr-\pi/4)}$ as $r \to \infty$, the outgoing cylindrical-wave amplitude from Corollary 4.16.

Exercises

Problem 4.1. *Derive the addition theorem $J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y)$ from the generating function $G(x+y,t) = G(x,t)G(y,t)$ and Cauchy product.*

Problem 4.2. *Prove $\sum_{n=-\infty}^{\infty} J_n(z)^2 = 1$ (equivalently, $J_0(z)^2 + 2\sum_{n=1}^{\infty} J_n(z)^2 = 1$) for real z. Hint: use $G(z,t)\cdot G(z,1/t) = 1$ and extract the constant term.*

Problem 4.3. *Use* (4.3) *to derive the series* (4.4) *directly by Laurent-expanding the exponential and applying the residue theorem (Theorem 1.25).*

Problem 4.4. *Compute* $\int_0^\infty J_0(bx)e^{-ax}\,dx$ *for* $a>0$, $b>0$, *using the integral representation* (4.45) *and Fubini.*

Problem 4.5. *Show* $J_{1/2}(z)=\sqrt{2/(\pi z)}\,\sin z$ *and* $J_{-1/2}(z)=\sqrt{2/(\pi z)}\,\cos z$ *by inserting* $\nu=\pm 1/2$ *into* (4.11) *and recognizing the resulting series. Cross-check against* (4.64).

Problem 4.6. *Verify the Wronskian* $W[J_\nu,J_{-\nu}](z)=-2\sin(\nu\pi)/(\pi z)$ *by computing the Wronskian at a specific point (e.g. small* z*) using the leading terms of each series, and by observing that* W *satisfies* $W'+W/z=0$.

Problem 4.7. *Starting from* (4.49)*, apply steepest descent to re-derive* (4.65) *with attention to the contribution of both saddles* $t=\pm i$*. Check every phase.*

Problem 4.8. *Rework Step 3 of the proof of Theorem 4.14 directly: show that the cubic contour* Γ *is the preimage of the standard Hankel contour under* $u=\sigma^3/3$*, compute* $\int_\Gamma e^{\sigma^3/3}\,d\sigma$*, and recover the constant in* (4.70) *from Theorem 3.16.*

Problem 4.9. *Verify* (4.91) *by a second route: use Abel's identity on Bessel's ODE to show* $zW[H_\nu^{(1)},H_\nu^{(2)}]$ *is constant, then evaluate the constant using the leading large-*z *asymptotics* (4.85) *directly, confirming it equals* $-4i/\pi$.

Problem 4.10. *Show that the Fourier–Bessel coefficients* (4.123) *of the constant function* $f(r)=1$ *on* $[0,a]$ *at* $\nu=0$ *are* $c_m=2/[\alpha_{0,m}J_1(\alpha_{0,m})]$*. Hint:* $\int_0^a J_0(\lambda r)\,r\,dr=(a/\lambda)J_1(\lambda a)$*; derive this from* (4.14) *at* $\nu=1$.

Problem 4.11. *(Numerical.) Compute the first three cutoff frequencies of a circular waveguide of radius* $a=2.5$ *mm. Identify the single-mode bandwidth. Determine which modes propagate at* $f=15$ *GHz.*

Problem 4.12. *Derive an explicit series for* Y_0*: show*

$$Y_0(z)=\tfrac{2}{\pi}\big[\ln(z/2)+\gamma\big]J_0(z)-\tfrac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{(z/2)^{2k}}{(k!)^2}H_k,$$

where $H_k=1+1/2+\cdots+1/k$ *and* γ *is the Euler–Mascheroni constant, by differentiating* J_ν *in* ν *at* $\nu=0$ *and using* $\Gamma'(1)=-\gamma$ *together with* $\Gamma'(k+1)/\Gamma(k+1)=\psi(k+1)=-\gamma+H_k$.

Problem 4.13. *(Modified Bessel integral representation.) Prove*

$$K_0(z)=\int_0^\infty e^{-z\cosh t}\,dt \qquad (\operatorname{Re} z>0)$$

by verifying that the right-hand side satisfies the modified Bessel ODE at $\nu=0$ *(differentiate under the integral), has the correct logarithmic singularity as* $z\to 0^+$*, and matches* (4.100) *via Laplace's method as* $z\to\infty$.

Problem 4.14. *(Transition-regime verification.) For* $\varphi(t)=\frac{1}{2}(t-1/t)-\log t$ *in* (4.72)*, verify explicitly* $\varphi(1)=0$, $\varphi'(1)=0$, $\varphi''(1)=0$*, and* $\varphi'''(1)=1$*. Expand to* $O(s^4)$ *with* $t=1+s$ *and compute the* s^4 *coefficient, confirming it is* $-1/4$ *(the next-order Airy correction).*

Problem 4.15. *(Drum overtones.) Using the numerical zeros in Example 4.20, show the ratios* $\omega_{0,1}:\omega_{0,2}:\omega_{0,3}\approx 1:2.295:3.598$ *for axisymmetric drum modes. Compare to a 1D string* $(1:2:3:\ldots)$*. Argue qualitatively why a drum is perceived as having an indefinite pitch while a string does not.*

5 Legendre Functions

Legendre polynomials organize angular dependence in spherical problems: electrostatic multipoles, the angular Schrödinger equation for a central potential, and partial-wave scattering. As with Bessel functions (Section 4), one can start from a power-series solution of the ODE. Here the cleaner route is the generating function: $P_n(x)$ appears as the t^n coefficient in the expansion of the physical kernel $1/|\mathbf{r}-\mathbf{r}'|$.

Prerequisites: Section 1 for Cauchy's formula (Thm. 1.13), its derivative form (Cor. 1.14), and the identity theorem (Thm. 2.2); Section 3 for Γ and factorials in the spherical-harmonic normalization. Section 4 gives a similar generating-function viewpoint but is not required.

5.1 The generating function

Start with the geometry. Place two points $\mathbf{r},\mathbf{r}'$ in space, with $|\mathbf{r}| = r$, $|\mathbf{r}'| = r'$, and angle θ between them. The law of cosines gives

$$|\mathbf{r}-\mathbf{r}'|^2 \;=\; r^2 + r'^2 - 2rr'\cos\theta. \tag{5.1}$$

If $r < r'$, factoring r'^2 out yields

$$|\mathbf{r}-\mathbf{r}'|^2 \;=\; r'^2\big[\,1-2(r/r')\cos\theta + (r/r')^2\,\big], \tag{5.2}$$

so the reciprocal distance is $1/r'$ times a function of the two dimensionless quantities $\cos\theta$ and r/r'. That function is the object we now name.

Definition 5.1 (Legendre generating function)**.** *For $x \in [-1,1]$ and $|t| < 1$,*

$$G(x,t) \;=\; \frac{1}{\sqrt{1-2xt+t^2}}. \tag{5.3}$$

The branch of the square root is the one that equals 1 at $t = 0$.

Proposition 5.2 (Physical origin)**.** *For $\mathbf{r},\mathbf{r}'$ with $r < r'$ and angle θ between them, setting $x = \cos\theta$ and $t = r/r'$,*

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \;=\; \frac{1}{r'}\,G(x,t). \tag{5.4}$$

Proof. Take the positive square root of (5.2):

$$|\mathbf{r}-\mathbf{r}'| \;=\; r'\sqrt{1-2(r/r')\cos\theta + (r/r')^2}.$$

Invert and set $x = \cos\theta$, $t = r/r'$. This gives $G(x,t)/r'$, which is (5.4). □

For $|x| \le 1$ and $|t| < 1$, the denominator of G does not vanish. If $|x| < 1$, its zeros are $t = x \pm i\sqrt{1-x^2}$, both on the unit circle; if $x = \pm 1$, the zero is the boundary point $t = x$. Thus, for each fixed $x \in [-1,1]$, $G(x,t)$ is holomorphic in t on $|t| < 1$ and has a convergent Taylor expansion there.

Definition 5.3 (Legendre polynomial)**.** *$P_n(x)$ is the coefficient of t^n in the Taylor expansion of $G(x,t)$ at $t = 0$:*

$$\frac{1}{\sqrt{1-2xt+t^2}} \;=\; \sum_{n=0}^{\infty} P_n(x)\,t^n. \tag{5.5}$$

By the Cauchy coefficient formula (Cor. 1.14) applied to the holomorphic function $t \mapsto G(x,t)$, for any positively oriented circle $|t| = \rho < 1$,

$$P_n(x) \;=\; \frac{1}{2\pi i}\oint_{|t|=\rho} \frac{dt}{t^{n+1}\sqrt{1-2xt+t^2}}. \tag{5.6}$$

Remark 5.4 (Uniform coefficient bound). *Fix ρ with $0<\rho<1$. For $x=\cos\theta$ with $\theta\in[0,\pi]$ and $|\zeta|=\rho$,*

$$1-2x\zeta+\zeta^2=\big(1-e^{i\theta}\zeta\big)\big(1-e^{-i\theta}\zeta\big).$$

Each factor has modulus at least $1-\rho$, so

$$|1-2x\zeta+\zeta^2|\;\ge\;(1-\rho)^2.$$

Therefore

$$|G(x,\zeta)|\;=\;\frac{1}{|1-2x\zeta+\zeta^2|^{1/2}}\;\le\;\frac{1}{1-\rho},$$

uniformly in $x\in[-1,1]$. Inserting this estimate into the contour formula (5.6) gives

$$|P_n(x)|\;\le\;\frac{1}{2\pi}\cdot(2\pi\rho)\cdot\frac{1}{\rho^{n+1}}\cdot\frac{1}{1-\rho}\;=\;\frac{1}{(1-\rho)\rho^n},\qquad x\in[-1,1].$$

In particular, if $|t|<\rho<1$, then $\sum_{n\ge0}P_n(x)t^n$ converges absolutely and uniformly in $x\in[-1,1]$. This is the estimate that will justify termwise integration later.

Proposition 5.5 (Polynomial nature, low orders). *$P_n(x)$ is a polynomial in x of degree n. Explicitly,*

$$P_0=1,\quad P_1=x,\quad P_2=\tfrac12(3x^2-1),\quad P_3=\tfrac12(5x^3-3x),\quad P_4=\tfrac18(35x^4-30x^2+3).\quad (5.7)$$

Proof. Write $1-2xt+t^2=1-u$ with $u=2xt-t^2$, and use the binomial series

$$(1-u)^{-1/2}\;=\;\sum_{k=0}^{\infty}\binom{-1/2}{k}(-u)^k\;=\;\sum_{k=0}^{\infty}\frac{(2k)!}{4^k(k!)^2}\,u^k,\qquad (5.8)$$

where the second equality uses $\binom{-1/2}{k}(-1)^k=(2k)!/(4^k(k!)^2)$, equivalently $(1/2)_k/k!$. To verify: by the definition of the generalized binomial coefficient,

$$\binom{-1/2}{k}=\frac{(-\frac12)(-\frac32)\cdots(-(2k-1)/2)}{k!}=\frac{(-1)^k\,(2k-1)!!}{2^k\,k!},$$

where $(2k-1)!!=1\cdot3\cdot5\cdots(2k-1)$ is the double factorial. Multiplying by $(-1)^k$ cancels the sign:

$$\binom{-1/2}{k}(-1)^k=\frac{(2k-1)!!}{2^k\,k!}=\frac{(2k)!}{4^k\,(k!)^2},$$

the last step using the identity $(2k)!=2^k\,k!\cdot(2k-1)!!$ (pair the even factors $2\cdot4\cdots2k=2^k\,k!$ with the remaining odd factors). Substituting $u=t(2x-t)$ gives

$$G(x,t)\;=\;\sum_{k=0}^{\infty}\frac{(2k)!}{4^k(k!)^2}\,t^k(2x-t)^k.\qquad (5.9)$$

For fixed k, $(2x-t)^k$ is a degree-k polynomial in t whose coefficients are polynomials in x. Hence $t^k(2x-t)^k$ contributes only to powers $t^k,\dots,t^{2k}$. The coefficient of t^n therefore receives terms only from $\lceil n/2\rceil\le k\le n$. In the k-th term, the largest possible power of x is x^k; the largest admissible k is n. Thus P_n is a polynomial of degree n.

Low orders. Expanding (5.9) through t^4 gives exactly the list in (5.7). □

Remark 5.6. *At $x=1$, the generating function collapses to $G(1,t)=1/(1-t)=\sum_n t^n$, so $P_n(1)=1$. At $x=-1$, it becomes $G(-1,t)=1/(1+t)=\sum_n(-t)^n$, so $P_n(-1)=(-1)^n$.*

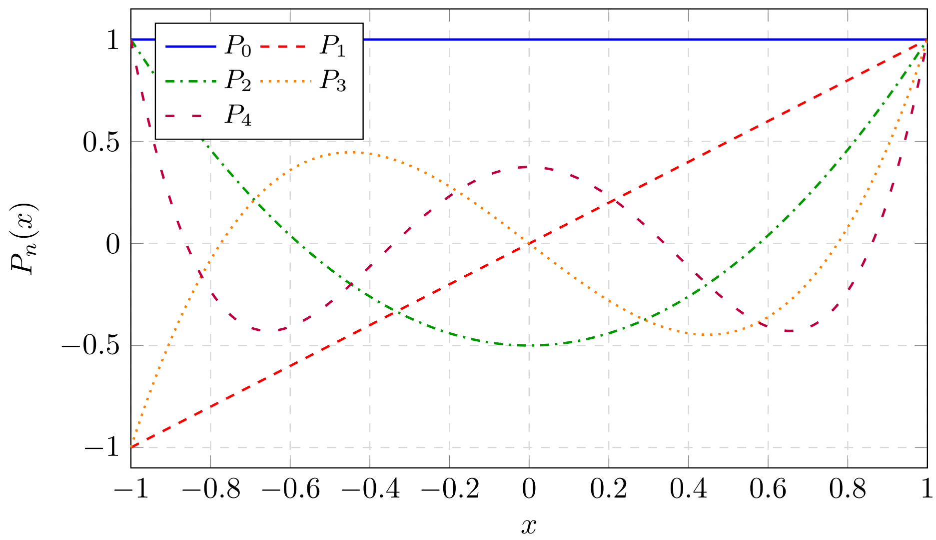

Figure 10: The first five Legendre polynomials $P_0, \ldots, P_4$ on $[-1, 1]$, as given by (5.7). All satisfy $P_n(\pm 1) = (\pm 1)^n$. The parity $P_n(-x) = (-1)^n P_n(x)$ is visible: even n are symmetric, odd n antisymmetric.

5.2 Recurrences

Three recurrences handle most calculations with P_n. All three come from differentiating the generating function.

Proposition 5.7 (Recurrence relations)**.** *For* $n \geq 1$*,*

$$(n+1)P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x), \tag{5.10}$$

$$(1-x^2)P_n'(x) = n\big[P_{n-1}(x) - xP_n(x)\big], \tag{5.11}$$

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)\,P_n(x). \tag{5.12}$$

Proof. Proof of (5.10). The strategy: differentiate the generating function identity $\sum P_n t^n = (1-2xt+t^2)^{-1/2}$ with respect to t, clear the denominator, substitute the series, and match coefficients of t^n. Differentiating (5.5) in t,

$$\frac{\partial G}{\partial t} = -\tfrac{1}{2}(1-2xt+t^2)^{-3/2} \cdot (-2x+2t) = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{(x-t)\,G}{1-2xt+t^2}, \tag{5.13}$$

where the last step uses $G = (1-2xt+t^2)^{-1/2}$ to replace $(1-2xt+t^2)^{-3/2}$ by $G/(1-2xt+t^2)$. Multiply both sides of (5.13) by $(1-2xt+t^2)$ to clear the denominator:

$$(1-2xt+t^2)\,\frac{\partial G}{\partial t} = (x-t)\,G. \tag{5.14}$$

Now substitute the series $G = \sum_{n\geq 0} P_n t^n$ and its termwise t-derivative $\partial G/\partial t = \sum_{n\geq 0} nP_n t^{n-1}$ (termwise differentiation is valid inside the radius of convergence):

$$(1-2xt+t^2)\sum_{n\geq 0} nP_n t^{n-1} = (x-t)\sum_{n\geq 0} P_n t^n. \tag{5.15}$$

Expand the left side:

$$\sum_{n\geq 0} nP_n t^{n-1} - 2x\sum_{n\geq 0} nP_n t^n + \sum_{n\geq 0} nP_n t^{n+1}.$$

Shift indices so that each sum is over t^n. In the first sum, set $n' = n-1$; it becomes $\sum_{n'\geq -1}(n'+1)P_{n'+1}t^{n'}$, and the $n'=-1$ term vanishes, so it starts at $n'=0$. In the third sum, set $n'=n+1$; it becomes $\sum_{n'\geq 1}(n'-1)P_{n'-1}t^{n'}$. Relabeling $n' \to n$ throughout:

$$\sum_{n\geq 0}(n+1)P_{n+1}t^n - 2x\sum_{n\geq 0} nP_n t^n + \sum_{n\geq 1}(n-1)P_{n-1}t^n. \tag{5.16}$$

Similarly the right side:

$$(x-t)\sum_{n\geq 0} P_n t^n \;=\; x\sum_{n\geq 0} P_n t^n - \sum_{n\geq 1} P_{n-1}t^n. \tag{5.17}$$

Equate coefficients of t^n for $n \geq 1$. From (5.16): $(n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1}$. From (5.17): $xP_n - P_{n-1}$. Therefore

$$(n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1} \;=\; xP_n - P_{n-1}. \tag{5.18}$$

Move everything to the left:

$$(n+1)P_{n+1} - 2xnP_n - xP_n + (n-1)P_{n-1} + P_{n-1} \;=\; 0,$$

and collect: the coefficient of P_n is $-(2n+1)x$, and the coefficient of P_{n-1} is n. Solving for $(n+1)P_{n+1}$:

$$(n+1)P_{n+1} \;=\; (2n+1)x\,P_n - n\,P_{n-1},$$

which is (5.10).

Proof of (5.11). Differentiate the generating function in x:

$$\frac{\partial G}{\partial x} \;=\; \frac{t}{(1-2xt+t^2)^{3/2}} \;=\; \frac{t\,G}{1-2xt+t^2}. \tag{5.19}$$

Comparing with (5.13),

$$(x-t)\,\frac{\partial G}{\partial x} \;=\; t\,\frac{\partial G}{\partial t}. \tag{5.20}$$

Substitute the series and match coefficients. Since $\partial G/\partial x = \sum P_n'(x)t^n$ and $t\,\partial G/\partial t = \sum nP_n t^n$, the left side is $x\sum P_n' t^n - \sum P_{n-1}' t^n$ after shifting. Hence

$$x\,P_n'(x) - P_{n-1}'(x) \;=\; nP_n(x). \tag{5.21}$$

Use (5.19) and (5.13) to combine the x- and t-derivatives in a way that produces the factor $1-x^2$:

$$\begin{aligned} tG - t(x-t)\,\frac{\partial G}{\partial t} &= \frac{t}{\sqrt{1-2xt+t^2}} - \frac{t(x-t)^2}{(1-2xt+t^2)^{3/2}} \\ &= \frac{t\big[(1-2xt+t^2)-(x-t)^2\big]}{(1-2xt+t^2)^{3/2}} \\ &= \frac{t(1-x^2)}{(1-2xt+t^2)^{3/2}} \\ &= (1-x^2)\,\frac{\partial G}{\partial x}. \end{aligned} \tag{5.22}$$

Substitute the series:

$$(1-x^2)\sum_{n\geq 0} P_n'(x)t^n \;=\; t\sum_{n\geq 0} P_n(x)t^n - x\sum_{n\geq 0} nP_n(x)t^n + \sum_{n\geq 0} nP_n(x)t^{n+1}. \tag{5.23}$$

Shift the two sums carrying an extra factor of t:

$$(1-x^2)\sum_{n\geq 0} P_n' t^n \;=\; \sum_{n\geq 1} P_{n-1} t^n - x\sum_{n\geq 0} nP_n t^n + \sum_{n\geq 1}(n-1)P_{n-1} t^n. \tag{5.24}$$

For $n \geq 1$, the coefficient of t^n on the right is

$$P_{n-1} - nxP_n + (n-1)P_{n-1} \;=\; nP_{n-1} - nxP_n.$$

Therefore

$$(1-x^2)P_n'(x) \;=\; nP_{n-1}(x) - nxP_n(x),$$

which is (5.11).

Proof of (5.12). Combine (5.19) and (5.13) in a second way:

$$\begin{aligned} 2t^2\frac{\partial G}{\partial t} + tG &= \frac{2t^2(x-t)}{(1-2xt+t^2)^{3/2}} + \frac{t(1-2xt+t^2)}{(1-2xt+t^2)^{3/2}} \\ &= \frac{t\big[(1-2xt+t^2)+2t(x-t)\big]}{(1-2xt+t^2)^{3/2}} \\ &= \frac{t(1-t^2)}{(1-2xt+t^2)^{3/2}} \\ &= (1-t^2)\frac{\partial G}{\partial x}. \end{aligned} \tag{5.25}$$

Substitute the series again:

$$(1-t^2)\sum_{n\geq 0} P_n'(x)t^n \;=\; 2t^2\sum_{n\geq 0} nP_n(x)t^{n-1} + t\sum_{n\geq 0} P_n(x)t^n. \tag{5.26}$$

Shift indices so every sum is in powers of t^n:

$$\sum_{n\geq 0} P_n' t^n - \sum_{n\geq 2} P_{n-2}' t^n \;=\; \sum_{n\geq 1} 2(n-1)P_{n-1} t^n + \sum_{n\geq 1} P_{n-1} t^n. \tag{5.27}$$

For $n \geq 1$, equating coefficients of t^n gives

$$P_n'(x) - P_{n-2}'(x) \;=\; \big[2(n-1)+1\big]P_{n-1}(x) \;=\; (2n-1)P_{n-1}(x).$$

Replace n by $n+1$:

$$P_{n+1}'(x) - P_{n-1}'(x) \;=\; (2n+1)P_n(x),$$

which is (5.12). □

Example 5.8 (Low-index check). *At* $n=1$*:* $(n+1)P_{n+1} = 2P_2 = 3x^2-1$ *from* (5.7)*. The right side of* (5.10) *is* $3x\cdot x - 1\cdot 1 = 3x^2-1$*. Agreement.*

5.3 Rodrigues formula and Schläfli integral

Two other representations of P_n are worth keeping close: a contour integral (Schläfli) and a compact differential formula (Rodrigues). They make orthogonality, differentiation identities, and coefficient formulas much easier to derive.

Proposition 5.9 (Schläfli integral). *For any sufficiently small* $\rho > 0$,

$$P_n(x) \;=\; \frac{1}{2\pi i}\oint_{|w-x|=\rho} \frac{(w^2-1)^n}{2^n(w-x)^{n+1}}\,dw. \tag{5.28}$$

Because the integrand is holomorphic away from $w = x$*, the same value is obtained on any simple positively oriented contour enclosing* x*.*

Proof. Start from the generating-function contour (5.6):

$$P_n(x) = \frac{1}{2\pi i}\oint_{|t|=\rho} \frac{dt}{t^{n+1}\sqrt{1-2xt+t^2}}. \tag{5.29}$$

The idea is to find a change of variable $t \longleftrightarrow w$ that makes the square root $\sqrt{1-2xt+t^2}$ rational. If we can write $dt/t^{n+1}\sqrt{\cdots}$ as a rational function times dw, the Cauchy derivative formula (Cor. 1.14) gives Rodrigues' formula directly. Set

$$t = \frac{2(w-x)}{w^2-1}, \qquad \text{equivalently} \qquad tw^2 - 2w + (2x-t) = 0. \tag{5.30}$$

Solving the quadratic for w gives

$$w_\pm = \frac{1 \pm \sqrt{1-2xt+t^2}}{t}. \tag{5.31}$$

Choose the $(-)$ branch,

$$w = \frac{1-\sqrt{1-2xt+t^2}}{t}, \tag{5.32}$$

because this branch sends $w \to x$ as $t \to 0$:

$$\sqrt{1-2xt+t^2} = 1 - xt + O(t^2),$$

so $w = (xt + O(t^2))/t = x + O(t)$.

Differentiate (5.30) implicitly:

$$w^2\,dt + 2tw\,dw - 2\,dw - dt = 0, \tag{5.33}$$

hence

$$(w^2-1)\,dt = 2(1-tw)\,dw. \tag{5.34}$$

From (5.32),

$$\sqrt{1-2xt+t^2} = 1 - tw, \tag{5.35}$$

so (5.34) becomes

$$dt = \frac{2\sqrt{1-2xt+t^2}}{w^2-1}\,dw. \tag{5.36}$$

Also (5.30) gives

$$t^{n+1} = \left[\frac{2(w-x)}{w^2-1}\right]^{n+1}. \tag{5.37}$$

Substitute (5.36) and (5.37) into (5.29):

$$\begin{aligned} P_n(x) &= \frac{1}{2\pi i}\oint \left(\frac{w^2-1}{2(w-x)}\right)^{n+1} \frac{1}{\sqrt{1-2xt+t^2}} \frac{2\sqrt{1-2xt+t^2}}{w^2-1}\,dw \\ &= \frac{1}{2\pi i}\oint \frac{(w^2-1)^n}{2^n(w-x)^{n+1}}\,dw, \end{aligned}$$

which is (5.28) for the small circle around $w = x$. For $x \neq \pm 1$, the chosen branch maps a small contour around $t = 0$ to one around $w = x$; the endpoint cases follow by continuity in x. Since the integrand is holomorphic on $\mathbb{C} \setminus \{x\}$, Cauchy's theorem then allows us to deform that circle to any simple positively oriented contour around x. □

Theorem 5.10 (Rodrigues)**.**

$$P_n(x) \;=\; \frac{1}{2^n n!}\,\frac{d^n}{dx^n}(x^2-1)^n. \tag{5.38}$$

Proof. The function $w \mapsto (w^2-1)^n$ is entire, so Cauchy's derivative formula gives

$$\frac{d^n}{dx^n}(x^2-1)^n \;=\; \frac{n!}{2\pi i}\oint_{|w-x|=\rho} \frac{(w^2-1)^n}{(w-x)^{n+1}}\,dw. \tag{5.39}$$

Divide by $2^n n!$ and compare with (5.28). The result is (5.38). □

Remark 5.11 (Logical order)**.** *The logical chain is*

$$\textit{generating function} \implies \textit{Schläfli integral} \implies \textit{Rodrigues},$$

while Legendre's ODE is proved independently in Theorem 5.13. This avoids using the ODE to prove Rodrigues and then using Rodrigues to prove the ODE.

Example 5.12 (Verification at $n=2$)**.** $(x^2-1)^2 = x^4-2x^2+1$*. First derivative:* $4x^3-4x$*. Second derivative:* $12x^2-4$*. Divide by* $2^2\cdot 2! = 8$*:* $(12x^2-4)/8 = (3x^2-1)/2 = P_2(x)$*, matching* (5.7).

5.4 Legendre's ODE

The generating function defines the Legendre polynomials algebraically. The same polynomials also arise as eigenfunctions of a natural second-order differential operator on $[-1,1]$: the operator sends P_n to a scalar multiple of P_n. This ODE is the source of the structural facts that follow: orthogonality, Sturm–Liouville form, and the link to Laplace's equation on S^2. Here Sturm–Liouville form means the second derivative is packaged as a single derivative of $p(x)y'$, which makes integration by parts produce orthogonality.

Theorem 5.13 (Legendre equation)**.** P_n *satisfies*

$$(1-x^2)\,y'' - 2x\,y' + n(n+1)\,y \;=\; 0, \tag{5.40}$$

equivalently in Sturm–Liouville form $\frac{d}{dx}\big[(1-x^2)y'\big] + n(n+1)y = 0$.

Proof. Differentiate (5.11) with respect to x:

$$\frac{d}{dx}\big[(1-x^2)P_n'(x)\big] \;=\; \frac{d}{dx}\Big(n\big[P_{n-1}(x) - xP_n(x)\big]\Big). \tag{5.41}$$

Differentiate both sides:

$$(1-x^2)P_n''(x) - 2xP_n'(x) \;=\; n\big[P_{n-1}'(x) - P_n(x) - xP_n'(x)\big]. \tag{5.42}$$

Now use (5.21), namely $xP_n'(x) - P_{n-1}'(x) = nP_n(x)$, or equivalently

$$P_{n-1}'(x) - xP_n'(x) \;=\; -nP_n(x). \tag{5.43}$$

Substituting (5.43) into (5.42) yields

$$(1-x^2)P_n'' - 2xP_n' \;=\; n\big[-nP_n - P_n\big] \;=\; -n(n+1)P_n.$$

Move all terms to the left:

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n \;=\; 0,$$

which is (5.40). Writing the left side as $\frac{d}{dx}\big[(1-x^2)P_n'\big] + n(n+1)P_n$ gives the Sturm–Liouville form. □

Remark 5.14 (Sturm–Liouville form). *In Sturm–Liouville theory, a second-order ODE is written as* $[p(x)y']' + q(x)y = \lambda w(x)y$*. Here the Legendre equation* (5.40) *has* $p(x) = 1-x^2$*,* $w(x) = 1$ *(the* weight function*), and eigenvalue* $\lambda = -n(n+1)$*. The word* eigenvalue *means the scalar multiplying the unknown function, and the corresponding nonzero solution is the eigenfunction. The leading coefficient* $1-x^2$ *vanishes at* $x = \pm 1$*, making these* singular endpoints *(where* $p(x) = 0$*). At such endpoints we do not impose fixed boundary conditions; instead we require* regularity *(finiteness) at* ± 1*. This condition selects the polynomial solutions* P_n *and excludes the logarithmically singular solutions* Q_n *(introduced in Section 5.8).*

5.5 Orthogonality and normalization

The integration-by-parts symmetry in the Sturm–Liouville form is often called *self-adjointness*: it lets the differential operator move from one factor in an integral to the other without leaving boundary terms. This is why different eigenvalues $n(n+1)$ give orthogonal functions in $L^2[-1,1]$, the space of square-integrable functions on $[-1,1]$. We now compute the normalization constant, which fixes the coefficients in Legendre series.

Theorem 5.15 (Orthogonality).

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\,\delta_{mn}. \tag{5.44}$$

Here δ_{mn} *is the Kronecker delta: it equals* 1 *when* $m = n$ *and* 0 *otherwise.*

Proof. Step 1: Orthogonality ($m \neq n$)*.* Legendre's equation in Sturm–Liouville form is $[(1-x^2)P_n']' = -n(n+1)P_n$. Multiply by P_m, and similarly the equation for P_m multiplied by P_n, and subtract:

$$[(1-x^2)P_n']'\,P_m - [(1-x^2)P_m']'\,P_n = [m(m+1)-n(n+1)]\,P_mP_n.$$

Integrate both sides from -1 to 1. The left side integrates by parts (Lagrange identity):

$$\begin{aligned}\int_{-1}^{1}\big([(1-x^2)P_n']'P_m - [(1-x^2)P_m']'P_n\big)dx &= \big[(1-x^2)(P_n'P_m - P_m'P_n)\big]_{-1}^{1} \\ &\quad - \int_{-1}^{1}(1-x^2)(P_n'P_m' - P_m'P_n')\,dx \\ &= 0-0 = 0,\end{aligned}$$

since $(1-x^2)$ vanishes at $x = \pm 1$ (killing the boundary term) and $P_n'P_m' - P_m'P_n' = 0$. Therefore

$$[m(m+1)-n(n+1)]\int_{-1}^{1} P_mP_n\,dx = 0.$$

For nonnegative integers m,n with $m \neq n$, $m(m+1)-n(n+1) = (m-n)(m+n+1) \neq 0$, forcing the integral to vanish.

Step 2: Normalization. Integrate the square of the generating function. From $G(x,t) = \sum_n P_n(x)t^n$,

$$G(x,t)^2 = \sum_{m,n\geq 0} P_m(x)P_n(x)\,t^{m+n}. \tag{5.45}$$

Fix t with $|t| < 1$, and choose ρ so that $|t| < \rho < 1$. By Remark 5.4,

$$|P_m(x)P_n(x)t^{m+n}| \leq \frac{1}{(1-\rho)^2}\left(\frac{|t|}{\rho}\right)^{m+n}, \qquad x \in [-1,1].$$

The majorant on the right is summable over $m, n \geq 0$, so the double series in (5.45) converges absolutely and uniformly on $[-1, 1]$. We may therefore integrate term by term. By Step 1, only the diagonal terms remain:

$$\int_{-1}^{1} G(x,t)^2 \, dx \;=\; \sum_{n\geq 0} t^{2n} \int_{-1}^{1} P_n(x)^2 \, dx. \tag{5.46}$$

For the left side, fix $0 < t < 1$ temporarily. Then $G^2 = 1/(1-2xt+t^2)$, so

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2}. \tag{5.47}$$

Use the antiderivative $\frac{d}{dx}\ln(1-2xt+t^2) = -2t/(1-2xt+t^2)$:

$$\int \frac{dx}{1-2xt+t^2} \;=\; -\frac{1}{2t}\ln(1-2xt+t^2) + C. \tag{5.48}$$

Evaluate from -1 to 1:

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} \;=\; -\frac{1}{2t}\Big[\ln(1-2t+t^2) - \ln(1+2t+t^2)\Big] \;=\; -\frac{1}{2t}\Big[\ln(1-t)^2 - \ln(1+t)^2\Big],$$

using $1 \mp 2t + t^2 = (1 \mp t)^2$. Then

$$= -\frac{1}{2t}\cdot 2\Big[\ln(1-t) - \ln(1+t)\Big] \;=\; \frac{1}{t}\ln\frac{1+t}{1-t},$$

because $0 < t < 1$ implies $1 \pm t > 0$. Now expand the logarithm. Subtracting the Taylor series for $\ln(1+t)$ and $\ln(1-t)$ gives

$$\ln\frac{1+t}{1-t} \;=\; \sum_{k\geq 1} \frac{(-1)^{k-1}+1}{k}\, t^k \;=\; 2\sum_{j\geq 0} \frac{t^{2j+1}}{2j+1}, \tag{5.49}$$

the second equality because only odd k survive (even k cancel: $(-1)^{k-1}+1 = 0$). Dividing by t:

$$\frac{1}{t}\ln\frac{1+t}{1-t} \;=\; \sum_{j\geq 0} \frac{2}{2j+1}\, t^{2j} \;=\; \sum_{n\geq 0} \frac{2}{2n+1}\, t^{2n} \tag{5.50}$$

(relabeling $j \to n$). Both sides are holomorphic in t on $|t| < 1$, so the identity just obtained for $0 < t < 1$ extends to the full disk by the identity theorem (Thm. 2.2). Combining (5.46) with (5.50):

$$\sum_{n\geq 0} t^{2n} \int_{-1}^{1} P_n^2 \, dx \;=\; \sum_{n\geq 0} \frac{2}{2n+1}\, t^{2n}. \tag{5.51}$$

Both sides are power series in t^2, convergent for $|t| < 1$. Matching the coefficient of t^{2n} yields

$$\int_{-1}^{1} P_n(x)^2 \, dx \;=\; \frac{2}{2n+1},$$

completing the normalization claim. □

5.6 Associated Legendre functions

Legendre's equation is the $m = 0$ angular equation on S^2. Full three-dimensional boundary-value problems also need the $m \neq 0$ modes, which appear when Laplace's equation is separated in spherical coordinates.

5.6.1 Spherical Laplacian and separation of variables

In spherical coordinates (r,θ,ϕ), with $(x,y,z)=(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, the scalar Laplacian is

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}. \tag{5.52}$$

For orthogonal coordinates with scale factors h_1, h_2, h_3 (the distance factors converting coordinate changes dq_j into physical lengths), the scalar Laplacian is

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\sum_{j=1}^{3}\frac{\partial}{\partial q_j}\left(\frac{h_1h_2h_3}{h_j^2}\frac{\partial f}{\partial q_j}\right).$$

Here $(q_1,q_2,q_3)=(r,\theta,\phi)$ and $(h_1,h_2,h_3)=(1,r,r\sin\theta)$, which gives (5.52).

Separation. Put $f(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$ in $\nabla^2 f=0$ and multiply by $r^2\sin^2\theta/(R\Theta\Phi)$:

$$\sin^2\theta\,\frac{(r^2R')'}{R} + \frac{\sin\theta}{\Theta}(\sin\theta\,\Theta')' + \frac{\Phi''}{\Phi} = 0.$$

The ϕ-term must be constant; write it as $-m^2$:

$$\Phi''(\phi)+m^2\,\Phi(\phi) = 0. \tag{5.53}$$

The angle ϕ and $\phi+2\pi$ represent the same physical point, so $\Phi(\phi+2\pi)=\Phi(\phi)$; this single-valuedness gives $m\in\mathbb{Z}$ and $\Phi(\phi)=e^{im\phi}$. Then

$$\frac{(r^2R')'}{R} = -\frac{1}{\Theta\sin\theta}(\sin\theta\,\Theta')' + \frac{m^2}{\sin^2\theta}.$$

The left side depends only on r, while the right side depends only on θ. Since r and θ can vary independently, both sides must be the same constant; write that constant as $\ell(\ell+1)$. Thus

$$\text{radial:}\quad (r^2R')'-\ell(\ell+1)R = 0, \tag{5.54}$$

$$\text{angular:}\quad \frac{1}{\sin\theta}(\sin\theta\,\Theta')' + \left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]\Theta = 0. \tag{5.55}$$

Radial solutions. Trying $R(r)=r^\alpha$ in (5.54) gives $\alpha(\alpha+1)=\ell(\ell+1)$, hence

$$R(r) = r^\ell \quad\text{and}\quad R(r) = r^{-\ell-1}. \tag{5.56}$$

These are the radial building blocks of the multipole expansion.

Angular change of variable. Set $x=\cos\theta$ and $y(x)=\Theta(\theta)$. Then

$$\frac{d\Theta}{d\theta} = \frac{dy}{dx}\cdot\frac{dx}{d\theta} = -\sin\theta\,y',$$

so $\sin\theta\,\Theta' = -(1-x^2)y'$. Differentiating once more,

$$\frac{d}{d\theta}\big[\sin\theta\,\Theta'\big] = \frac{dx}{d\theta}\frac{d}{dx}\big[-(1-x^2)y'\big] = -\sin\theta\cdot\frac{d}{dx}\big[-(1-x^2)y'\big] = \sin\theta\cdot\big[(1-x^2)y'\big]'.$$

Divide by $\sin\theta$ in (5.55):

$$\big[(1-x^2)y'\big]' + \left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]y = 0,$$

Expanding the derivative gives the *associated Legendre equation*

$$(1-x^2)\,y''-2x\,y' + \left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]y = 0. \tag{5.57}$$

At $m=0$ this reduces to Legendre's equation (5.40).

5.6.2 The associated Legendre functions

Definition 5.16 (Associated Legendre function). *For $\ell \in \mathbb{N}_0$ and $m \in \{0,1,\dots,\ell\}$, where $\mathbb{N}_0 = \{0,1,2,\dots\}$,*

$$P_\ell^m(x) \;=\; (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x). \tag{5.58}$$

For negative m the Condon–Shortley *convention is*

$$P_\ell^{-m}(x) \;=\; (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x), \qquad m > 0. \tag{5.59}$$

The factor $(1-x^2)^{m/2} P_\ell^{(m)}(x)$ is what turns the $m=0$ equation into the $m \neq 0$ one. The sign $(-1)^m$ is the *Condon–Shortley phase*; some texts place it in Y_ℓ^m instead.

Proposition 5.17 (P_ℓ^m solves the associated Legendre equation). *For $\ell \ge m \ge 0$, the function P_ℓ^m of* (5.58) *satisfies* (5.57).

Proof. Write $P := P_\ell$ and $w := P^{(m)}$.

Step 1: ODE for w. By Thm. 5.13, P satisfies

$$(1-x^2)P'' - 2xP' + \ell(\ell+1)P \;=\; 0. \tag{5.60}$$

Differentiate (5.60) m times. By the generalized Leibniz rule $(fg)^{(m)} = \sum_{j=0}^m \binom{m}{j} f^{(j)} g^{(m-j)}$, and using that $(1-x^2)'' = -2$ with all higher derivatives of $1-x^2$ vanishing, and $(-2x)' = -2$ with all higher derivatives of $-2x$ vanishing:

$$\frac{d^m}{dx^m}[(1-x^2)P''] \;=\; (1-x^2)P^{(m+2)} + m(-2x)P^{(m+1)} + \binom{m}{2}(-2)\,P^{(m)},$$

$$\frac{d^m}{dx^m}[-2x\,P'] \;=\; -2x\,P^{(m+1)} - 2m\,P^{(m)}.$$

Thus

$$(1-x^2)P^{(m+2)} - 2(m+1)x\,P^{(m+1)} + [\ell(\ell+1) - m(m-1) - 2m]\,P^{(m)} \;=\; 0. \tag{5.61}$$

With $w = P^{(m)}$, this becomes

$$(1-x^2)w'' - 2(m+1)x\,w' + [\ell(\ell+1) - m(m+1)]\,w \;=\; 0. \tag{5.62}$$

Step 2: Substitution. Let $y = (1-x^2)^{m/2} w$, so $w = (1-x^2)^{-m/2} y$. Then

$$w' \;=\; (1-x^2)^{-m/2}\,y' + y \cdot \frac{d}{dx}(1-x^2)^{-m/2} \;=\; (1-x^2)^{-m/2}\,y' + mx(1-x^2)^{-m/2-1}\,y, \tag{5.63}$$

and differentiating once more gives

$$\begin{aligned} w'' = {} & (1-x^2)^{-m/2}\,y'' + 2mx(1-x^2)^{-m/2-1}\,y' \\ & + \Big[m(1-x^2)^{-m/2-1} \\ & \quad + m(m+2)x^2(1-x^2)^{-m/2-2}\Big]y. \end{aligned} \tag{5.64}$$

Substitute (5.63) and (5.64) into (5.62) and cancel the common factor $(1-x^2)^{-m/2}$. The y'' coefficient is plainly $(1-x^2)$. The y' terms come from

$$(1-x^2)w'' \quad \text{and} \quad -2(m+1)xw',$$

and their coefficient is

$$2mx-2(m+1)x=-2x.$$

For the y coefficient, keep the three sources visible:

$$\left[m+\frac{m(m+2)x^2}{1-x^2}\right]-\frac{2m(m+1)x^2}{1-x^2}+\ell(\ell+1)-m(m+1)$$
$$=\ell(\ell+1)-m^2-\frac{m^2x^2}{1-x^2}.$$

Combine the last two terms over a common denominator:

$$-m^2-\frac{m^2x^2}{1-x^2}=\frac{-m^2(1-x^2)-m^2x^2}{1-x^2}=\frac{-m^2}{1-x^2}.$$

Thus

$$(1-x^2)y''-2x\,y'+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]y=0,$$

which is (5.57). Multiplying by the constant sign $(-1)^m$ gives P_ℓ^m. □

Proposition 5.18 (Orthogonality of associated Legendre functions). *For integers m,ℓ,ℓ' with $|m|\le\min(\ell,\ell')$,*

$$\int_{-1}^{1}P_\ell^m(x)\,P_{\ell'}^m(x)\,dx=\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\ell'}. \tag{5.65}$$

Proof. It is enough to prove the formula for $m\ge0$, because the case $m<0$ then follows from the Condon–Shortley relation (5.59). Fix $m\ge0$.

Step 1: Orthogonality for $\ell\ne\ell'$. Rewrite (5.57) in Sturm–Liouville form:

$$\frac{d}{dx}\big[(1-x^2)y'(x)\big]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]y(x)=0. \tag{5.66}$$

Let $u=P_\ell^m$ and $v=P_{\ell'}^m$. Multiply the equation for u by v, the equation for v by u, and subtract:

$$v\frac{d}{dx}\big[(1-x^2)u'\big]-u\frac{d}{dx}\big[(1-x^2)v'\big]+\big[\ell(\ell+1)-\ell'(\ell'+1)\big]uv=0. \tag{5.67}$$

Integrating from -1 to 1 gives

$$\big[\ell(\ell+1)-\ell'(\ell'+1)\big]\int_{-1}^{1}uv\,dx=\big[(1-x^2)(uv'-vu')\big]_{-1}^{1}. \tag{5.68}$$

We claim the boundary term is zero. If $m=0$, then u,v are polynomials, so u,v,u',v' are bounded and the factor $1-x^2$ forces the endpoint value to vanish. If $m\ge1$, then by (5.58) we can write

$$u(x)=(1-x^2)^{m/2}R(x),\qquad v(x)=(1-x^2)^{m/2}S(x),$$

with R,S polynomials. Therefore $u'=O((1-x^2)^{m/2-1})$ and $v'=O((1-x^2)^{m/2-1})$ near $x=\pm1$, so

$$(1-x^2)uv'=O((1-x^2)^m),\qquad(1-x^2)vu'=O((1-x^2)^m),$$

and both vanish at the endpoints. Thus the right-hand side of (5.68) is zero. Since $\ell(\ell+1)\ne\ell'(\ell'+1)$ when $\ell\ne\ell'$, we obtain

$$\int_{-1}^{1}P_\ell^m(x)P_{\ell'}^m(x)\,dx=0\qquad(\ell\ne\ell').$$

Step 2: A norm recursion. Define

$$I_{\ell m} := \int_{-1}^{1} \left[P_\ell^m(x)\right]^2 dx. \tag{5.69}$$

Differentiate (5.58) once:

$$\begin{aligned}\frac{d}{dx}P_\ell^m(x) &= \frac{d}{dx}\left[(-1)^m(1-x^2)^{m/2}P_\ell^{(m)}(x)\right]\\ &= (-1)^m\left[-mx(1-x^2)^{m/2-1}P_\ell^{(m)}(x)+(1-x^2)^{m/2}P_\ell^{(m+1)}(x)\right].\end{aligned} \tag{5.70}$$

Multiply (5.70) by $(1-x^2)^{1/2}$ and add $\frac{mx}{(1-x^2)^{1/2}}P_\ell^m(x)$. The terms involving $P_\ell^{(m)}$ cancel, leaving

$$(1-x^2)^{1/2}\frac{d}{dx}P_\ell^m(x)+\frac{mx}{(1-x^2)^{1/2}}P_\ell^m(x) \;=\; -P_\ell^{m+1}(x). \tag{5.71}$$

Square (5.71) and integrate:

$$\begin{aligned}I_{\ell,m+1} = \int_{-1}^{1}(1-x^2)\left[(P_\ell^m)'(x)\right]^2 dx + 2m\int_{-1}^{1} xP_\ell^m(x)(P_\ell^m)'(x)\,dx\\ +m^2\int_{-1}^{1}\frac{x^2}{1-x^2}\left[P_\ell^m(x)\right]^2 dx.\end{aligned} \tag{5.72}$$

When $m=0$, the middle term is absent. When $m\ge 1$, $P_\ell^m(\pm 1)=0$, so integrating $2xP_\ell^m(P_\ell^m)' = \frac{d}{dx}\left[x(P_\ell^m)^2\right]-(P_\ell^m)^2$ gives

$$2m\int_{-1}^{1} xP_\ell^m(P_\ell^m)'\,dx \;=\; -m\int_{-1}^{1}\left[P_\ell^m(x)\right]^2 dx \;=\; -mI_{\ell m}. \tag{5.73}$$

Thus (5.73) is valid for all $m\ge 0$.

Next multiply (5.57) by P_ℓ^m and integrate from -1 to 1. One integration by parts gives

$$\int_{-1}^{1}(1-x^2)\left[(P_\ell^m)'(x)\right]^2 dx + m^2\int_{-1}^{1}\frac{\left[P_\ell^m(x)\right]^2}{1-x^2}dx \;=\; \ell(\ell+1)I_{\ell m}, \tag{5.74}$$

and the boundary term vanishes for the same endpoint reasons used in Step 1.

Use (5.73) in (5.72), and rewrite

$$\frac{x^2}{1-x^2} \;=\; \frac{1}{1-x^2}-1.$$

Then (5.74) yields

$$\begin{aligned}I_{\ell,m+1} &= \ell(\ell+1)I_{\ell m}-m^2I_{\ell m}-mI_{\ell m}\\ &= \left[\ell(\ell+1)-m(m+1)\right]I_{\ell m}\\ &= (\ell-m)(\ell+m+1)I_{\ell m}.\end{aligned} \tag{5.75}$$

Starting from $I_{\ell 0}=\frac{2}{2\ell+1}$ from Theorem 5.15, iterate (5.75):

$$\begin{aligned}I_{\ell m} &= \prod_{j=0}^{m-1}(\ell-j)(\ell+j+1)\,I_{\ell 0}\\ &= \frac{(\ell+m)!}{(\ell-m)!}\,\frac{2}{2\ell+1}.\end{aligned} \tag{5.76}$$

This proves (5.65) for $m \geq 0$.

For $m < 0$, write $m = -s$ with $s > 0$. Then (5.59) gives

$$\begin{aligned}\int_{-1}^{1} P_\ell^{-s}(x)P_{\ell'}^{-s}(x)\,dx &= \frac{(\ell-s)!}{(\ell+s)!}\frac{(\ell'-s)!}{(\ell'+s)!}\int_{-1}^{1} P_\ell^{s}(x)P_{\ell'}^{s}(x)\,dx \\ &= \frac{(\ell-s)!}{(\ell+s)!}\frac{(\ell'-s)!}{(\ell'+s)!}\cdot\frac{2}{2\ell+1}\frac{(\ell+s)!}{(\ell-s)!}\delta_{\ell\ell'} \\ &= \frac{2}{2\ell+1}\frac{(\ell-s)!}{(\ell+s)!}\delta_{\ell\ell'}.\end{aligned}$$

Since $m = -s$, this is exactly (5.65). The proposition follows for all integers m. □

5.7 Spherical harmonics

The separated angular solutions $\Phi(\phi) = e^{im\phi}$ and $\Theta(\theta) = P_\ell^m(\cos\theta)$ combine into the spherical harmonics.

For normalization, start with

$$Y_\ell^m(\theta,\phi) = N_{\ell m}\,P_\ell^m(\cos\theta)\,e^{im\phi} \tag{5.77}$$

with real $N_{\ell m} > 0$ chosen so that $\int_{S^2} |Y_\ell^m|^2\,d\Omega = 1$. Using $d\Omega = \sin\theta\,d\theta\,d\phi$ and substituting $x = \cos\theta$ so that $dx = -\sin\theta\,d\theta$ in the θ-integral,

$$\begin{aligned}\int_{S^2} |Y_\ell^m|^2\,d\Omega &= N_{\ell m}^2 \int_0^{2\pi} |e^{im\phi}|^2\,d\phi \int_0^{\pi} |P_\ell^m(\cos\theta)|^2 \sin\theta\,d\theta \\ &= N_{\ell m}^2\cdot 2\pi\cdot\int_{-1}^{1} |P_\ell^m(x)|^2\,dx.\end{aligned}$$

The ϕ-integral is 2π because $|e^{im\phi}| = 1$. The x-integral is supplied by Proposition 5.18 at $\ell = \ell'$:

$$\int_{-1}^{1} [P_\ell^m(x)]^2\,dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}.$$

Setting the product equal to 1 and solving for $N_{\ell m}$,

$$N_{\ell m}^2 = \frac{1}{2\pi}\cdot\frac{2\ell+1}{2}\cdot\frac{(\ell-m)!}{(\ell+m)!} = \frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}. \tag{5.78}$$

Taking the positive square root yields the definition below.

Definition 5.19 (Spherical harmonic). *For $\ell \in \mathbb{N}_0$, $m \in \{-\ell,\dots,\ell\}$, and spherical coordinates $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$ on the unit sphere S^2,*

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\phi}, \tag{5.79}$$

normalized so that $\int_{S^2} |Y_\ell^m(\theta,\phi)|^2\,d\Omega = 1$, where $d\Omega = \sin\theta\,d\theta\,d\phi$. A bar or star, as in $\overline{Y_\ell^m}$ or Y_ℓ^{m*}, denotes complex conjugation.

This normalization gives orthonormality.

Proposition 5.20 (Orthonormality of spherical harmonics: verifying our choice)**.**

$$\int_{S^2} Y_\ell^m(\theta,\phi)\,\overline{Y_{\ell'}^{m'}(\theta,\phi)}\,d\Omega = \delta_{\ell\ell'}\delta_{mm'}. \tag{5.80}$$

Proof. Let

$$N_{\ell m} := \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}. \tag{5.81}$$

With $x=\cos\theta$ and $dx=-\sin\theta\,d\theta$,

$$\int_{S^2} Y_\ell^m \overline{Y_{\ell'}^{m'}}\,d\Omega = N_{\ell m}N_{\ell' m'}\int_0^{2\pi} e^{i(m-m')\phi}\,d\phi \int_{-1}^{1} P_\ell^m(x)P_{\ell'}^{m'}(x)\,dx. \tag{5.82}$$

The ϕ-integral equals $2\pi\,\delta_{mm'}$. Therefore only the case $m=m'$ remains, and then Proposition 5.18 gives

$$\begin{aligned}\int_{S^2} Y_\ell^m \overline{Y_{\ell'}^{m'}}\,d\Omega &= 2\pi\,\delta_{mm'}\,N_{\ell m}N_{\ell' m}\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\ell'}\\ &= \delta_{mm'}\delta_{\ell\ell'},\end{aligned}$$

because when $\ell=\ell'$ the normalization constants satisfy

$$2\pi N_{\ell m}^2\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!} = 2\pi\cdot\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\cdot\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!} = 1.$$

This proves (5.80). In particular, the constant in (5.79) is exactly the $L^2(S^2)$ normalization. □

Theorem 5.21 (Addition theorem). *For two unit vectors $\hat{n},\hat{n}'$ on S^2 with angle γ between them (so $\cos\gamma=\hat{n}\cdot\hat{n}'$),*

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\hat{n}')\,Y_\ell^m(\hat{n}). \tag{5.83}$$

Proof. Define

$$F_\ell(\hat{n},\hat{n}') = \sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\hat{n}')\,Y_\ell^m(\hat{n}).$$

Let $V_\ell := \operatorname{span}\{Y_\ell^m : -\ell\le m\le \ell\}$, the set of all finite linear combinations of the degree-ℓ spherical harmonics. An *orthonormal basis* of this finite-dimensional space means a list of functions that are mutually orthogonal, each have L^2 norm 1, and span V_ℓ. The key point is that the kernel

$$K_\ell(\hat{n},\hat{n}') = \sum_{j=1}^{2\ell+1} e_j(\hat{n})\,\overline{e_j(\hat{n}')}$$

is independent of the chosen orthonormal basis $\{e_j\}$ of V_ℓ. A finite-dimensional way to read this formula is: evaluate every basis function at $\hat{n}$ to get a coordinate vector, evaluate every basis function at $\hat{n}'$ to get another coordinate vector, and take their complex inner product. Changing orthonormal bases multiplies both coordinate vectors by the same unitary matrix, so the inner product does not change. Indeed, if $\{\tilde{e}_i\}$ is another orthonormal basis, then $\tilde{e}_i=\sum_j U_{ij}e_j$ for a unitary matrix U (a change-of-orthonormal-basis matrix satisfying $U^*U=I$), so

$$\begin{aligned}\sum_i \tilde{e}_i(\hat{n})\,\overline{\tilde{e}_i(\hat{n}')} &= \sum_{i,j,k} U_{ij}\,\overline{U_{ik}}\,e_j(\hat{n})\,\overline{e_k(\hat{n}')}\\ &= \sum_{j,k}\left(\sum_i U_{ij}\,\overline{U_{ik}}\right) e_j(\hat{n})\,\overline{e_k(\hat{n}')}\\ &= \sum_j e_j(\hat{n})\,\overline{e_j(\hat{n}')},\end{aligned}$$

because $U^*U = I$. In particular, our F_ℓ is this basis-independent kernel. Rotating coordinates sends degree-ℓ spherical harmonics to degree-ℓ spherical harmonics. The reason is physical as well as algebraic: rotating the axes does not change the angular Laplacian, so a rotated degree-ℓ solution is still a degree-ℓ solution. Equivalently, V_ℓ is the eigenspace of the spherical Laplacian Δ_{S^2} with eigenvalue $-\ell(\ell+1)$. A rotation R replaces a function $Y(\hat{n})$ by $Y(R^{-1}\hat{n})$, and Δ_{S^2} gives the same answer before and after this change of axes. Thus the rotated functions still lie in V_ℓ. This is the only rotation-theory input used here: rotation changes the orthonormal basis inside V_ℓ, but the kernel above does not depend on which orthonormal basis we use.

Now choose spherical coordinates (γ, ψ) whose north pole is the fixed vector $\hat{n}'$. In those rotated coordinates the degree-ℓ spherical harmonics

$$\widetilde{Y}_\ell^m(\gamma, \psi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\gamma)\, e^{im\psi}$$

form another orthonormal basis of V_ℓ. By basis-independence,

$$F_\ell(\hat{n}, \hat{n}') = \sum_{m=-\ell}^{\ell} \widetilde{Y}_\ell^m(\hat{n})\, \overline{\widetilde{Y}_\ell^m(\hat{n}')}.$$

But in these coordinates $\hat{n}'$ is the north pole, so $\gamma(\hat{n}') = 0$ and $\cos\gamma(\hat{n}') = 1$. For $m > 0$, (5.58) gives $P_\ell^m(1) = 0$ because of the factor $(1-x^2)^{m/2}$; for $m < 0$, (5.59) gives the same conclusion. Therefore only the $m = 0$ term survives:

$$F_\ell(\hat{n}, \hat{n}') = \widetilde{Y}_\ell^0(\hat{n})\, \overline{\widetilde{Y}_\ell^0(\hat{n}')}.$$

Using $P_\ell(1) = 1$ and Definition (5.79),

$$\widetilde{Y}_\ell^0(\hat{n}') = \sqrt{\frac{2\ell+1}{4\pi}}, \qquad \widetilde{Y}_\ell^0(\hat{n}) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\gamma),$$

where γ is exactly the angle between $\hat{n}$ and $\hat{n}'$ because (γ, ψ) was chosen with polar axis $\hat{n}'$. Hence

$$F_\ell(\hat{n}, \hat{n}') = \frac{2\ell+1}{4\pi}\, P_\ell(\cos\gamma).$$

Rearranging gives (5.83). □

Before stating completeness, fix the terminology. Here $L^2(S^2)$ means square-integrable functions on the sphere, with inner product $\langle f, g\rangle = \int_{S^2} f\,\overline{g}\, d\Omega$. An *orthonormal basis* means two things: the functions are mutually orthogonal and normalized to length 1, and finite linear combinations of them approximate every L^2 function in mean-square error.

Theorem 5.22 (Completeness). *$\{Y_\ell^m : \ell \in \mathbb{N}_0, |m| \le \ell\}$ is an orthonormal basis of $L^2(S^2)$.*

Proof. Orthonormality is Proposition 5.20. Let $\mathcal{Y}$ denote the linear span of the spherical harmonics. We prove density, meaning approximation by elements of $\mathcal{Y}$, by a constructive Poisson-kernel argument.

For $0 < r < 1$ and $t \in [-1, 1]$, define

$$\mathcal{P}_r(t) := \frac{1-r^2}{4\pi(1-2rt+r^2)^{3/2}}. \tag{5.84}$$

Start from the Legendre generating function (5.5),

$$G(t,r) = \frac{1}{\sqrt{1-2rt+r^2}} = \sum_{\ell=0}^{\infty} r^\ell P_\ell(t),$$

valid for $|r|<1$. Differentiate with respect to r:

$$\partial_r G(t,r) = \frac{t-r}{(1-2rt+r^2)^{3/2}} = \sum_{\ell=1}^{\infty} \ell r^{\ell-1} P_\ell(t).$$

Therefore

$$G + 2r\,\partial_r G = \frac{1-r^2}{(1-2rt+r^2)^{3/2}} = \sum_{\ell=0}^{\infty} (2\ell+1) r^\ell P_\ell(t). \tag{5.85}$$

Now set $t = \hat{n}\cdot\hat{n}' = \cos\gamma$. Using the addition theorem (5.83), we obtain the grouped expansion

$$\mathcal{P}_r(\hat{n}\cdot\hat{n}') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} r^\ell\, Y_\ell^m(\hat{n})\, Y_\ell^{m*}(\hat{n}'). \tag{5.86}$$

For fixed $r<1$ this grouped series converges uniformly on $S^2\times S^2$: choose ρ with $r<\rho<1$. By Remark 5.4,

$$|P_\ell(\hat{n}\cdot\hat{n}')| \le \frac{1}{(1-\rho)\rho^\ell},$$

so the Legendre series in (5.85) is dominated by

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi(1-\rho)}\left(\frac{r}{\rho}\right)^\ell,$$

which converges.

Integrating (5.86) in $\hat{n}'$ and using orthonormality,

$$\int_{S^2} \mathcal{P}_r(\hat{n}\cdot\hat{n}')\, d\Omega' = 1. \tag{5.87}$$

Also $\mathcal{P}_r \ge 0$ by the closed form (5.84).

These two facts say that, for each fixed $\hat{n}$, $\mathcal{P}_r(\hat{n}\cdot\hat{n}')\,d\Omega'$ is a probability weight on the sphere. As $r\uparrow 1$, the closed form becomes sharply peaked near $\hat{n}'=\hat{n}$. Thus the Poisson integral below is a weighted average of f near $\hat{n}$, and the proof makes that concentration estimate precise.

Now let $f\in C(S^2)$ and define its Poisson integral

$$u_r(\hat{n}) := \int_{S^2} \mathcal{P}_r(\hat{n}\cdot\hat{n}') f(\hat{n}')\, d\Omega'. \tag{5.88}$$

By the uniform convergence of (5.86), we may interchange sum and integral:

$$u_r(\hat{n}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} r^\ell c_{\ell m} Y_\ell^m(\hat{n}), \qquad c_{\ell m} := \int_{S^2} f(\hat{n}')\, Y_\ell^{m*}(\hat{n}')\, d\Omega'. \tag{5.89}$$

Hence each u_r is a uniform limit of finite linear combinations from $\mathcal{Y}$; this is what it means to lie in the uniform closure of $\mathcal{Y}$.

We next show $u_r \to f$ uniformly as $r\uparrow 1$. Fix $\varepsilon>0$. Since f is uniformly continuous on the compact sphere, choose $\delta>0$ so that

$$\angle(\hat{n},\hat{n}') < \delta \quad\Longrightarrow\quad |f(\hat{n}')-f(\hat{n})| < \varepsilon.$$

Using (5.87),

$$\begin{aligned}|u_r(\hat n)-f(\hat n)| &= \left|\int_{S^2}\mathcal{P}_r(\hat n\cdot\hat n')[f(\hat n')-f(\hat n)]\,d\Omega'\right|\\ &\le \varepsilon\int_{\angle(\hat n,\hat n')<\delta}\mathcal{P}_r\,d\Omega' + 2\|f\|_\infty\int_{\angle(\hat n,\hat n')\ge\delta}\mathcal{P}_r\,d\Omega'\\ &\le \varepsilon + 2\|f\|_\infty\int_{\angle(\hat n,\hat n')\ge\delta}\mathcal{P}_r\,d\Omega'.\end{aligned}$$

If $\angle(\hat n,\hat n')\ge\delta$, then $\hat n\cdot\hat n'\le\cos\delta$, so

$$1-2r\,\hat n\cdot\hat n'+r^2\ge 1-2r\cos\delta+r^2.$$

Therefore

$$0\le\mathcal{P}_r(\hat n\cdot\hat n')\le\frac{1-r^2}{4\pi(1-2r\cos\delta+r^2)^{3/2}},$$

and the right-hand side tends to 0 as $r\uparrow 1$ because $1-2\cos\delta+1=2(1-\cos\delta)>0$. Since $\mathrm{area}(S^2)=4\pi$, the “far” integral is bounded by 4π times a quantity tending to 0, uniformly in $\hat n$. Thus

$$\sup_{\hat n\in S^2}|u_r(\hat n)-f(\hat n)|\longrightarrow 0\qquad (r\uparrow 1).$$

The supremum here is the largest error over the whole sphere. So every continuous function lies in the uniform, hence L^2, closure of $\mathcal{Y}$.

Continuous functions are dense in $L^2(S^2)$: every square-integrable function on the sphere can be approximated in mean-square error by continuous functions. Since every continuous function lies in the L^2-closure of $\mathcal{Y}$, the span $\mathcal{Y}$ is dense in $L^2(S^2)$. Together with orthonormality, this proves that the spherical harmonics form an orthonormal basis. □

Corollary 5.23 (Legendre series). *Any $f\in L^2[-1,1]$ has a Legendre expansion*

$$f(x)=\sum_{n=0}^{\infty}c_nP_n(x),\qquad c_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)\,dx, \tag{5.90}$$

with convergence in $L^2[-1,1]$.

Proof. Define an axisymmetric, meaning ϕ-independent, function on the sphere by

$$F(\theta,\phi):=\frac{1}{\sqrt{2\pi}}f(\cos\theta).$$

Using $x=\cos\theta$ and $dx=-\sin\theta\,d\theta$,

$$\int_{S^2}|F|^2\,d\Omega=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^{\pi}|f(\cos\theta)|^2\sin\theta\,d\theta=\int_{-1}^{1}|f(x)|^2\,dx.$$

So $f\mapsto F$ is an isometry, meaning it preserves the L^2 norm, from $L^2[-1,1]$ onto the axisymmetric subspace of $L^2(S^2)$.

By Theorem 5.22, F has an $L^2(S^2)$ expansion in spherical harmonics. Because F is independent of ϕ, orthogonality of the factors $e^{im\phi}$ forces all coefficients with $m\neq 0$ to vanish. Hence

$$F(\theta,\phi)=\sum_{\ell=0}^{\infty}a_{\ell 0}\,Y_\ell^0(\theta,\phi)\qquad\text{in } L^2(S^2).$$

Now $Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)$ by (5.79), so multiplying by $\sqrt{2\pi}$ and writing $x = \cos\theta$ gives

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x) \qquad \text{in } L^2[-1,1],$$

with

$$c_\ell = \sqrt{2\pi}\,a_{\ell 0}\sqrt{\frac{2\ell+1}{4\pi}}.$$

Finally,

$$\begin{aligned} a_{\ell 0} &= \int_{S^2} F(\theta,\phi)\,\overline{Y_\ell^0(\theta,\phi)}\,d\Omega \\ &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2\ell+1}{4\pi}}\int_0^{2\pi} d\phi \int_0^{\pi} f(\cos\theta)P_\ell(\cos\theta)\sin\theta\,d\theta \\ &= \sqrt{\frac{2\ell+1}{2}}\int_{-1}^{1} f(x)P_\ell(x)\,dx. \end{aligned}$$

Therefore

$$c_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} f(x)P_\ell(x)\,dx,$$

which is exactly (5.90). □

Proposition 5.24 (Multipole expansion in spherical harmonics). *Let* $r_< = \min(r,r')$ *and* $r_> = \max(r,r')$*. Then*

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\gamma) = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_\ell^{m*}(\hat{r}')Y_\ell^m(\hat{r}), \tag{5.91}$$

where γ *is the angle between* $\mathbf{r}$ *and* $\mathbf{r}'$*.*

Proof. Apply Proposition 5.2 to whichever of $\mathbf{r},\mathbf{r}'$ has smaller radius, so that the expansion parameter is $r_</r_> < 1$; this yields the first equality. Then insert Theorem 5.21 to split $P_\ell(\cos\gamma)$ into separable $\hat{r}$, $\hat{r}'$ factors, giving the second. □

This is the *multipole expansion*: the angular dependence of a two-point function factorizes as a sum of separable $\hat{r}$-, $\hat{r}'$- pieces.

5.8 Legendre Q_n: the second kind

The polynomial solutions P_n are the solutions regular on all of $[-1,1]$. When the endpoints are excluded, the second solution Q_n can also matter; it is logarithmically singular at $x = \pm 1$. We only need Q_0 and Q_1. For each n, we find Q_n by *reduction of order*: given one solution $y_1 = P_n$ of the linear second-order ODE, write $y = y_1 u$, substitute into the ODE, and solve the resulting first-order equation for u'.

Derivation of Q_0. For $n=0$, Legendre's equation is $(1-x^2)y'' - 2xy' = 0$. Set $v = y'$:

$$(1-x^2)v' - 2xv = 0.$$

Separate variables:

$$\frac{dv}{v} = \frac{2x\,dx}{1-x^2}.$$

Integrating gives

$$\ln|v| = -\ln|1-x^2|+C_1, \qquad v(x) = \frac{C_2}{1-x^2}.$$

Thus

$$y(x) = C_2\int \frac{dx}{1-x^2} = C_2\cdot\frac{1}{2}\ln\frac{1+x}{1-x}+C_3.$$

Dropping the additive constant, which is the P_0 solution, and taking $C_2=1$,

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \qquad |x|<1. \tag{5.92}$$

Derivation of Q_1. For $n=1$, use the known solution $P_1=x$ and write $y=xu$. Then $y'=u+xu'$ and $y''=2u'+xu''$. Substitution gives

$$(1-x^2)(2u'+xu'')-2x(u+xu')+2xu = 0.$$

The u-terms cancel, leaving $x(1-x^2)u''+2(1-2x^2)u'=0$. With $p=u'$,

$$\frac{dp}{p} = -\frac{2(1-2x^2)}{x(1-x^2)}\,dx.$$

Using partial fractions,

$$-\frac{2(1-2x^2)}{x(1-x^2)} = -\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x},$$

so

$$\ln|p| = -\big[2\ln|x|+\ln|1-x|+\ln|1+x|\big]+C = -\ln[x^2(1-x^2)]+C,$$

so $u'=C_2/[x^2(1-x^2)]$. Since $1/[x^2(1-x^2)]=1/x^2+1/(1-x^2)$,

$$u = C_2\left[-\frac{1}{x}+\frac{1}{2}\ln\frac{1+x}{1-x}\right]+C_3.$$

Thus, with $C_2=1$ and the P_1 term dropped,

$$Q_1(x) = \frac{x}{2}\ln\frac{1+x}{1-x}-1. \tag{5.93}$$

The logarithm in (5.92) and (5.93) shows the endpoint singularity. Higher Q_n are part of the general Legendre-function theory; here Q_0 and Q_1 are enough.

5.9 Physical applications

In the electrostatic examples below, Φ denotes the scalar potential. This keeps the potential distinct from the azimuthal coordinate ϕ in $Y_\ell^m(\theta,\phi)$.

Example 5.25 (Grounded sphere in a uniform field). *Problem: a conducting sphere of radius R is grounded ($\Phi=0$ on $r=R$) and placed in a uniform external field $\mathbf{E}_0=E_0\hat{z}$. Find the electrostatic potential Φ outside the sphere and the induced dipole moment.*

Setup. *In the absence of the sphere, the external potential giving $\mathbf{E}_0$ is $\Phi_0=-E_0z=-E_0r\cos\theta$, so the asymptotic boundary condition is $\Phi(\mathbf{r})\to -E_0r\cos\theta$ as $r\to\infty$. The region $r>R$ has $\nabla^2\Phi=0$ and $\Phi(R,\theta)=0$.*

Separation of variables. *In spherical coordinates with azimuthal symmetry ($m=0$), general solutions of Laplace's equation are*

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\big(A_\ell r^\ell+B_\ell r^{-\ell-1}\big)P_\ell(\cos\theta). \tag{5.94}$$

(Each $(r^\ell, r^{-\ell-1})$ pair is the radial-solution family (5.56)*, and each $P_\ell(\cos\theta)$ is the axisymmetric $m=0$ angular solution; the derivation of both was carried out in Section 5.6.1.)*

Asymptotic matching. *As $r\to\infty$, $\Phi\to -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$, using $P_1(\cos\theta) = \cos\theta$. Matching the growing modes: $A_1 = -E_0$, and $A_\ell = 0$ for $\ell\neq 1$.*

Boundary condition at $r=R$. *$\Phi(R,\theta) = 0$ term-by-term in P_ℓ (since $\{P_\ell\}$ are linearly independent on $[-1,1]$):*

$$A_\ell R^\ell + B_\ell R^{-\ell-1} = 0 \quad\Rightarrow\quad B_\ell = -A_\ell R^{2\ell+1}.$$

Only $\ell=1$ contributes: $B_1 = -A_1R^3 = E_0R^3$, $B_\ell = 0$ for $\ell\neq 1$. Therefore

$$\Phi(r,\theta) = -E_0 r\cos\theta + \frac{E_0R^3\cos\theta}{r^2} = -E_0 r\cos\theta + \frac{E_0R^3}{r^2}\cos\theta. \tag{5.95}$$

Induced dipole. *The second term has the form of a dipole potential $\Phi_{\rm dip}(\mathbf{r}) = \mathbf{p}\cdot\hat r/(4\pi\epsilon_0 r^2) = p\cos\theta/(4\pi\epsilon_0 r^2)$ for a dipole along $\hat z$. Matching,*

$$\frac{p}{4\pi\epsilon_0} = E_0R^3 \quad\Rightarrow\quad p = 4\pi\epsilon_0 R^3 E_0.$$

The sphere responds to the external field with an induced dipole of polarizability $\alpha = 4\pi\epsilon_0 R^3$.

Theorem 5.26 (Multipole expansion of a charge distribution). *Let $\rho\in L^1(\mathbb{R}^3)$ be a charge distribution (so $|\rho|$ is integrable) supported in $|\mathbf{r}'|\le a$ (so $\rho(\mathbf{r}')=0$ outside that ball), and let $\mathbf{r}$ be an exterior point with $|\mathbf{r}| = r > a$. Then*

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r' = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{q_{\ell m}}{(2\ell+1)\,r^{\ell+1}}\,Y_\ell^m(\theta,\phi), \tag{5.96}$$

with multipole moments

$$q_{\ell m} := \int r'^\ell\,\overline{Y_\ell^m(\theta',\phi')}\,\rho(\mathbf{r}')\,d^3r'. \tag{5.97}$$

For each fixed ℓ the inner sum over m is finite, and the outer sum over ℓ converges absolutely and uniformly on compact subsets of $\{r>a\}$, meaning on regions where $r\ge a+\delta$ for some fixed $\delta>0$.

Proof. For every $\mathbf{r}'$ in the support of ρ, we have $r'\le a<r$, so the scalar multipole expansion in (5.91) reads

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\infty}\frac{r'^\ell}{r^{\ell+1}}P_\ell(\cos\gamma),$$

where γ is the angle between $\hat r$ and $\hat r'$. Multiply by $\rho(\mathbf{r}')/(4\pi\epsilon_0)$ and integrate over $\mathbf{r}'$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{r}')\sum_{\ell=0}^{\infty}\frac{r'^\ell}{r^{\ell+1}}P_\ell(\cos\gamma)\,d^3r'.$$

It remains to justify interchanging the sum and the integral.

Convergence / interchange. Fix $r_0>a$ and restrict to $r\ge r_0$. Then

$$q := \frac{a}{r_0} < 1, \qquad \frac{r'}{r}\le q$$

uniformly on $\{r \ge r_0\} \times \{|\mathbf{r}'| \le a\}$. Choose ϱ with $q < \varrho < 1$. Remark 5.4 gives

$$|P_\ell(\cos\gamma)| \;\le\; \frac{1}{(1-\varrho)\varrho^\ell}.$$

Therefore the tail of the scalar series satisfies

$$\left|\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \sum_{\ell=0}^{L} \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\gamma)\right| \le \frac{1}{r_0(1-\varrho)} \sum_{\ell>L} \left(\frac{q}{\varrho}\right)^\ell,$$

which tends to 0 uniformly in $(\mathbf{r},\mathbf{r}')$. In particular,

$$\sum_{\ell=0}^{\infty} \left|\frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\gamma)\right| \le \frac{1}{r_0(1-\varrho)} \sum_{\ell=0}^{\infty} \left(\frac{q}{\varrho}\right)^\ell$$

uniformly on the same set. Since $\rho \in L^1(\mathbb{R}^3)$, dominated convergence now permits us to swap the sum and the $\mathbf{r}'$-integral:

$$\Phi(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \int r'^\ell P_\ell(\cos\gamma)\,\rho(\mathbf{r}')\,\frac{d^3r'}{r^{\ell+1}}.$$

Finally insert the second equality in (5.91),

$$P_\ell(\cos\gamma) \;=\; \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \overline{Y_\ell^m(\hat{r}')}\,Y_\ell^m(\hat{r}),$$

to obtain

$$\Phi(\mathbf{r}) \;=\; \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_\ell^m(\hat{r})}{(2\ell+1)r^{\ell+1}} \int r'^\ell\,\overline{Y_\ell^m(\hat{r}')}\,\rho(\mathbf{r}')\,d^3r'.$$

The integral is exactly $q_{\ell m}$ from (5.97), so this is (5.96). □

Example 5.27 (Potential of a uniformly charged sphere via the multipole expansion)**.** *A uniform charge density ρ_0 fills a ball of radius a, centered at the origin. Compute the electrostatic potential $\Phi(\mathbf{r})$ at an exterior point $r > a$.*

Using the first equality in (5.91) *with $r_< = r' \le a$, $r_> = r$, and $d^3r' = r'^2\,dr'\,d\Omega'$,*

$$\Phi(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0} \int \frac{\rho_0}{|\mathbf{r}-\mathbf{r}'|} d^3r' \;=\; \frac{\rho_0}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \int_0^a r'^{\ell+2}\,dr' \int_{S^2} \frac{P_\ell(\cos\gamma)}{r^{\ell+1}}\,d\Omega'.$$

By (5.83)*,*

$$\int_{S^2} P_\ell(\cos\gamma)\,d\Omega' \;=\; \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{r}) \int_{S^2} Y_\ell^{m*}(\hat{r}')\,d\Omega'.$$

The integral of Y_ℓ^{m*} over S^2 is $\sqrt{4\pi}\,\delta_{\ell 0}\delta_{m0}$ (orthogonality against $Y_0^0 = 1/\sqrt{4\pi}$, which picks out $\ell = m = 0$). So only $\ell = 0$ contributes:

$$\Phi(\mathbf{r}) \;=\; \frac{\rho_0}{4\pi\epsilon_0} \cdot \frac{4\pi}{r} \int_0^a r'^2\,dr' \;=\; \frac{\rho_0}{3\epsilon_0} \cdot \frac{a^3}{r} \;=\; \frac{Q}{4\pi\epsilon_0 r},$$

with total charge $Q = (4\pi/3)a^3\rho_0$. Exactly the potential of a point charge at the origin, as expected from symmetry — with the multipole expansion making the vanishing of all $\ell \ge 1$ terms a direct consequence of the angular integral.

Example 5.28 (Hydrogen angular wavefunction)**.** *The angular part of the stationary Schrödinger equation with any central potential is* $Y_\ell^m(\theta,\phi)$*; the radial equation, for the Coulomb case, reduces to the Laguerre equation (Section 6).*

For the first few angular states, use $P_0 = 1$*,* $P_1(x) = x$*, and* $P_1^1(x) = -(1-x^2)^{1/2}$ *from Definition 5.16. Formula* (5.79) *gives*

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta,$$

and

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}, \qquad Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}.$$

Thus the real $2p_z$ *angular shape is proportional to* $Y_1^0 \propto \cos\theta$*: positive on the northern hemisphere, negative on the southern hemisphere, and zero on the equatorial plane. The other two p-orbitals are obtained from real linear combinations of* Y_1^1 *and* Y_1^{-1}*.*

Exercises

Problem 5.1. *Compute* $P_5(x)$ *by expanding the generating function through* t^5*, and verify via Rodrigues' formula* (5.38).

Problem 5.2. *Show* $P_n(1) = 1$ *and* $P_n(-1) = (-1)^n$ *by evaluating the generating function at* $x = \pm 1$.

Problem 5.3. *Evaluate* $\int_{-1}^{1} x^k P_n(x)\,dx$ *for* $k < n$ *(hint: orthogonality) and for* $k = n$ *(hint: Rodrigues and repeated integration by parts).*

Problem 5.4. *Derive* (5.11) *from* (5.10) *by differentiating in* x *and using* (5.21).

Problem 5.5. *Expand* $f(x) = |x|$ *on* $[-1,1]$ *in a Legendre series (Cor. 5.23); compute the first three nonzero coefficients.*

Problem 5.6. *Verify the addition theorem* (5.83) *at* $\ell = 1$ *using the explicit forms of* $Y_1^0, Y_1^{\pm 1}$ *and* $P_1(\cos\gamma) = \cos\gamma = \hat{n}\cdot\hat{n}'$.

Problem 5.7. *A conducting sphere of radius* R *carries total charge* Q *and is placed in a uniform external field* $\mathbf{E}_0 = E_0\hat{z}$*. Find the electrostatic potential* Φ *outside the sphere and the induced dipole moment. (Extension of Example 5.25.)*

Problem 5.8. *Give a second proof of Prop. 5.18: derive the orthogonality and normalization of* P_ℓ^m *directly from the Rodrigues form by repeated integration by parts.*

Problem 5.9. *Prove that* P_n *has exactly* n *real simple zeros in the open interval* $(-1,1)$*.* Hint*: apply Rolle's theorem to successive derivatives of* $(x^2-1)^n$*, which has zeros of multiplicity* n *at* ± 1.

Problem 5.10. *Starting from the Schläfli integral* (5.28)*, parametrize the contour* $|w-x| = \sqrt{x^2-1}$ *(for* $x > 1$*) by* $w = x + \sqrt{x^2-1}\,e^{i\varphi}$*,* $\varphi \in [0,2\pi)$*, and simplify* $(w^2-1)/(2(w-x))$ *to derive the* Laplace integral representation

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \left(x + \sqrt{x^2-1}\cos\varphi\right)^n d\varphi, \qquad x > 1.$$

Problem 5.11. *(*Unsöld's theorem.*) Using the addition theorem at $\hat{n} = \hat{n}'$, show that $\sum_{m=-\ell}^{\ell} |Y_\ell^m(\hat{n})|^2 = (2\ell+1)/(4\pi)$, a constant on S^2 (independent of $\hat{n}$). Interpret physically: the probability density of a state of fixed ℓ summed over magnetic sublevels is isotropic.*

Problem 5.12. *(*Point charge near a grounded sphere.*) A point charge q is placed at $\mathbf{r}_0$ outside a grounded sphere of radius R with $|\mathbf{r}_0| = d > R$. Expand the potential of the point charge via* (5.91)*, impose the boundary condition $\Phi = 0$ at $r = R$, and compare to the method of images: show that the induced charge density on the sphere produces the same exterior field as an image charge $q' = -qR/d$ placed at $\mathbf{r}_0' = (R^2/d^2)\mathbf{r}_0$.*

6 More Special Functions

Hermite, Laguerre, Chebyshev, and hypergeometric functions appear throughout quantum mechanics, numerical analysis, and approximation theory. Like Bessel and Legendre functions, they are often best introduced by generating functions. The hypergeometric viewpoint then shows why these families are related.

Prerequisites. This section leans on the Gamma function's meromorphic continuation (Thm. 3.3) and functional equation (Prop. 3.2, (3.2)), the Beta integral (Prop. 3.9), the Pochhammer symbol (Def. 3.22), and the Cauchy formula for n-th derivatives (Cor. 1.14).

6.1 Hermite polynomials

Motivation. In quantum mechanics, stationary states of the harmonic oscillator involve $H_n(x)e^{-x^2/2}$. The cleanest way to produce the whole family at once is through a generating function, exactly as for Bessel (J_n) and Legendre (P_n).

Definition 6.1 (Hermite generating function)**.** *The (physicist) Hermite polynomials $H_n(x)$ are defined by*

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}. \tag{6.1}$$

The left side is entire in t, so the coefficients determine H_n uniquely. Throughout this section, H_n means the physicists' Hermite polynomial; the probabilists' convention is related by $H_n(x) = 2^{n/2}He_n(x\sqrt{2})$. Equivalently, by Cauchy's coefficient formula (Cor. 1.14),

$$H_n(x) = \frac{n!}{2\pi i}\oint_{|t|=\rho} e^{2xt-t^2}\,t^{-n-1}\,dt, \tag{6.2}$$

for any $\rho > 0$.

Proposition 6.2 (First values, parity)**.** $H_0 = 1$, $H_1 = 2x$, $H_2 = 4x^2 - 2$. *More generally, H_n is a polynomial of degree n with leading coefficient 2^n and parity*

$$H_n(-x) = (-1)^n H_n(x).$$

Proof. Expand (6.1) at $t = 0$:

$$e^{2xt-t^2} = 1 + (2xt - t^2) + \tfrac{1}{2}(2xt - t^2)^2 + \tfrac{1}{6}(2xt - t^2)^3 + O(t^4). \tag{6.3}$$

Expand the squared factor:

$$(2xt - t^2)^2 = 4x^2t^2 - 4xt^3 + t^4. \tag{6.4}$$

The cubic term contributes $8x^3t^3 + O(t^4)$. Thus

$$\begin{aligned}
t^0 &: \ 1,\\
t^1 &: \ 2x,\\
t^2 &: \ -1 + \tfrac{1}{2}\cdot 4x^2 = 2x^2 - 1,\\
t^3 &: \ 0 + \tfrac{1}{2}\cdot(-4x) + \tfrac{1}{6}\cdot 8x^3 = -2x + \tfrac{4}{3}x^3.
\end{aligned}$$

Matching coefficients with $\sum_n H_n(x)t^n/n!$ gives

$$\begin{aligned}
H_0(x) &= 0!\cdot 1 = 1,\\
H_1(x) &= 1!\cdot 2x = 2x,\\
H_2(x) &= 2!\cdot(2x^2-1) = 4x^2-2,\\
H_3(x) &= 3!\cdot(-2x+\tfrac{4}{3}x^3) = 8x^3-12x.
\end{aligned}$$

Degree, leading coefficient, parity. Replace $x\to -x$, $t\to -t$ in (6.1). The left side is unchanged, so coefficient comparison gives $H_n(-x) = (-1)^n H_n(x)$.

Also $e^{2xt-t^2} = e^{-t^2}e^{2xt}$. The top power of x in the coefficient of t^n comes from $(2x)^n/n!$, so $H_n(x) = 2^n x^n + (\text{lower})$.

Closed form. Finally, expand $e^{2xt}e^{-t^2}$ and collect the terms with $p+2q=n$:

$$H_n(x) = n!\sum_{q=0}^{\lfloor n/2\rfloor}\frac{(-1)^q\,(2x)^{n-2q}}{(n-2q)!\,q!}. \tag{6.5}$$

□

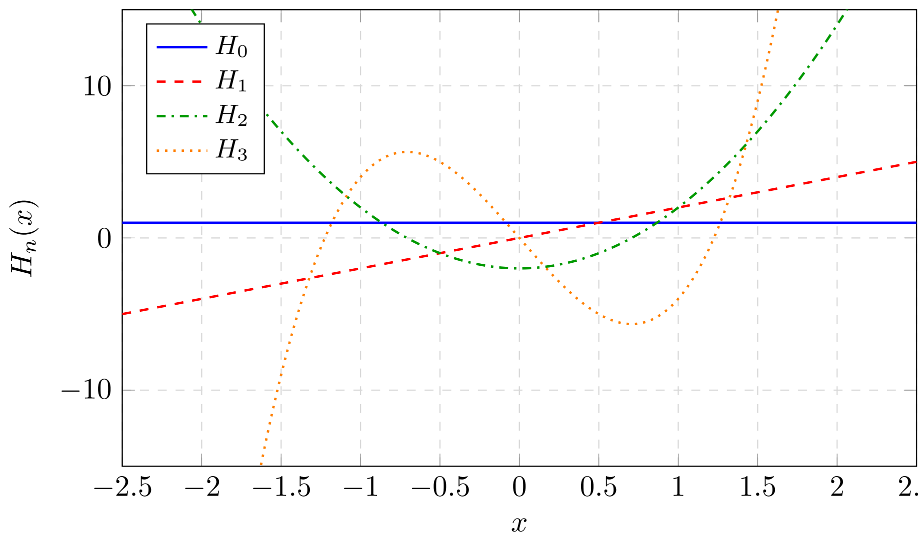

Figure 11: The first four (physicist-convention) Hermite polynomials on $[-2.5, 2.5]$, as given by Proposition 6.2. Parity $H_n(-x) = (-1)^n H_n(x)$ is visible, as is the 2^n leading coefficient. Each plotted H_n has n real zeros.

Proposition 6.3 (Recurrences and ODE). *From* (6.1)*, for all* $n\geq 1$*,*

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \tag{6.6}$$

$$H_n'(x) = 2nH_{n-1}(x), \tag{6.7}$$

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{6.8}$$

Proof. Write $G(x,t) = e^{2xt-t^2} = \sum_n H_n(x)t^n/n!$.

Step 1 (t-derivative gives the three-term recurrence). Differentiate G in t:

$$\partial_t G = (2x-2t)\,e^{2xt-t^2} = (2x-2t)\,G(x,t). \tag{6.9}$$

Expand both sides in t. Left side:

$$\partial_t \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty} H_n(x)\frac{nt^{n-1}}{n!} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} H_{m+1}(x)\frac{t^m}{m!},$$

after the index shift $m = n-1$. Right side:

$$(2x-2t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = 2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!}.$$

In the second sum, set $m = n+1$: $\sum_{m=1}^{\infty} H_{m-1}(x)\, t^m/(m-1)! = \sum_{m=1}^{\infty} m\cdot H_{m-1}(x)\, t^m/m!$ (multiplying numerator and denominator by m). So the right side is

$$\sum_{m=0}^{\infty} 2xH_m(x)\frac{t^m}{m!} - \sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!}.$$

Matching the coefficient of $t^m/m!$ on both sides, for $m \geq 1$,

$$H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x),$$

which is (6.6) after renaming $m \to n$.

Step 2 (x-derivative gives the lowering relation). Differentiate G in x:

$$\partial_x G = 2t\, e^{2xt-t^2} = 2t\, G(x,t). \tag{6.10}$$

Left side:

$$\partial_x \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!}.$$

Right side:

$$2t\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} = \sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!},$$

with the same index shift $m = n+1$ as above. Match coefficients of $t^n/n!$:

$$H_n'(x) = 2nH_{n-1}(x) \quad (n \geq 1), \qquad H_0'(x) = 0,$$

which is (6.7).

Step 3 (ODE). For $n = 1$, Proposition 6.2 gives $H_1(x) = 2x$, so

$$H_1''(x) - 2xH_1'(x) + 2H_1(x) = 0 - 4x + 4x = 0.$$

Now assume $n \geq 2$. Differentiate (6.7) once in x:

$$H_n''(x) = 2nH_{n-1}'(x) \overset{(6.7)}{=} 2n\cdot 2(n-1)H_{n-2}(x). \tag{6.11}$$

From (6.6) (shifting $n \to n-1$): $H_n = 2xH_{n-1} - 2(n-1)H_{n-2}$, i.e. $2(n-1)H_{n-2} = 2xH_{n-1} - H_n$. Substitute into (6.11):

$$H_n''(x) = 2n\big[2xH_{n-1}(x) - H_n(x)\big] = 2x\cdot 2nH_{n-1}(x) - 2nH_n(x).$$

Using $2nH_{n-1}(x) = H_n'(x)$ from (6.7):

$$H_n''(x) = 2xH_n'(x) - 2nH_n(x),$$

which rearranges to (6.8). □

Theorem 6.4 (Rodrigues).

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{6.12}$$

Proof. Start from (6.2). Completing the square gives $2xt - t^2 = x^2 - (x-t)^2$, so

$$H_n(x) = \frac{n!}{2\pi i} \oint_{|t|=\rho} e^{x^2} e^{-(x-t)^2}\, t^{-n-1}\, dt = \frac{n! e^{x^2}}{2\pi i} \oint_{|t|=\rho} e^{-(x-t)^2}\, t^{-n-1}\, dt.$$

Substitute $s = x - t$. Then $t = x - s$, $dt = -ds$, and the image contour is again the counter-clockwise circle $|s - x| = \rho$:

$$H_n(x) = -\frac{n! e^{x^2}}{2\pi i} \oint_{|s-x|=\rho} e^{-s^2} (x-s)^{-n-1}\, ds.$$

Factor $(x-s)^{-n-1} = (-1)^{n+1}(s-x)^{-n-1}$:

$$H_n(x) = -\frac{(-1)^{n+1} n! e^{x^2}}{2\pi i} \oint_{|s-x|=\rho} \frac{e^{-s^2}}{(s-x)^{n+1}}\, ds \;=\; \frac{(-1)^n n! e^{x^2}}{2\pi i} \oint_{|s-x|=\rho} \frac{e^{-s^2}}{(s-x)^{n+1}}\, ds. \tag{6.13}$$

By Cauchy's formula for the n-th derivative (Cor. 1.14) applied to the entire function $f(s) = e^{-s^2}$ at $s = x$,

$$\frac{1}{2\pi i} \oint_{|s-x|=\rho} \frac{e^{-s^2}}{(s-x)^{n+1}}\, ds = \frac{1}{n!} \frac{d^n}{dx^n} e^{-x^2}.$$

Substituting into (6.13):

$$H_n(x) = (-1)^n n! e^{x^2} \cdot \frac{1}{n!} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

which is (6.12). □

Theorem 6.5 (Orthogonality).

$$\int_{-\infty}^{\infty} H_m(x) H_n(x)\, e^{-x^2}\, dx = 2^n n! \sqrt{\pi}\, \delta_{mn}. \tag{6.14}$$

Here δ_{mn} is the Kronecker delta: it equals 1 when $m = n$ and 0 otherwise.

Proof. Assume $m \le n$, since the integral is symmetric in m, n. Substitute Rodrigues' formula (6.12) for H_n:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx = \int_{-\infty}^{\infty} H_m(x) \cdot (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n} \cdot e^{-x^2}\, dx.$$

The two exponentials cancel:

$$= (-1)^n \int_{-\infty}^{\infty} H_m(x) \frac{d^n}{dx^n} e^{-x^2}\, dx. \tag{6.15}$$

Integrate by parts once, with $u = H_m(x)$, $dv = (d^n/dx^n) e^{-x^2}\, dx$:

$$\int H_m \frac{d^n e^{-x^2}}{dx^n}\, dx = \left[H_m \frac{d^{n-1} e^{-x^2}}{dx^{n-1}} \right]_{-\infty}^{\infty} - \int H_m'(x) \frac{d^{n-1} e^{-x^2}}{dx^{n-1}}\, dx.$$

The boundary term vanishes: every derivative of e^{-x^2} is a polynomial times e^{-x^2}, and the Gaussian dominates polynomial growth. Repeating the integration by parts n times gives

$$\int_{-\infty}^{\infty} H_m \frac{d^n e^{-x^2}}{dx^n}\, dx = (-1)^n \int_{-\infty}^{\infty} H_m^{(n)}(x)\, e^{-x^2}\, dx. \tag{6.16}$$

Substituting (6.16) into (6.15):

$$\int_{-\infty}^{\infty} H_m H_n\, e^{-x^2}\, dx = (-1)^n \cdot (-1)^n \int_{-\infty}^{\infty} H_m^{(n)}(x)\, e^{-x^2}\, dx = \int_{-\infty}^{\infty} H_m^{(n)}(x)\, e^{-x^2}\, dx, \tag{6.17}$$

since $(-1)^{2n} = 1$.

Case $m < n$. H_m is a polynomial of degree $m < n$ (Prop. 6.2), so $H_m^{(n)} \equiv 0$, and (6.17) gives zero.

Case $m = n$. By Prop. 6.2, $H_n(x) = 2^n x^n + (\text{lower})$. Differentiate n times: the lower-degree terms vanish (all have degree $< n$) and the leading monomial $2^n x^n$ produces $2^n \cdot n!$. Thus $H_n^{(n)}(x) = 2^n n!$, a constant. Substituting into (6.17):

$$\int_{-\infty}^{\infty} H_n(x)^2\, e^{-x^2}\, dx = 2^n n! \int_{-\infty}^{\infty} e^{-x^2}\, dx = 2^n n! \sqrt{\pi},$$

using the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (which follows from $\Gamma(1/2) = \sqrt{\pi}$, Cor. 3.14). Both cases combine into (6.14). □

Example 6.6 (Quantum harmonic oscillator). *For the one-dimensional quantum harmonic oscillator, use units $\hbar = m = \omega = 1$. The* Hamiltonian *is the energy operator whose eigenvalues are the allowed energies. Here it is*

$$\hat{H} = \tfrac{1}{2}(\hat{p}^2 + \hat{x}^2), \qquad \hat{p} = -i\, d/dx. \tag{6.18}$$

We seek square-integrable solutions of $\hat{H}\psi = E\psi$, i.e.

$$-\tfrac{1}{2}\psi''(x) + \tfrac{1}{2}x^2\psi(x) = E\psi(x). \tag{6.19}$$

Use the ansatz $\psi(x) = H(x)\, e^{-x^2/2}$. The derivatives are

$$\begin{aligned} \psi'(x) &= \big[H'(x) - xH(x)\big]e^{-x^2/2}, \\ \psi''(x) &= \big[H''(x) - 2xH'(x) + (x^2-1)H(x)\big]e^{-x^2/2}, \end{aligned}$$

by the product rule.

Substitute into (6.19) and divide by $e^{-x^2/2}$:

$$-\tfrac{1}{2}\big[H'' - 2xH' + (x^2-1)H\big] + \tfrac{1}{2}x^2 H = EH.$$

The $\tfrac{1}{2}x^2 H$ on the left cancels against the $-\tfrac{1}{2}x^2 H$ inside the brackets:

$$-\tfrac{1}{2}H'' + xH' + \tfrac{1}{2}H = EH,$$

or equivalently

$$H''(x) - 2xH'(x) + (2E-1)H(x) = 0. \tag{6.20}$$

Compare with Hermite's equation (6.8): H_n solves $H'' - 2xH' + 2nH = 0$. This suggests the candidate energies

$$E_n = n + \tfrac{1}{2}, \qquad n = 0, 1, 2, \ldots,$$

with $H = H_n$. The next recurrence isolates the polynomial branch.

Why the energies are discrete. *Write a power series $H(x) = \sum_{k=0}^{\infty} a_k x^k$ and substitute it into* (6.20). *Using*

$$H''(x) = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k, \qquad xH'(x) = \sum_{k=0}^{\infty} k a_k x^k,$$

we get

$$\sum_{k=0}^{\infty} \big[(k+2)(k+1)a_{k+2} - (2k+1-2E)a_k\big]x^k = 0,$$

for every $k \geq 0$, hence

$$a_{k+2} = \frac{2k+1-2E}{(k+2)(k+1)} a_k.$$

The even coefficients propagate from a_0, and the odd coefficients from a_1. If $2E-1 \neq 2n$ for every integer $n \geq 0$, neither chain terminates. For large k, the recurrence behaves like $a_{k+2} \sim 2a_k/k$, the same leading ratio as the Taylor series of e^{x^2}. Thus a non-terminating branch gives a non-square-integrable large-x behavior for $\psi = He^{-x^2/2}$. The square-integrable states come from forcing one chain to terminate and setting the other initial coefficient to zero. Termination happens exactly when $2E-1 = 2n$; the surviving branch is the Hermite polynomial H_n.

Here is the comparison in a little more detail. The even Taylor series of e^{x^2} is

$$e^{x^2} = \sum_{j=0}^{\infty} \frac{x^{2j}}{j!}, \qquad \frac{b_{2j+2}}{b_{2j}} = \frac{1}{j+1} \sim \frac{2}{2j}.$$

The odd series xe^{x^2} has the same large-index ratio along odd powers. Our recurrence has

$$\frac{a_{k+2}}{a_k} = \frac{2k+1-2E}{(k+2)(k+1)} \sim \frac{2}{k}.$$

So any nonzero chain that keeps going has the same tail scale as one of e^{x^2} or xe^{x^2}. Multiplying by the Gaussian in $\psi = He^{-x^2/2}$ then leaves an $e^{x^2/2}$-type growth, which cannot be square-integrable on the real line.

Normalization. *By Thm. 6.5,*

$$\int_{-\infty}^{\infty} \big[H_n(x)e^{-x^2/2}\big]^2 dx = \int_{-\infty}^{\infty} H_n(x)^2 e^{-x^2} dx = 2^n n! \sqrt{\pi},$$

so the normalized stationary states are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x)\, e^{-x^2/2}, \qquad E_n = n + \tfrac{1}{2}, \qquad n = 0, 1, 2, \ldots \tag{6.21}$$

Orthogonality $\langle \psi_m | \psi_n \rangle = \delta_{mn}$ follows from (6.14) *with the same weight.*

6.2 Laguerre polynomials

Motivation. The radial part of the hydrogen wavefunction reduces to Laguerre polynomials. They are the orthogonal family on $[0, \infty)$ with weight $x^{\alpha} e^{-x}$.

Definition 6.7 (Laguerre generating function)**.** *The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by*

$$\frac{1}{(1-t)^{\alpha+1}}\exp\Big[-\frac{xt}{1-t}\Big] = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,t^n, \qquad |t|<1, \tag{6.22}$$

with $\alpha>-1$ when the orthogonality weight $x^\alpha e^{-x}$ is used. The generating-function identity itself may be read for complex α after choosing the branch of $(1-t)^{-\alpha-1}$ with value 1 at $t=0$; equivalently, the finite series below extends $L_n^{(\alpha)}$ as a polynomial in x and α. When $\alpha=0$ we write $L_n(x)=L_n^{(0)}(x)$, the ordinary *Laguerre polynomial.*

Proposition 6.8 (Series and first values)**.**

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}. \tag{6.23}$$

In particular, $L_n^{(\alpha)}$ is a polynomial of degree n with leading coefficient $(-1)^n/n!$. The first ordinary Laguerres are

$$L_0(x)=1, \qquad L_1(x)=1-x, \qquad L_2(x)=\tfrac{1}{2}(x^2-4x+2).$$

Proof of the series (6.23). Expand the two factors in (6.22) separately. The binomial series gives

$$(1-t)^{-\alpha-1} = \sum_{j=0}^{\infty}\binom{-\alpha-1}{j}(-t)^j = \sum_{j=0}^{\infty}\binom{\alpha+j}{j}t^j, \tag{6.24}$$

where $\binom{-\alpha-1}{j}(-1)^j=\binom{\alpha+j}{j}$. Also,

$$\exp\Big(-\frac{xt}{1-t}\Big) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\,\frac{t^k}{(1-t)^k}. \tag{6.25}$$

Multiply (6.24) by (6.25):

$$\frac{1}{(1-t)^{\alpha+1}}e^{-xt/(1-t)} = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\,\frac{t^k}{(1-t)^{\alpha+1+k}}.$$

Apply (6.24) again with α replaced by $\alpha+k$:

$$\frac{1}{(1-t)^{\alpha+k+1}} = \sum_{j=0}^{\infty}\binom{\alpha+k+j}{j}t^j.$$

Hence

$$\frac{1}{(1-t)^{\alpha+1}}e^{-xt/(1-t)} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-x)^k}{k!}\binom{\alpha+k+j}{j}t^{k+j}.$$

Collect the coefficient of t^n by setting $n=k+j$, i.e. $j=n-k$ with $0\le k\le n$:

$$[t^n]\mathrm{LHS} = \sum_{k=0}^{n}\binom{\alpha+n}{n-k}\frac{(-x)^k}{k!},$$

which, matched against $L_n^{(\alpha)}(x)$ on the right of (6.22), is (6.23).

Degree and first values. The top-degree term in (6.23) is the $k=n$ term,

$$\binom{n+\alpha}{0}\frac{(-x)^n}{n!} = \frac{(-1)^n}{n!}x^n,$$

so $L_n^{(\alpha)}$ has degree n and leading coefficient $(-1)^n/n!$.

Worked case $n=2$, $\alpha=0$. Here

$$\begin{aligned} k=0 &: \binom{2}{2}\cdot 1 = 1, \\ k=1 &: \binom{2}{1}\cdot(-x) = -2x, \\ k=2 &: \binom{2}{0}\cdot x^2/2 = x^2/2. \end{aligned}$$

so $L_2(x) = 1-2x+x^2/2 = (x^2-4x+2)/2$. The cases $n=0,1$ give $L_0=1$ and $L_1 = 1-x$. □

Theorem 6.9 (Laguerre Rodrigues formula)**.**

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\big[e^{-x}x^{n+\alpha}\big]. \tag{6.26}$$

Proof. Apply Leibniz's rule:

$$\frac{d^n}{dx^n}\big[e^{-x}x^{n+\alpha}\big] = \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}}{dx^{n-k}}e^{-x}\,\frac{d^k}{dx^k}x^{n+\alpha}.$$

Use

$$\frac{d^{n-k}}{dx^{n-k}}e^{-x} = (-1)^{n-k}e^{-x},$$

and

$$\frac{d^k}{dx^k}x^{n+\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-k+1)}x^{n+\alpha-k},$$

Multiply by $x^{-\alpha}e^x/n!$:

$$\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\big[e^{-x}x^{n+\alpha}\big] = \sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!(n-k)!}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-k+1)}x^{n-k}.$$

Now set $j=n-k$. Then $k=n-j$, and the sum becomes

$$\sum_{j=0}^{n}\frac{(-1)^j}{j!(n-j)!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}x^j.$$

Recognize

$$\binom{n+\alpha}{n-j} = \frac{\Gamma(n+\alpha+1)}{(n-j)!\,\Gamma(\alpha+j+1)},$$

so the last expression is exactly

$$\sum_{j=0}^{n}\binom{n+\alpha}{n-j}\frac{(-x)^j}{j!}.$$

By Proposition 6.8, this is $L_n^{(\alpha)}(x)$. □

Proposition 6.10 (Laguerre ODE, orthogonality)**.** *$L_n^{(\alpha)}$ satisfies*

$$x(L_n^{(\alpha)})'' + (\alpha+1-x)(L_n^{(\alpha)})' + nL_n^{(\alpha)} = 0, \tag{6.27}$$

and, for $\alpha > -1$,

$$\int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)\,x^\alpha e^{-x}\,dx = \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{mn}. \tag{6.28}$$

Proof of the ODE.. Let $G(x,t) = (1-t)^{-\alpha-1}e^{-xt/(1-t)}$. Compute:

$$\begin{aligned}\partial_x G &= -\frac{t}{1-t}G,\\ \partial_x^2 G &= \frac{t^2}{(1-t)^2}G,\\ \partial_t G &= \left(\frac{\alpha+1}{1-t} - \frac{x}{(1-t)^2}\right)G.\end{aligned} \tag{6.29}$$

The last line comes from logarithmic differentiation.

Consider the combination

$$x\,\partial_x^2 G + (\alpha+1-x)\,\partial_x G = x\,\frac{t^2}{(1-t)^2}G - (\alpha+1-x)\,\frac{t}{1-t}G.$$

Factor $t/(1-t)$:

$$= \frac{t}{1-t}\left[\frac{xt}{1-t} - (\alpha+1) + x\right]G = \frac{t}{1-t}\left[\frac{xt + x(1-t)}{1-t} - (\alpha+1)\right]G,$$

which simplifies (since $xt + x(1-t) = x$) to

$$x\,\partial_x^2 G + (\alpha+1-x)\,\partial_x G = -\frac{t}{1-t}\left[(\alpha+1) - \frac{x}{1-t}\right]G. \tag{6.30}$$

Compare to $-t\,\partial_t G$ from (6.29):

$$-t\,\partial_t G = -\frac{t}{1-t}\left[(\alpha+1) - \frac{x}{1-t}\right]G,$$

which is the same expression as (6.30). Therefore

$$x\,\partial_x^2 G + (\alpha+1-x)\,\partial_x G + t\,\partial_t G = 0. \tag{6.31}$$

Substitute $G = \sum_n L_n^{(\alpha)}(x)t^n$: the term $t\,\partial_t G = \sum_n nL_n^{(\alpha)}(x)t^n$. Matching the coefficient of t^n in (6.31):

$$x(L_n^{(\alpha)})'' + (\alpha+1-x)(L_n^{(\alpha)})' + nL_n^{(\alpha)} = 0,$$

which is (6.27). □

Proof of orthogonality. The integral is symmetric in m,n, so assume $m \le n$. Substitute the Rodrigues formula (6.26) for one copy of $L_n^{(\alpha)}$:

$$\begin{aligned}I_{mn} &:= \int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)\,x^\alpha e^{-x}\,dx\\ &= \int_0^\infty L_m^{(\alpha)}(x)\cdot\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}[e^{-x}x^{n+\alpha}]\cdot x^\alpha e^{-x}\,dx\\ &= \frac{1}{n!}\int_0^\infty L_m^{(\alpha)}(x)\,\frac{d^n}{dx^n}\left[e^{-x}x^{n+\alpha}\right]dx,\end{aligned}$$

since $x^{-\alpha}\cdot x^\alpha = 1$ and $e^x\cdot e^{-x} = 1$. Integrate by parts n times. At step j, where $0 \le j \le n-1$, the boundary term has the form

$$(\text{polynomial})\cdot e^{-x}x^{n+\alpha-j}\Big|_0^\infty.$$

At $x = \infty$, the factor e^{-x} kills every polynomial. At $x = 0$, Leibniz's rule shows that the lowest surviving power in $\frac{d^{n-1-j}}{dx^{n-1-j}}[e^{-x}x^{n+\alpha}]$ is $x^{\alpha+1+j}$. Since $\alpha > -1$, this vanishes at 0. Thus every boundary term is zero. After n integrations:

$$I_{mn} = \frac{(-1)^n}{n!}\int_0^\infty (L_m^{(\alpha)})^{(n)}(x)\, e^{-x}x^{n+\alpha}\, dx. \tag{6.32}$$

Case $m < n$. $L_m^{(\alpha)}$ has degree $m < n$ (Prop. 6.8), so $(L_m^{(\alpha)})^{(n)} \equiv 0$ and $I_{mn} = 0$.

Case $m = n$. From (6.23), the top-degree term of $L_n^{(\alpha)}$ is $(-x)^n/n! = (-1)^n x^n/n!$. Differentiate n times: $(L_n^{(\alpha)})^{(n)}(x) = (-1)^n$. Substitute into (6.32):

$$I_{nn} = \frac{(-1)^n}{n!}\cdot(-1)^n\int_0^\infty e^{-x}x^{n+\alpha}\, dx = \frac{1}{n!}\Gamma(n+\alpha+1),$$

using the definition (3.1) of Γ (Def. 3.1). This is (6.28). □

Proposition 6.11 (Laguerre three-term recurrence). *For $n \geq 1$,*

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x). \tag{6.33}$$

Proof. Start from the generating function $G(x,t) = \sum_{n=0}^\infty L_n^{(\alpha)}(x)t^n$. Rearranging (6.29) gives

$$(1-t)^2\,\partial_t G = \big[(\alpha+1)(1-t) - x\big]G.$$

Expand each side in powers of t. Since

$$\partial_t G = \sum_{n=1}^\infty nL_n^{(\alpha)}(x)t^{n-1},$$

the left side is

$$\begin{aligned}(1-t)^2\partial_t G &= (1-2t+t^2)\sum_{n=1}^\infty nL_n^{(\alpha)}(x)t^{n-1}\\ &= \sum_{n=0}^\infty (n+1)L_{n+1}^{(\alpha)}(x)t^n - 2\sum_{n=1}^\infty nL_n^{(\alpha)}(x)t^n + \sum_{n=2}^\infty (n-1)L_{n-1}^{(\alpha)}(x)t^n.\end{aligned}$$

The right side is

$$\big[(\alpha+1)(1-t)-x\big]G = (\alpha+1-x)\sum_{n=0}^\infty L_n^{(\alpha)}(x)t^n - (\alpha+1)\sum_{n=1}^\infty L_{n-1}^{(\alpha)}(x)t^n.$$

Match the coefficient of t^n for $n \geq 1$:

$$(n+1)L_{n+1}^{(\alpha)} - 2nL_n^{(\alpha)} + (n-1)L_{n-1}^{(\alpha)} = (\alpha+1-x)L_n^{(\alpha)} - (\alpha+1)L_{n-1}^{(\alpha)}.$$

Move the last two terms on the right to the left and collect coefficients of $L_n^{(\alpha)}$ and $L_{n-1}^{(\alpha)}$:

$$(n+1)L_{n+1}^{(\alpha)} = (2n+\alpha+1-x)L_n^{(\alpha)} - (n+\alpha)L_{n-1}^{(\alpha)},$$

which is (6.33). □

Example 6.12 (Hydrogen radial wavefunction). *Use atomic units $\hbar = m_e = e = a_0 = 1$, so the physical constants do not clutter the formulas. The radial Schrödinger equation for an electron in the Coulomb field of a nucleus of charge Z is*

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[2E+\frac{2Z}{r}-\frac{\ell(\ell+1)}{r^2}\right]R=0,$$

for $R = R_{N\ell}(r)$ with angular-momentum quantum number ℓ. The term $\ell(\ell+1)$ comes from the spherical-harmonic separation in Section 5: writing $\psi(r,\theta,\phi) = R(r)Y_\ell^m(\theta,\phi)$ diagonalizes the angular Laplacian with eigenvalue $-\ell(\ell+1)$. Bound states have $E < 0$; write $E = -\kappa^2/2$ and set $\rho = 2\kappa r$:

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\left[-\tfrac{1}{4}+\frac{Z/\kappa}{\rho}-\frac{\ell(\ell+1)}{\rho^2}\right]R=0.$$

Use the ansatz $R(\rho) = \rho^\ell e^{-\rho/2} f(\rho)$. The factor ρ^ℓ handles the short-distance centrifugal behavior, while $e^{-\rho/2}$ handles the large-distance decay. Let

$$A(\rho)=\rho^\ell e^{-\rho/2}.$$

Then

$$\frac{A'(\rho)}{A(\rho)}=\frac{\ell}{\rho}-\frac{1}{2},\qquad \frac{A''(\rho)}{A(\rho)}=\left(\frac{\ell}{\rho}-\frac{1}{2}\right)^2-\frac{\ell}{\rho^2}=\frac{\ell(\ell-1)}{\rho^2}-\frac{\ell}{\rho}+\frac{1}{4}.$$

Since $R = Af$,

$$\begin{aligned}R' &= A\left[f'+\left(\frac{\ell}{\rho}-\frac{1}{2}\right)f\right],\\ R'' &= A\left[f''+2\left(\frac{\ell}{\rho}-\frac{1}{2}\right)f'+\left(\frac{\ell(\ell-1)}{\rho^2}-\frac{\ell}{\rho}+\frac{1}{4}\right)f\right].\end{aligned}$$

Also,

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=R''+\frac{2}{\rho}R'.$$

Substitute the formulas for R' and R'':

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=A\left[f''+\frac{2\ell+2-\rho}{\rho}f'+\left(\frac{\ell(\ell+1)}{\rho^2}-\frac{\ell+1}{\rho}+\frac{1}{4}\right)f\right].$$

Insert this into the ρ-equation. The ρ^{-2} terms cancel against $-\ell(\ell+1)R/\rho^2$, and the constants $\pm 1/4$ cancel as well, leaving

$$f''+\frac{2\ell+2-\rho}{\rho}f'+\frac{Z/\kappa-\ell-1}{\rho}f=0.$$

Multiplying by ρ gives

$$\rho f''+(2\ell+2-\rho)f'+\left(\frac{Z}{\kappa}-\ell-1\right)f=0.$$

Compare with Laguerre's equation (6.27). Here $\alpha = 2\ell + 1$ and the Laguerre degree is $n_r = Z/\kappa - \ell - 1$. The bound-state branch is the terminating Laguerre branch, so n_r must be a non-negative integer. The reason is the same square-integrability test used to choose the decaying exponential: the nonterminating Kummer solution grows like an e^ρ factor at infinity, and after multiplying by the prefactor $e^{-\rho/2}$ in $R(\rho) = e^{-\rho/2}\rho^\ell f(\rho)$ it still behaves like $e^{\rho/2}$ times a power of ρ. That is not normalizable. Termination removes this growing branch and leaves a polynomial

Laguerre factor. Thus $\kappa = Z/(n_r+\ell+1)$. *Defining the* principal quantum number $N := n_r+\ell+1$ *gives*

$$E_N = -\frac{Z^2}{2N^2}, \qquad N = 1,2,3,\dots, \qquad \ell = 0,1,\dots,N-1,$$

This recovers Bohr's spectrum. The radial function is

$$R_{N\ell}(r) = C_{N\ell}\,\rho^{\ell}\,e^{-\rho/2}\,L^{(2\ell+1)}_{N-\ell-1}(\rho), \qquad \rho = \frac{2Zr}{N}, \tag{6.34}$$

with $C_{N\ell}$ *determined by*

$$\int_0^{\infty} |R_{N\ell}(r)|^2 r^2\,dr = 1.$$

Since $r = N\rho/(2Z)$ *and* $dr = (N/2Z)\,d\rho$, *we have*

$$r^2\,dr = \left(\frac{N}{2Z}\right)^3 \rho^2\,d\rho.$$

Substituting (6.34) *gives*

$$1 = |C_{N\ell}|^2 \left(\frac{N}{2Z}\right)^3 \int_0^{\infty} \rho^{2\ell+2} e^{-\rho}\Big[L^{(2\ell+1)}_{N-\ell-1}(\rho)\Big]^2\,d\rho. \tag{6.35}$$

Set

$$n_r = N-\ell-1, \qquad \alpha = 2\ell+1.$$

For a general Laguerre degree n, set

$$J^{(\alpha)}_n := \int_0^{\infty} \rho^{\alpha+1} e^{-\rho}\Big[L^{(\alpha)}_n(\rho)\Big]^2\,d\rho.$$

The remaining integral in (6.35) *is* $J^{(\alpha)}_{n_r}$. *Use the three-term recurrence* (6.33)

$$(n+1)L^{(\alpha)}_{n+1} = (2n+\alpha+1-\rho)L^{(\alpha)}_n - (n+\alpha)L^{(\alpha)}_{n-1}.$$

Rearranging,

$$\rho L^{(\alpha)}_n = (2n+\alpha+1)L^{(\alpha)}_n - (n+1)L^{(\alpha)}_{n+1} - (n+\alpha)L^{(\alpha)}_{n-1}.$$

Multiply by $L^{(\alpha)}_n(\rho)\rho^{\alpha}e^{-\rho}$ *and integrate over* $[0,\infty)$. *We will then set* $n = n_r$:

$$\begin{aligned} J^{(\alpha)}_n = (2n+\alpha+1)&\int_0^{\infty}\Big[L^{(\alpha)}_n(\rho)\Big]^2 \rho^{\alpha} e^{-\rho}\,d\rho \\ -(n+1)&\int_0^{\infty} L^{(\alpha)}_n(\rho)L^{(\alpha)}_{n+1}(\rho)\rho^{\alpha}e^{-\rho}\,d\rho \\ -(n+\alpha)&\int_0^{\infty} L^{(\alpha)}_n(\rho)L^{(\alpha)}_{n-1}(\rho)\rho^{\alpha}e^{-\rho}\,d\rho. \end{aligned}$$

The last two integrals vanish by orthogonality (6.28), *so*

$$J^{(\alpha)}_n = (2n+\alpha+1)\frac{\Gamma(n+\alpha+1)}{n!}.$$

For $n = n_r = N-\ell-1$ *and* $\alpha = 2\ell+1$,

$$2n+\alpha+1 = 2(N-\ell-1)+(2\ell+1)+1 = 2N,$$

and

$$\Gamma(n+\alpha+1) = \Gamma(N+\ell+1) = (N+\ell)!.$$

Therefore

$$J^{(2\ell+1)}_{N-\ell-1} = 2N\,\frac{(N+\ell)!}{(N-\ell-1)!}.$$

Substitute this into (6.35)*:*

$$1 = |C_{N\ell}|^2 \left(\frac{N}{2Z}\right)^3 2N\,\frac{(N+\ell)!}{(N-\ell-1)!}.$$

Choosing $C_{N\ell} > 0$*,*

$$C_{N\ell} = \left(\frac{2Z}{N}\right)^{3/2} \sqrt{\frac{(N-\ell-1)!}{2N\,(N+\ell)!}}.$$

Hence the normalized radial wavefunction is

$$R_{N\ell}(r) = \left(\frac{2Z}{N}\right)^{3/2} \sqrt{\frac{(N-\ell-1)!}{2N\,(N+\ell)!}}\,\rho^\ell\, e^{-\rho/2}\, L^{(2\ell+1)}_{N-\ell-1}(\rho).$$

Ground state. $N=1$, $\ell=0$, $\rho=2Zr$, $L_0^{(1)}=1$*:*

$$C_{10} = (2Z)^{3/2}\sqrt{\frac{0!}{2\cdot 1\cdot 1!}} = 2Z^{3/2}.$$

$$R_{10}(r) = 2Z^{3/2}\,e^{-Zr},$$

reproducing the usual ground state.

6.3 Chebyshev polynomials

Motivation. Chebyshev polynomials have a minimax property on $[-1,1]$: after the degree and leading coefficient are fixed, they minimize the maximum absolute size on the interval. They are therefore a natural basis for uniform approximation. Their key identities are $T_n(\cos\theta) = \cos n\theta$ and $U_n(\cos\theta)\sin\theta = \sin(n+1)\theta$.

Why the trigonometric substitution. The change of variable $x = \cos\theta$ is suggested by the factorization

$$1 - 2xt + t^2 = (1 - te^{i\theta})(1 - te^{-i\theta}).$$

Each linear factor is a geometric-series kernel, so the Chebyshev generating function splits into two simple series in $e^{\pm i\theta}$. In the θ variable, the recurrences and orthogonality relations become ordinary Fourier identities.

Definition 6.13 (Chebyshev generating functions)**.** *The Chebyshev polynomials of the first and second kinds are defined by*

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)\,t^n, \tag{6.36}$$

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)\,t^n, \tag{6.37}$$

for $|t|<1$ *and* $x \in [-1,1]$*. Equivalently, these identities may be read as formal power series in* t*; their coefficients are polynomials in* x*, so the resulting* T_n *and* U_n *extend to all complex* x*.*

Proposition 6.14 (Trigonometric representation). *For $x \in [-1,1]$, setting $x = \cos\theta$ with $\theta \in [0,\pi]$,*

$$T_n(\cos\theta) = \cos(n\theta), \qquad U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{6.38}$$

At $\theta = 0$ and $\theta = \pi$, the formula for U_n is understood by continuity:

$$U_n(1) = n+1, \qquad U_n(-1) = (-1)^n(n+1).$$

Proof. Set $x = \cos\theta$. Factor the denominator:

$$1 - 2xt + t^2 = 1 - (e^{i\theta} + e^{-i\theta})t + t^2 = (1 - te^{i\theta})(1 - te^{-i\theta}),$$

since $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and expanding the product reproduces the LHS.

First kind. Decompose the left side of (6.36):

$$\frac{1 - xt}{(1 - te^{i\theta})(1 - te^{-i\theta})} = \frac{A}{1 - te^{i\theta}} + \frac{B}{1 - te^{-i\theta}}.$$

Clear denominators: $1 - xt = A(1 - te^{-i\theta}) + B(1 - te^{i\theta})$. Evaluate at $t = e^{-i\theta}$ (zeroing the first denominator's partner): $1 - \cos\theta \cdot e^{-i\theta} = A(1 - e^{-2i\theta})$. Compute $1 - \cos\theta \cdot e^{-i\theta} = 1 - \frac{1}{2}(1 + e^{-2i\theta}) = \frac{1}{2}(1 - e^{-2i\theta})$, so $A = 1/2$. By symmetry (swap $\theta \longleftrightarrow -\theta$), $B = 1/2$. Hence

$$\frac{1 - xt}{1 - 2xt + t^2} = \frac{1}{2}\left[\frac{1}{1 - te^{i\theta}} + \frac{1}{1 - te^{-i\theta}}\right].$$

Expand each geometric series ($|te^{\pm i\theta}| = |t| < 1$):

$$= \frac{1}{2}\sum_{n=0}^{\infty} t^n e^{in\theta} + \frac{1}{2}\sum_{n=0}^{\infty} t^n e^{-in\theta} = \sum_{n=0}^{\infty} \frac{e^{in\theta} + e^{-in\theta}}{2} t^n = \sum_{n=0}^{\infty} \cos(n\theta)\, t^n.$$

Matching with (6.36): $T_n(\cos\theta) = \cos(n\theta)$.

Second kind. Similarly,

$$\frac{1}{(1 - te^{i\theta})(1 - te^{-i\theta})} = \frac{C}{1 - te^{i\theta}} + \frac{D}{1 - te^{-i\theta}}.$$

Clear denominators: $1 = C(1 - te^{-i\theta}) + D(1 - te^{i\theta})$. At $t = e^{-i\theta}$: $1 = C(1 - e^{-2i\theta})$, so $C = 1/(1 - e^{-2i\theta}) = e^{i\theta}/(e^{i\theta} - e^{-i\theta}) = e^{i\theta}/(2i\sin\theta)$. By symmetry, $D = -e^{-i\theta}/(2i\sin\theta)$. Hence

$$\frac{1}{1 - 2xt + t^2} = \frac{1}{2i\sin\theta}\left[\frac{e^{i\theta}}{1 - te^{i\theta}} - \frac{e^{-i\theta}}{1 - te^{-i\theta}}\right].$$

Expand geometrically:

$$\begin{aligned}
&= \frac{1}{2i\sin\theta}\sum_{n=0}^{\infty}\left[e^{i\theta} \cdot e^{in\theta} - e^{-i\theta} \cdot e^{-in\theta}\right]t^n \\
&= \frac{1}{2i\sin\theta}\sum_{n=0}^{\infty}\left[e^{i(n+1)\theta} - e^{-i(n+1)\theta}\right]t^n \\
&= \sum_{n=0}^{\infty} \frac{\sin((n+1)\theta)}{\sin\theta}\, t^n,
\end{aligned}$$

using $(e^{i\phi} - e^{-i\phi})/(2i) = \sin\phi$. Matching with (6.37): $U_n(\cos\theta) = \sin((n+1)\theta)/\sin\theta$. □

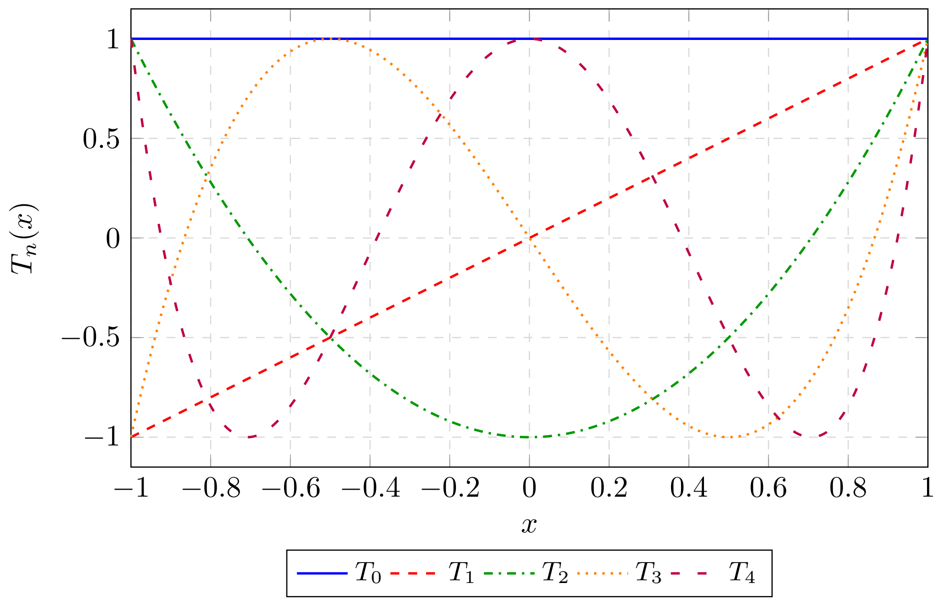

Figure 12: The first five Chebyshev polynomials of the first kind on $[-1,1]$. Via $T_n(\cos\theta) = \cos(n\theta)$ (Proposition 6.14), for $n \geq 1$ the polynomial T_n oscillates n times between $+1$ and -1 with extrema at $x_k^* = \cos(k\pi/n)$, $k = 0, \dots, n$.

Proposition 6.15 (Chebyshev ODE, orthogonality)**.** *T_n satisfies*

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0. \tag{6.39}$$

Orthogonality on $[-1,1]$*:*

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} \pi, & m = n = 0, \\ \pi/2, & m = n \geq 1, \\ 0, & m \neq n, \end{cases} \tag{6.40}$$

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\delta_{mn}. \tag{6.41}$$

Derivation of the ODE from (6.38)*.* Let $y(x) = T_n(x)$ and set $x = \cos\theta$. Then $y(\cos\theta) = \cos(n\theta)$ by (6.38). Apply the chain rule, treating θ as a function of x via $x = \cos\theta$, so $dx/d\theta = -\sin\theta$ and $d\theta/dx = -1/\sin\theta$:

$$y'(x) = \frac{dy/d\theta}{dx/d\theta} = \frac{-n\sin(n\theta)}{-\sin\theta} = \frac{n\sin(n\theta)}{\sin\theta}. \tag{6.42}$$

Differentiate again. Write $y' = n\sin(n\theta)/\sin\theta$ and differentiate in x:

$$\begin{aligned} y'' &= \frac{d}{dx}\left[\frac{n\sin(n\theta)}{\sin\theta}\right] = \frac{d\theta}{dx}\cdot\frac{d}{d\theta}\left[\frac{n\sin(n\theta)}{\sin\theta}\right] \\ &= -\frac{1}{\sin\theta}\cdot\frac{n^2\cos(n\theta)\sin\theta - n\sin(n\theta)\cos\theta}{\sin^2\theta} \\ &= -\frac{n^2\cos(n\theta)}{\sin^2\theta} + \frac{n\sin(n\theta)\cos\theta}{\sin^3\theta}. \end{aligned}$$

Multiply by $(1-x^2) = \sin^2\theta$:

$$(1-x^2)y'' = -n^2\cos(n\theta) + \frac{n\sin(n\theta)\cos\theta}{\sin\theta}.$$

Using $\cos\theta = x$ and (6.42): the second term is $x \cdot y'$. And $\cos(n\theta) = y$. Hence

$$(1-x^2)y'' = -n^2 y + xy',$$

which rearranges to (6.39). □

Proof of T_n orthogonality. Substitute $x = \cos\theta$, so $dx = -\sin\theta\, d\theta$ and $\sqrt{1-x^2} = \sin\theta$ for $\theta \in (0,\pi)$. The limits map $x : -1 \to 1$ to $\theta : \pi \to 0$, so the extra sign from dx cancels against the flipped limits:

$$\int_{-1}^{1} \frac{T_m T_n}{\sqrt{1-x^2}}\, dx = \int_0^{\pi} \cos(m\theta)\cos(n\theta)\, d\theta.$$

Use $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$:

$$= \tfrac{1}{2}\int_0^{\pi} \cos((m-n)\theta)\, d\theta + \tfrac{1}{2}\int_0^{\pi} \cos((m+n)\theta)\, d\theta.$$

For integers $k \neq 0$, $\int_0^{\pi} \cos(k\theta)\, d\theta = \sin(k\pi)/k = 0$; for $k = 0$, it equals π. Apply case-by-case: if $m \neq n$ (so $m - n \neq 0$ and $m + n \neq 0$ unless $m = n = 0$, already excluded), both integrals vanish. If $m = n > 0$, the first integral is π and the second vanishes, giving $\pi/2$. If $m = n = 0$, both integrals are π, giving π. This is (6.40). □

Proof of U_n orthogonality. Again set $x = \cos\theta$, so $dx = -\sin\theta\, d\theta$ and $\sqrt{1-x^2} = \sin\theta$. Using (6.38),

$$\begin{aligned}\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\, dx &= \int_0^{\pi} \frac{\sin((m+1)\theta)}{\sin\theta}\,\frac{\sin((n+1)\theta)}{\sin\theta}\,\sin^2\theta\, d\theta \\ &= \int_0^{\pi} \sin((m+1)\theta)\sin((n+1)\theta)\, d\theta.\end{aligned}$$

Use $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$:

$$= \tfrac{1}{2}\int_0^{\pi} \cos((m-n)\theta)\, d\theta - \tfrac{1}{2}\int_0^{\pi} \cos((m+n+2)\theta)\, d\theta.$$

If $m \neq n$, then both cosine frequencies are nonzero integers, so both integrals vanish. If $m = n$, the first integral is π and the second still vanishes because $2n + 2 \neq 0$. Hence

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\, dx = \frac{\pi}{2}\delta_{mn},$$

which is (6.41). □

Example 6.16 (Chebyshev spectral methods and the minimax property)**.** Chebyshev nodes. *The zeros of T_N are*

$$x_k = \cos\left(\frac{(2k-1)\pi}{2N}\right), \qquad k = 1, \ldots, N, \tag{6.43}$$

since $T_N(\cos\theta) = \cos(N\theta) = 0$ if and only if $N\theta = (2k-1)\pi/2$.

Minimax property. *For $N \geq 1$, among all monic polynomials p of degree N on* $[-1,1]$,

$$\min_{p \text{ monic}, \deg p = N}\ \max_{x\in[-1,1]} |p(x)| = 2^{1-N}, \tag{6.44}$$

with the minimum achieved by $2^{1-N}T_N(x)$*. A polynomial is* monic *when its leading coefficient is* 1.

Proof. *The leading coefficient of* T_N *is* 2^{N-1}*. This follows from the recurrence* $T_{n+1} = 2xT_n - T_{n-1}$*, obtained by multiplying* (6.36) *through by* $1-2xt+t^2$*; the leading coefficient doubles at each step after* $T_1 = x$*. Thus* $2^{1-N}T_N$ *is monic. Its maximum on* $[-1,1]$ *is* 2^{1-N}*, because* $|T_N(\cos\theta)| = |\cos(N\theta)| \le 1$*. Equality holds at* $\theta = k\pi/N$*, i.e. at* $x_k^* = \cos(k\pi/N)$*,* $k = 0,1,\ldots,N$*, where* $T_N(x_k^*) = (-1)^k$*.*

Suppose a monic p *of degree* N *satisfied* $\max|p(x)| < 2^{1-N}$*. Consider* $q(x) = 2^{1-N}T_N(x) - p(x)$*. Since both* $2^{1-N}T_N$ *and* p *are monic of degree* N*,* q *has degree* $\le N-1$*. At the* $N+1$ *equioscillation points* x_k^**,*

$$q(x_k^*) = 2^{1-N}(-1)^k - p(x_k^*).$$

At even k*:* $q(x_k^*) = 2^{1-N} - p(x_k^*) > 2^{1-N} - 2^{1-N} = 0$ *(using* $|p| < 2^{1-N}$*). At odd* k*:* $q(x_k^*) < 0$*. So* q *changes sign at least* N *times on* $[-1,1]$*, hence has* $\ge N$ *zeros. But* $\deg q \le N-1$ *— contradiction. Thus* $\max|p| \ge 2^{1-N}$*.*

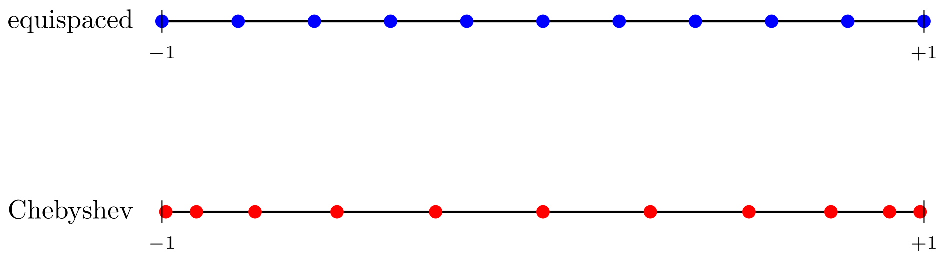

Figure 13: Comparison of $N = 11$ equispaced nodes (top) and Chebyshev nodes $x_k = \cos((2k-1)\pi/22)$ (bottom) on $[-1,1]$. Chebyshev nodes cluster near the endpoints, which suppresses the monic product $\prod_k(x-x_k)$ uniformly: this is the node-dependent part of the interpolation bound (6.45).

Interpolation error. *For* $f \in C^N[-1,1]$*, the polynomial interpolant* p *of degree* $\le N-1$ *through* N *nodes* $\{x_k\}$ *has error*

$$f(x) - p(x) = \frac{f^{(N)}(\xi)}{N!}\prod_{k=1}^{N}(x-x_k), \tag{6.45}$$

for some $\xi \in (-1,1)$ *(the interpolation remainder from one-variable calculus). The product* $\prod_k(x-x_k)$ *is monic of degree* N*; choosing* x_k *to be the Chebyshev nodes* (6.43) *makes this product* $2^{1-N}T_N(x)$*, minimizing the worst-case error bound.*

Spectral expansion. *For* $f \in L^2([-1,1],(1-x^2)^{-1/2}dx)$*, meaning* $|f|^2$ *is integrable with weight* $(1-x^2)^{-1/2}$*, define*

$$S_N f(x) = \sum_{n=0}^{N} a_n T_n(x), \qquad a_n = \frac{2-\delta_{n0}}{\pi}\int_{-1}^{1}\frac{f(x)T_n(x)}{\sqrt{1-x^2}}\,dx.$$

Then $S_N f \to f$ *in the weighted* L^2 *norm, with coefficients determined by Prop. 6.15. Pointwise or uniform convergence requires additional regularity on* f*.*

The Bernstein ellipse. *For* $r > 1$*, define the* Bernstein ellipse $E_r \subset \mathbb{C}$ *as the ellipse with foci at* ± 1 *and semi-axes* $a = (r+r^{-1})/2$ *and* $b = (r-r^{-1})/2$*. Thus* $a+b = r$ *and* $a^2 - b^2 = 1$*. As* $r \to 1^+$ *the ellipse collapses to* $[-1,1]$*; as* $r \to \infty$ *it expands without bound.*

An equivalent description uses the Joukowski map $z = (w + w^{-1})/2$. *Parametrizing the circle* $|w| = r$ *by* $w = re^{i\phi}$ *(so* $w^{-1} = r^{-1}e^{-i\phi}$*),*

$$\begin{aligned} z &= \tfrac{1}{2}\big(re^{i\phi} + r^{-1}e^{-i\phi}\big) \\ &= \tfrac{1}{2}\big[(r + r^{-1})\cos\phi + i(r - r^{-1})\sin\phi\big] \\ &= a\cos\phi + ib\sin\phi, \end{aligned}$$

using $e^{\pm i\phi} = \cos\phi \pm i\sin\phi$ *and grouping real and imaginary parts. This is the standard parametric form of the ellipse with semi-axes* a, b*, so it traces exactly the boundary of* E_r*; thus* E_r *is the image of* $\{|w| \le r\} \setminus \{|w| < r^{-1}\}$ *under Joukowski (the two circles* $|w| = r$ *and* $|w| = r^{-1}$ *both map to* ∂E_r*, reflecting the* $w \longleftrightarrow w^{-1}$ *symmetry of the map). Inverting,* $w = z + \sqrt{z^2 - 1}$ *on the branch normalized so that* $|w| \ge 1$ *for* $z \in \mathbb{C} \setminus [-1, 1]$*. In particular,*

$$z \in E_r \quad \textit{if and only if} \quad |z + \sqrt{z^2 - 1}| \le r,$$

which is how the ellipse was written on first sight.

Core estimate. *The Joukowski coordinate* w *is the right variable because* T_n *behaves like* w^n*. The identity* $T_n(\cos\theta) = \cos n\theta$ *becomes* $T_n(z) = (w^n + w^{-n})/2$ *when* $z = (w + w^{-1})/2$*. On* $|w| = r > 1$*, this formula gives the simple bound* $|T_n(z)| \le (r^n + r^{-n})/2 \le r^n$*.*

For the coefficient estimate, write $F(w) = f((w + w^{-1})/2)$*. If* $f(x) = \sum_{n \ge 0} a_n T_n(x)$*, then for* $n \ge 1$ *the coefficient of* w^n *in the Laurent series of* F *is* $a_n/2$*. If* f *is analytic near* E_r *and* $|f| \le M$ *on* ∂E_r*, then* F *is analytic in the annulus* $r^{-1} < |w| < r$ *and bounded by* M *on* $|w| = r$*. Cauchy's coefficient estimate on that circle gives* $|a_n|/2 \le Mr^{-n}$*, hence* $|a_n| \le 2Mr^{-n}$*. The ellipse parameter* r *sets the geometric decay rate, just as the disk radius sets Taylor coefficient decay.*

Consequently, if f *is analytic in some Bernstein ellipse* E_r *with* $r > 1$*, then* $|a_n| \le Cr^{-n}$*: the expansion converges geometrically. This is the basis for exponential convergence in Chebyshev spectral methods for analytic boundary-value problems.*

6.4 The hypergeometric function ${}_2F_1$

Motivation. The hypergeometric function is the common source of many classical special functions. Negative-integer parameters make the series terminate, which is why orthogonal polynomials appear inside this framework.

Definition 6.17 (Gauss hypergeometric)**.** *For complex parameters* a, b, c *with* $c \notin \{0, -1, -2, \dots\}$*, and for* $|z| < 1$*,*

$${}_2F_1(a, b; c; z) \;=\; \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!}\, z^n, \tag{6.46}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ *is the Pochhammer symbol (Def. 3.22). The restriction on* c *prevents the denominator* $(c)_n$ *from becoming zero.*

Unless a or b is a non-positive integer, the ratio of successive terms tends to z, so the radius of convergence is 1. In the exceptional terminating cases, the series is a polynomial. The Euler integral below continues the non-terminating function to $\mathbb{C} \setminus [1, \infty)$.
A *regular singular point* is a controlled singular point of an ODE: after factoring out possible powers such as z^λ, solutions still have convergent series expansions.

Theorem 6.18 (Hypergeometric ODE)**.** $y = {}_2F_1(a, b; c; z)$ *satisfies*

$$z(1-z)y'' + [c - (a+b+1)z]y' - ab\,y = 0. \tag{6.47}$$

This is the Gauss hypergeometric equation, a second-order linear ODE with regular singular points at 0*,* 1*, and* ∞*.*

Proof. Write $y = \sum_{n=0}^{\infty} a_n z^n$ with

$$a_n = \frac{(a)_n (b)_n}{(c)_n \, n!}. \tag{6.48}$$

Term by term, $y' = \sum_{n\ge1} n a_n z^{n-1}$ and $y'' = \sum_{n\ge2} n(n-1) a_n z^{n-2}$. Substitute into $Ly := z(1-z)y'' + [c-(a+b+1)z]y' - ab\,y$:

$$\begin{aligned}
zy'' &= \sum_{n\ge2} n(n-1)a_n z^{n-1},\\
-z^2 y'' &= -\sum_{n\ge2} n(n-1)a_n z^n,\\
cy' &= c\sum_{n\ge1} n a_n z^{n-1},\\
-(a+b+1)zy' &= -(a+b+1)\sum_{n\ge1} n a_n z^n,\\
-ab\,y &= -ab\sum_{n\ge0} a_n z^n.
\end{aligned}$$

In the first and third lines, shift index $m = n-1$ (so $n = m+1$):

$$\begin{aligned}
zy'' &= \sum_{m\ge1} (m+1)m\, a_{m+1} z^m,\\
cy' &= c\sum_{m\ge0} (m+1) a_{m+1} z^m.
\end{aligned}$$

The coefficient of z^m is

$$[z^m](Ly) = (m+1)m\,a_{m+1} + c(m+1)a_{m+1} - m(m-1)a_m - (a+b+1)m\,a_m - ab\,a_m,$$

Group terms:

$$[z^m](Ly) = (m+1)(m+c)\,a_{m+1} - [m(m-1) + m(a+b+1) + ab]\,a_m.$$

Simplify the bracket:

$$m(m-1) + m(a+b+1) + ab = m^2 + m(a+b) + ab = (m+a)(m+b).$$

So

$$(m+1)(m+c)\,a_{m+1} - (m+a)(m+b)\,a_m = 0. \tag{6.49}$$

From (6.48), the ratio is

$$\frac{a_{m+1}}{a_m} = \frac{(m+a)(m+b)}{(m+1)(m+c)},$$

so (6.49) vanishes for all $m \ge 0$. Therefore $Ly \equiv 0$. □

Theorem 6.19 (Euler integral). *For* $\operatorname{Re} c > \operatorname{Re} b > 0$ *and* $z \in \mathbb{C} \setminus [1, \infty)$,

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt. \tag{6.50}$$

Here t^{b-1} *and* $(1-t)^{c-b-1}$ *use the ordinary real logarithms of* $t > 0$ *and* $1-t > 0$. *The factor* $(1-zt)^{-a}$ *uses the principal logarithm; the restriction* $z \notin [1, \infty)$ *keeps the path* $1-zt$, $0 \le t \le 1$, *away from the principal branch cut* $(-\infty, 0]$.

Proof. Write the right-hand side of (6.50) as

$$I(z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt.$$

Step 1: identify the series for $|z| < 1$. Expand

$$(1-zt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}(zt)^n,$$

using $\binom{-a}{n}(-1)^n = (a)_n/n!$. For $|z| < 1$, dominated convergence justifies exchanging the sum and integral. To see the domination explicitly, choose r with $|z| < r < 1$. Since $0 \le t \le 1$,

$$|zt| \le |z| < r,$$

and the absolute binomial majorant

$$C_{a,r} := \sum_{n=0}^{\infty} \left|\frac{(a)_n}{n!}\right| r^n$$

is finite because the binomial series has radius 1. Therefore every partial sum is bounded in absolute value by $C_{a,r}$. The remaining absolute weight is

$$t^{\operatorname{Re} b-1}(1-t)^{\operatorname{Re}(c-b)-1},$$

which is integrable on $[0,1]$ because $\operatorname{Re} b > 0$ and $\operatorname{Re}(c-b) > 0$. Thus dominated convergence gives

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\, dt.$$

The t-integral is a Beta integral:

$$\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\, dt = B(b+n, c-b) = \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(b+n+c-b)} = \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}.$$

Multiply by $\Gamma(c)/(\Gamma(b)\Gamma(c-b))$:

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} = \frac{\Gamma(c)}{\Gamma(b)} \cdot \frac{\Gamma(b+n)}{\Gamma(c+n)} = \frac{\Gamma(b+n)/\Gamma(b)}{\Gamma(c+n)/\Gamma(c)}.$$

By $\Gamma(s+n) = \Gamma(s)(s)_n$,

$$\frac{\Gamma(b+n)}{\Gamma(b)} = (b)_n, \qquad \frac{\Gamma(c+n)}{\Gamma(c)} = (c)_n.$$

Thus the coefficient of $z^n/n!$ is

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} \cdot (a)_n = \frac{(a)_n (b)_n}{(c)_n},$$

matching (6.46). Hence

$$I(z) = {}_2F_1(a,b;c;z), \qquad |z| < 1.$$

Step 2: continue to the slit plane. For $z \in \mathbb{C} \setminus [1, \infty)$ and $0 \le t \le 1$, the point $1 - zt$ avoids the branch cut $(-\infty, 0]$. Use the principal branch and define

$$(1-zt)^{-a} := \exp\bigl(-a \operatorname{Log}(1-zt)\bigr).$$

On compact subsets of the slit plane the integrand and its z-derivative are bounded by integrable multiples of the Beta weight. Hence $I(z)$ is holomorphic there. Since it agrees with ${}_2F_1(a,b;c;z)$ for $|z| < 1$, the identity theorem gives the continuation. □

Proposition 6.20 (Gauss summation). *For* $\operatorname{Re}(c-a-b)>0$ *and* $c\notin\{0,-1,-2,\dots\}$*,*

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{6.51}$$

Proof. Step 1: evaluate the Euler integral at $z=1$*.* Assume first that

$$\operatorname{Re}c>\operatorname{Re}b>0,\qquad \operatorname{Re}(c-a-b)>0.$$

For real $0\le z<1$, the Euler integral gives

$$ {}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt.$$

As $z\uparrow 1$, the integrand converges pointwise to

$$t^{b-1}(1-t)^{c-a-b-1}.$$

The displayed hypotheses give an integrable endpoint bound, so dominated convergence yields

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,dt.$$

Now evaluate the Beta integral:

$$\int_0^1 t^{b-1}(1-t)^{c-b-a-1}\,dt=B(b,c-a-b)=\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)},$$

by Proposition 3.9. Substituting gives

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\cdot\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Step 2: extend in the parameters. Both sides are meromorphic in (a,b,c). They agree on the nonempty open set

$$\{\operatorname{Re}c>\operatorname{Re}b>0,\ \operatorname{Re}(c-a-b)>0\},$$

so analytic continuation extends the identity to $\operatorname{Re}(c-a-b)>0$, with $c\notin\{0,-1,-2,\dots\}$. In practical terms, two analytic formulas that agree on a region with room to move must continue to agree wherever both formulas are defined. □

6.5 The confluent hypergeometric ${}_1F_1$

Definition 6.21 (Confluent hypergeometric). *For* $c\notin\{0,-1,-2,\dots\}$*,*

$$ {}_1F_1(a;c;z)=M(a,c,z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}. \tag{6.52}$$

This series is entire in z *and is obtained from* ${}_2F_1$ *by a* confluent limit*. In* ${}_2F_1(a,b;c;z/b)$*, the* n*-th coefficient is*

$$\frac{(a)_n(b)_n}{(c)_n\,n!}\cdot b^{-n}=\frac{(a)_n}{(c)_n\,n!}\cdot\frac{(b)_n}{b^n}.$$

Since $(b)_n/b^n\to 1$ *as* $b\to\infty$*, the coefficients tend to those of* ${}_1F_1$*. In this limit, singular points merge; this is the reason for the name* confluent*.*

Proposition 6.22 (Kummer equation)**.** *The function* $y(z) = M(a,c,z)$ *satisfies*

$$zy'' + (c-z)y' - ay = 0. \tag{6.53}$$

Proof. Write $y = \sum_{n=0}^{\infty} a_n z^n$ with

$$a_n = \frac{(a)_n}{(c)_n\, n!}.$$

Then

$$y' = \sum_{n\ge1} n a_n z^{n-1}, \qquad y'' = \sum_{n\ge2} n(n-1)a_n z^{n-2}.$$

Substitute into $Ly := zy'' + (c-z)y' - ay$:

$$\begin{aligned}
zy'' &= \sum_{n\ge2} n(n-1)a_n z^{n-1} = \sum_{m\ge1} (m+1)m\, a_{m+1} z^m,\\
cy' &= c\sum_{n\ge1} n a_n z^{n-1} = c\sum_{m\ge0} (m+1)a_{m+1} z^m,\\
-zy' &= -\sum_{m\ge1} m a_m z^m,\\
-ay &= -\sum_{m\ge0} a a_m z^m.
\end{aligned}$$

Therefore the coefficient of z^m is

$$[z^m](Ly) = (m+1)(m+c)a_{m+1} - (m+a)a_m.$$

From the coefficient formula,

$$\frac{a_{m+1}}{a_m} = \frac{(a)_{m+1}}{(a)_m}\frac{(c)_m}{(c)_{m+1}}\frac{1}{m+1} = \frac{m+a}{(m+c)(m+1)},$$

so $(m+1)(m+c)a_{m+1} = (m+a)a_m$. Hence every coefficient of Ly is zero, and (6.53) follows. □

Remark 6.23 (Singularity structure)**.** *Kummer's equation has a regular singular point at* 0 *and an irregular singular point at* ∞*. Roughly, a regular singular point allows power-law-type local behavior, while an irregular singular point can produce exponential asymptotics.*

6.6 Hypergeometric catalog

Remark 6.24 (Special-function catalog)**.** *We use the generalized hypergeometric notation*

$$_0F_1(-;c;w) = \sum_{k=0}^{\infty} \frac{w^k}{(c)_k k!}, \qquad c \ne 0,-1,-2,\dots.$$

The dash means that there are no numerator parameters.

Function	*Hypergeometric form*
$P_n(x)$	$_2F_1(-n,n+1;1;(1-x)/2)$
$T_n(x)$	$_2F_1(-n,n;1/2;(1-x)/2)$
$U_n(x)$	$(n+1)\,_2F_1(-n,n+2;3/2;(1-x)/2)$
$L_n^{(\alpha)}(x)$	$\binom{n+\alpha}{n}\,_1F_1(-n;\alpha+1;x)$
$H_{2n}(x)$	$(-1)^n\dfrac{(2n)!}{n!}\,_1F_1(-n;1/2;x^2)$
$H_{2n+1}(x)$	$(-1)^n\dfrac{(2n+1)!}{n!}\,2x\,_1F_1(-n;3/2;x^2)$
$J_\nu(z)$	$\dfrac{(z/2)^\nu}{\Gamma(\nu+1)}\,_0F_1(-;\nu+1;-z^2/4)$

We verify the entries by expanding both sides and matching coefficients.

6.6.1 Verification (i): Legendre P_n

Claim. $P_n(x) = {}_2F_1(-n, n+1; 1; (1-x)/2)$.

Let $u = (1-x)/2$. The right side, using $(1)_k = k!$,

$$ {}_2F_1(-n, n+1; 1; u) = \sum_{k=0}^{\infty} \frac{(-n)_k (n+1)_k}{k!\,k!} u^k. $$

Since $(-n)_k = 0$ for $k > n$, the sum terminates at $k = n$. Use

$$ (-n)_k = (-1)^k \frac{n!}{(n-k)!}, \qquad (n+1)_k = \frac{(n+k)!}{n!}, $$

(the first by $(-n)(-n+1)\cdots(-n+k-1) = (-1)^k n(n-1)\cdots(n-k+1)$; the second by $(n+1)(n+2)\cdots(n+k)$). Therefore

$$ {}_2F_1(-n, n+1; 1; u) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} u^k. \tag{6.54} $$

General argument. From Rodrigues (5.38), $P_n(x) = (2^n n!)^{-1}(d/dx)^n (x^2-1)^n$. Write $(x^2-1)^n = [(x-1)(x+1)]^n$. With $x = 1-2u$, so $x-1 = -2u$ and $x+1 = 2-2u = 2(1-u)$:

$$ (x^2-1)^n = (-2u)^n \cdot [2(1-u)]^n = (-1)^n 2^{2n} u^n (1-u)^n. $$

Since $d/dx = -\frac{1}{2} d/du$,

$$ \frac{d^n}{dx^n}(x^2-1)^n = \left(-\tfrac{1}{2}\right)^n \frac{d^n}{du^n}\left[(-1)^n 2^{2n} u^n (1-u)^n\right] = 2^n \frac{d^n}{du^n}[u^n(1-u)^n]. $$

Leibniz's rule:

$$ \frac{d^n}{du^n}[u^n(1-u)^n] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k u^n}{du^k} \cdot \frac{d^{n-k}(1-u)^n}{du^{n-k}} = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} u^{n-k} \cdot (-1)^{n-k} \frac{n!}{k!} (1-u)^k. $$

Divide by $2^n n!$:

$$ P_n = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)! \cdot k!} (-1)^{n-k} u^{n-k} (1-u)^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}^2 u^{n-k} (1-u)^k, $$

using $n!/[(n-k)!k!] = \binom{n}{k}$. Relabel $j = n-k$:

$$ P_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j}^2 u^j (1-u)^{n-j}. $$

Expand $(1-u)^{n-j} = \sum_\ell \binom{n-j}{\ell}(-u)^\ell$ and collect $[u^k]$:

$$ [u^k]P_n = \sum_{j=0}^{k} (-1)^j \binom{n}{j}^2 (-1)^{k-j} \binom{n-j}{k-j} = (-1)^k \sum_{j=0}^{k} \binom{n}{j}^2 \binom{n-j}{k-j}. $$

The only combinatorial identity needed is a Vandermonde convolution. First,

$$ \binom{n}{j}\binom{n-j}{k-j} = \binom{n}{k}\binom{k}{j}; $$

both sides choose k objects from n and then mark j of those chosen objects. Therefore

$$\sum_{j=0}^{k}\binom{n}{j}^2\binom{n-j}{k-j}=\binom{n}{k}\sum_{j=0}^{k}\binom{n}{j}\binom{k}{j}.$$

Using $\binom{n}{j}=\binom{n}{n-j}$ and Vandermonde's identity gives

$$\sum_{j=0}^{k}\binom{n}{j}\binom{k}{j}=\sum_{j=0}^{k}\binom{n}{n-j}\binom{k}{j}=\binom{n+k}{n}=\binom{n+k}{k}.$$

Thus

$$\sum_{j=0}^{k}\binom{n}{j}^2\binom{n-j}{k-j}=\binom{n}{k}\binom{n+k}{k}.$$

So $[u^k]P_n=(-1)^k\binom{n}{k}\binom{n+k}{k}$, matching (6.54) term-by-term.

6.6.2 Verification (ii): Chebyshev T_n

Claim. $T_n(x)={}_2F_1(-n,n;1/2;(1-x)/2)$.

Let $u=(1-x)/2=\sin^2(\theta/2)$ with $x=\cos\theta$. The right side, with $(-n)_k=(-1)^k n!/(n-k)!$, $(n)_k=(n+k-1)!/(n-1)!$ (for $n\geq 1$, else terminate at $k=0$), and $(1/2)_k=(2k)!/(4^k k!)$:

$${}_2F_1(-n,n;1/2;u)=\sum_{k=0}^{n}\frac{(-n)_k(n)_k}{(1/2)_k k!}u^k. \tag{6.55}$$

General argument. Let $F(u)=T_n(1-2u)$, so $x=1-2u$. Then

$$F'(u)=-2T_n'(x),\qquad F''(u)=4T_n''(x),$$

and

$$1-x^2=1-(1-2u)^2=4u(1-u).$$

Substitute these relations into Chebyshev's ODE (6.39):

$$0=(1-x^2)T_n''(x)-xT_n'(x)+n^2T_n(x)=u(1-u)F''(u)+\left(\tfrac{1}{2}-u\right)F'(u)+n^2F(u).$$

This is exactly the hypergeometric equation (6.47) with

$$a=-n,\qquad b=n,\qquad c=\tfrac{1}{2},$$

because $a+b+1=1$ and $-ab=n^2$. Also $F(0)=T_n(1)=1$ by (6.38). The Taylor-series recurrence for the hypergeometric ODE determines every later coefficient from this initial value, so the unique holomorphic solution at $u=0$ with value 1 is ${}_2F_1(-n,n;1/2;u)$, proving the claim.

6.6.3 Verification (iii): Chebyshev U_n

Claim. $U_n(x)=(n+1)\,{}_2F_1(-n,n+2;3/2;(1-x)/2)$.

Use $(3/2)_k=(2k+1)!/(4^k k!)$ and let $u=(1-x)/2$. The RHS:

$$(n+1)\sum_{k=0}^{n}\frac{(-n)_k(n+2)_k}{(3/2)_k k!}u^k.$$

General argument. Differentiate $T_{n+1}(\cos\theta) = \cos((n+1)\theta)$ from (6.38) with respect to x:

$$T'_{n+1}(x) = \frac{d(\cos((n+1)\theta))/d\theta}{dx/d\theta} = \frac{-(n+1)\sin((n+1)\theta)}{-\sin\theta} = (n+1)U_n(x).$$

Now differentiate the hypergeometric form of T_{n+1} proved above. For

$$F(z) = {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} z^k,$$

termwise differentiation gives

$$\begin{aligned}
F'(z) &= \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!}\, kz^{k-1} \\
&= \sum_{m=0}^{\infty} \frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}\, m!} z^m \\
&= \frac{ab}{c} \sum_{m=0}^{\infty} \frac{(a+1)_m(b+1)_m}{(c+1)_m\, m!} z^m \\
&= \frac{ab}{c}\, {}_2F_1(a+1, b+1; c+1; z).
\end{aligned}$$

Apply this with $a = -n-1$, $b = n+1$, $c = 1/2$, and $u = (1-x)/2$:

$$T'_{n+1}(x) = -\frac{1}{2} \cdot \frac{(-n-1)(n+1)}{1/2}\, {}_2F_1(-n, n+2; 3/2; u) = (n+1)^2\, {}_2F_1(-n, n+2; 3/2; u).$$

Divide by $n+1$ and use $T'_{n+1}(x) = (n+1)U_n(x)$ to obtain

$$U_n(x) = (n+1)\, {}_2F_1(-n, n+2; 3/2; u),$$

which is the claim.

6.6.4 Verification (iv): Laguerre $L_n^{(\alpha)}$

Claim. For $\alpha \notin \{-1, -2, \dots\}$, $L_n^{(\alpha)}(x) = \binom{n+\alpha}{n}\, {}_1F_1(-n; \alpha+1; x)$. Values at the excluded parameters are obtained by analytic continuation, or directly from the finite series (6.23).

Expand the RHS:

$$\binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}.$$

Use $(-n)_k = (-1)^k n!/(n-k)!$ and $(\alpha+1)_k = \Gamma(\alpha+1+k)/\Gamma(\alpha+1)$:

$$\binom{n+\alpha}{n} \cdot (-1)^k \frac{n!}{(n-k)!} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+k)} \cdot \frac{x^k}{k!}.$$

Substitute $\binom{n+\alpha}{n} = \Gamma(n+\alpha+1)/[n!\Gamma(\alpha+1)]$; the $n!$'s cancel and the $\Gamma(\alpha+1)$'s cancel:

$$= \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1+k)(n-k)!} \cdot \frac{(-x)^k}{k!} = \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!},$$

since $\binom{n+\alpha}{n-k} = \Gamma(n+\alpha+1)/[(n-k)!\Gamma(k+\alpha+1)]$ (for complex α, this is the Gamma-function version of the binomial coefficient). Summing over $k = 0, \dots, n$ reproduces (6.23).

6.6.5 Verification (v): Hermite H_{2n}

Claim. $H_{2n}(x) = (-1)^n (2n)!/n! \cdot {}_1F_1(-n;1/2;x^2)$.

From (6.5), $H_{2n}(x) = (2n)! \sum_{q=0}^{n} (-1)^q (2x)^{2n-2q}/[(2n-2q)!q!]$. Relabel $q = n-k$ (so $k = n-q$ runs $0,\ldots,n$):

$$H_{2n}(x) = (2n)! \sum_{k=0}^{n} \frac{(-1)^{n-k}(2x)^{2k}}{(2k)!(n-k)!} = (-1)^n \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \cdot \frac{4^k x^{2k}}{(2k)!}.$$

Now the RHS of the claim:

$$(-1)^n \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(1/2)_k} \frac{(x^2)^k}{k!}.$$

Use $(-n)_k = (-1)^k n!/(n-k)!$ and $(1/2)_k = (2k)!/(4^k k!)$. Hence

$$\frac{(-n)_k}{(1/2)_k\, k!} = \frac{(-1)^k n!/(n-k)!}{(2k)!/(4^k k!) \cdot k!} = \frac{(-1)^k n!\, 4^k}{(n-k)!(2k)!}.$$

So the RHS becomes

$$(-1)^n \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-1)^k n!\, 4^k x^{2k}}{(n-k)!(2k)!} = (-1)^n \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \cdot \frac{4^k x^{2k}}{(2k)!},$$

matching the expansion for H_{2n} term-by-term.

6.6.6 Verification (vi): Hermite H_{2n+1}

Claim. $H_{2n+1}(x) = (-1)^n (2n+1)!/n! \cdot 2x\, {}_1F_1(-n;3/2;x^2)$.

Similarly, $H_{2n+1} = (2n+1)! \sum_{q=0}^{n} (-1)^q (2x)^{2n+1-2q}/[(2n+1-2q)!q!]$. Relabel $q = n-k$:

$$H_{2n+1}(x) = (2n+1)! \sum_{k=0}^{n} \frac{(-1)^{n-k}(2x)^{2k+1}}{(2k+1)!(n-k)!} = (-1)^n \frac{(2n+1)!}{n!} \cdot 2x \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} \frac{4^k x^{2k}}{(2k+1)!}.$$

RHS: $(3/2)_k = (2k+1)!/(4^k k!)$. So

$$\frac{(-n)_k}{(3/2)_k k!} = \frac{(-1)^k n! 4^k}{(n-k)!(2k+1)!}.$$

Therefore the RHS equals

$$(-1)^n \frac{(2n+1)!}{n!} \cdot 2x \sum_k \frac{(-1)^k n!\, 4^k x^{2k}}{(n-k)!(2k+1)!},$$

matching H_{2n+1}.

6.6.7 Verification (vii): Bessel J_ν

Claim. $J_\nu(z) = (z/2)^\nu/\Gamma(\nu+1) \cdot {}_0F_1(-;\nu+1;-z^2/4)$, where ${}_0F_1(-;c;w) = \sum_{k\geq 0} w^k/[(c)_k k!]$.

Expand the RHS:

$$\frac{(z/2)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k\, k!} \left(-\frac{z^2}{4}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^\nu (z/2)^{2k}}{\Gamma(\nu+1)(\nu+1)_k\, k!}.$$

Use $\Gamma(\nu+1)(\nu+1)_k = \Gamma(\nu+1)\cdot\Gamma(\nu+k+1)/\Gamma(\nu+1) = \Gamma(\nu+k+1)$:

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{\nu+2k},$$

which is precisely the Bessel series (Def. 4.4, Eq. (4.11)).

Takeaway. The classical polynomial families are terminating hypergeometric series. Bessel is a confluent limit.

6.7 Asymptotic and physical applications

Two themes matter here: truncation gives bound states in physics, and off-interval growth controls approximation error in numerics.

Remark 6.25 (Polynomial truncation and discrete spectra)**.** *Remark 6.24 rewrites the Laguerre and Hermite families as confluent hypergeometric functions:*

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n}\,{}_1F_1(-n;\alpha+1;x),$$

and

$$H_{2n}(x),\ H_{2n+1}(x) \propto {}_1F_1(-n;\cdots;x^2).$$

The first parameter is the negative integer $-n$, so the series terminates. In the oscillator and hydrogen examples, the physical bound-state branch is selected by this termination condition. Quantization appears as hypergeometric truncation.

Example 6.26 (Chebyshev growth off the interval)**.** *For $x>1$, write $x=\cosh\mu$ with $\mu>0$. Since $\cos(i\mu)=\cosh\mu$, the trigonometric formulas* (6.38) *analytically continue to*

$$T_n(\cosh\mu) = \cosh(n\mu), \qquad U_n(\cosh\mu) = \frac{\sinh((n+1)\mu)}{\sinh\mu}.$$

Hence

$$T_n(x) \sim \frac{e^{n\mu}}{2}, \qquad U_n(x) \sim \frac{e^{(n+1)\mu}}{2\sinh\mu},$$

for fixed $x>1$. Inside $[-1,1]$, by contrast, $|T_n(x)|\le 1$ and $|U_n(x)|\le n+1$. This inside/outside contrast is the real-variable version of the Bernstein-ellipse estimate in Example 6.16.

Remark 6.27 (Bound states versus scattering states)**.** *The same equations also describe scattering states. For positive-energy Coulomb scattering, the radial equation still reduces to Kummer's equation, but the acceptable solution is non-terminating. Bound states occur at the exceptional parameter values where it truncates to Laguerre.*

Exercises

Problem 6.1. *Derive the generating function* (6.1) *from Rodrigues formula* (6.12) *by summing the Taylor series of e^{2xt-t^2} in t.*

Problem 6.2. *Show that the normalized eigenfunctions of the quantum harmonic oscillator $\psi_n(x) = (2^n n!\sqrt{\pi})^{-1/2} H_n(x) e^{-x^2/2}$ are orthonormal on $\mathbb{R}$.*

Problem 6.3. *Verify $T_n(\cos\theta) = \cos(n\theta)$ for $n=0,1,2,3$ by expanding the right side and matching to $T_n(x)$ from* (6.36).

Problem 6.4. *Prove the Chebyshev minimax property: for monic polynomials p of degree n,* $\max_{[-1,1]}|p(x)| \geq 2^{1-n}$*, with equality for* $p = 2^{1-n}T_n$.

Problem 6.5. *Derive the hypergeometric ODE* (6.47) *by substituting the series* (6.46) *and verifying the coefficient relation.*

Problem 6.6. *Use the Euler integral* (6.50) *to prove Gauss's summation formula (Prop. 6.20) at* $z = 1$.

Problem 6.7. *Verify the catalog entry for P_n in Remark 6.24: compute* ${}_2F_1(-n, n+1; 1; (1-x)/2)$ *explicitly for* $n = 0, 1, 2$ *and match.*

Problem 6.8. *Show* $J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1(-; \nu+1; -z^2/4)$ *by expanding both sides as series in* z.

7 Calculus of Variations

Ordinary calculus finds extrema of a function $f : \mathbb{R}^n \to \mathbb{R}$ by solving $\nabla f = 0$. The *calculus of variations* does the same job when the input is a whole function. Its basic object is a *functional* $\mathcal{L}[y]$, a real-valued map whose argument is a curve y. The model case is the integral functional

$$\mathcal{L}[y] = \int_a^b L\big(x, y(x), y'(x)\big)\,dx,$$

and the analogue of $\nabla f = 0$ is an equation for the unknown curve, called the Euler–Lagrange equation and derived below. Variational principles organize classical mechanics, geometric optics, relativity, and field theory because many physical laws say: among all admissible paths, the realized path makes an action stationary. We start with the Euler–Lagrange equation, then use it on the classical examples before moving to constraints, second variation, Noether's theorem (the link between continuous symmetries and conserved quantities), and Hamiltonian mechanics (a momentum-based reformulation of the same dynamics).

Why this section sits inside notes on complex analysis and special functions. The chapter looks like a detour from the complex-analytic theme, but it closes the loop with the rest of the notes in three concrete ways. (i) Every classical ODE we met in Sections 4–6 — Bessel, Legendre, Hermite, Laguerre, Chebyshev — is the Euler–Lagrange equation of a quadratic action with an appropriate weight; the orthogonality relations are then the consequence of a single integration-by-parts identity. The generating-function route used in those sections produces the functions; the variational route says *why* those particular ODEs are the right ones to solve. (ii) The brachistochrone transit time of Example 7.8 evaluates in closed form as a Beta function (Section 3), so the cycloid is, in one stroke, a variational extremal and an explicit Beta integral. (iii) Stationary action is the classical limit of an oscillatory integral $\int e^{iS[\gamma]/\hbar}\mathcal{D}\gamma$; the saddle points of the phase are exactly the Euler–Lagrange solutions, and the saddle-point asymptotics of Section 2 (Laplace, stationary phase, steepest descent) is the asymptotic machinery that turns a quantum amplitude into a classical trajectory plus a Gaussian fluctuation. Section 7.11 collects these links explicitly.

Prerequisites. This section is logically self-contained: it assumes multivariable calculus, integration by parts, elementary ODEs, and the linear-algebra notation used in classical mechanics. The connections to earlier chapters are pulled in at the end. The short sphere/geodesic discussion in Example 7.11 only uses the geometry introduced there.

7.1 The functional and its variation

Fix $a < b \in \mathbb{R}$ and boundary values $y_a, y_b \in \mathbb{R}$. Consider the space of admissible curves

$$\mathcal{A} = \{y \in C^2[a,b] : y(a) = y_a,\ y(b) = y_b\}.$$

Here $C^2[a,b]$ denotes the set of functions y on $[a,b]$ whose first and second derivatives y', y'' exist and are continuous on $[a,b]$. Let $L = L(x,y,p)$ be a C^2 (twice continuously differentiable) *Lagrangian* on an open set containing the relevant curve data. Here $p \in \mathbb{R}$ is a placeholder for the derivative y'. The *action functional* is

$$\mathcal{L}[y] = \int_a^b L\big(x,\ y(x),\ y'(x)\big)\,dx. \tag{7.1}$$

We write $L_y = \partial L/\partial y$, $L_{y'} = \partial L/\partial p|_{p=y'}$, $L_{yy} = \partial^2 L/\partial y^2$, and similarly for other partial derivatives.

Definition 7.1 (Variation)**.** *For $y \in \mathcal{A}$ and a perturbation $\eta \in C^2[a,b]$ with $\eta(a) = \eta(b) = 0$ (so that $y + \varepsilon\eta \in \mathcal{A}$ for sufficiently small ε), the* first variation *of $\mathcal{L}$ at y in the direction η is*

$$\delta\mathcal{L}[y;\eta] = \frac{d}{d\varepsilon}\mathcal{L}[y+\varepsilon\eta]\bigg|_{\varepsilon=0}. \tag{7.2}$$

We call y a stationary point *(or critical point, or* extremal*) of $\mathcal{L}$ if $\delta\mathcal{L}[y;\eta] = 0$ for every admissible η.*

The variation $\delta\mathcal{L}[y;\eta]$ is the directional derivative of $\mathcal{L}$ in the space of admissible curves. Stationary does not automatically mean minimum; it only means the first-order change is zero for every allowed perturbation. Physics texts often write the perturbation as $\delta y := \varepsilon\eta$.

7.2 The Euler–Lagrange equation

Lemma 7.2 (Fundamental lemma of the calculus of variations)**.** *Let $g \in C[a,b]$. If $\int_a^b g(x)\eta(x)\,dx = 0$ for every $\eta \in C^2[a,b]$ satisfying $\eta(a) = \eta(b) = 0$, then $g \equiv 0$ on $[a,b]$.*

Proof. Suppose instead that g is not identically zero. If the only point first noticed is an endpoint, continuity gives a nearby interior point where g is still nonzero, so choose $x_0 \in (a,b)$ with $g(x_0) \neq 0$. Replacing g by $-g$ if needed, assume $g(x_0) > 0$. By continuity there is $\delta > 0$ with $[x_0-\delta, x_0+\delta] \subset (a,b)$ and $g(x) \geq g(x_0)/2 > 0$ on this interval. Choose the C^2 bump

$$\eta(x) = \begin{cases} \big((x-(x_0-\delta))((x_0+\delta)-x)\big)^3, & x \in [x_0-\delta, x_0+\delta], \\ 0, & \text{otherwise}. \end{cases}$$

The cubing is only for smoothness at the two joining points: the polynomial and its first two derivatives vanish at $x_0 \pm \delta$, so extending by zero gives a C^2 function. Thus $\eta \in C^2[a,b]$, $\eta(a) = \eta(b) = 0$, and $\eta > 0$ on $(x_0-\delta, x_0+\delta)$. Consequently

$$\int_a^b g\eta\,dx \geq \frac{g(x_0)}{2}\int_{x_0-\delta}^{x_0+\delta}\eta(x)\,dx > 0,$$

contradicting the hypothesis. Thus $g \equiv 0$ on (a,b), and by continuity on $[a,b]$. ☐

Theorem 7.3 (Euler–Lagrange)**.** *Let $L \in C^2$ and $y \in \mathcal{A}$. Then y is stationary for $\mathcal{L}$ if and only if*

$$\frac{\partial L}{\partial y}\big(x, y(x), y'(x)\big) - \frac{d}{dx}\frac{\partial L}{\partial y'}\big(x, y(x), y'(x)\big) = 0, \qquad x \in [a,b]. \tag{7.3}$$

Proof. Differentiate (7.2) under the integral. Since $L \in C^2$ and $[a,b]$ is compact (closed and bounded), the Leibniz rule applies:

$$\begin{aligned} \delta\mathcal{L}[y;\eta] &= \frac{d}{d\varepsilon}\int_a^b L\big(x, y+\varepsilon\eta, y'+\varepsilon\eta'\big)dx\bigg|_{\varepsilon=0} \\ &= \int_a^b \frac{d}{d\varepsilon}L\big(x, y+\varepsilon\eta, y'+\varepsilon\eta'\big)\bigg|_{\varepsilon=0} dx \\ &= \int_a^b \big[L_y(x,y,y')\eta + L_{y'}(x,y,y')\eta'\big]\,dx. \end{aligned} \tag{7.4}$$

The last step is the chain rule, with x fixed.

Integrate the second term in (7.4) by parts. Let $u = L_{y'}$ and $dv = \eta'\,dx$, so $du = (d/dx)L_{y'}\,dx$ and $v = \eta$:

$$\begin{aligned}\int_a^b L_{y'}\,\eta'\,dx &= \big[L_{y'}\,\eta\big]_a^b - \int_a^b \frac{d}{dx}L_{y'}\,\eta\,dx \\ &= L_{y'}\big(b, y(b), y'(b)\big)\eta(b) - L_{y'}\big(a, y(a), y'(a)\big)\eta(a) - \int_a^b \frac{d}{dx}L_{y'}\,\eta\,dx \\ &= 0 - 0 - \int_a^b \frac{d}{dx}L_{y'}\,\eta\,dx,\end{aligned}$$

the last equality because $\eta(a) = \eta(b) = 0$. Substitute back into (7.4):

$$\delta\mathcal{L}[y;\eta] \;=\; \int_a^b \left[L_y - \frac{d}{dx}L_{y'}\right]\eta(x)\,dx. \tag{7.5}$$

(⇒) Suppose y is stationary. Then (7.5) vanishes for every admissible η. The bracket $g(x) := L_y - (d/dx)L_{y'}$ is continuous on $[a,b]$ (since $y \in C^2$ and $L \in C^2$). By Lem. 7.2, $g \equiv 0$, which is (7.3).

(⇐) If (7.3) holds, the bracket in (7.5) vanishes identically, so $\delta\mathcal{L}[y;\eta] = 0$ for every admissible η. □

Remark 7.4 (Alternative form). *Expanding the total x-derivative in* (7.3) *with the chain rule,*

$$\frac{d}{dx}L_{y'} \;=\; L_{y'x} + L_{y'y}\,y' + L_{y'y'}\,y'',$$

so the Euler–Lagrange equation is the (generally nonlinear) second-order ODE

$$L_{y'y'}\,y'' \;+\; L_{y'y}\,y' \;+\; L_{y'x} \;-\; L_y \;=\; 0. \tag{7.6}$$

When $L_{y'y'} \neq 0$, this can be solved for y'' as a function of (x, y, y').

7.3 The Beltrami identity

When L has no explicit x-dependence ($L_x = 0$), the EL equation has a conserved quantity. This is the same mechanism as energy conservation when a mechanical system has no explicit time dependence.

Proposition 7.5 (Beltrami identity). *If $L_x = 0$ and y satisfies the EL equation* (7.3)*, then*

$$L - y'\,\frac{\partial L}{\partial y'} \;=\; C \quad \textit{(constant along the extremal).} \tag{7.7}$$

Proof. We compute $d(L - y'L_{y'})/dx$ and show it vanishes. First,

$$\begin{aligned}\frac{dL}{dx} &= L_x + L_y\,y' + L_{y'}\,y'' \\ &= 0 + L_y\,y' + L_{y'}\,y'',\end{aligned} \tag{7.8}$$

using $L_x = 0$. Second,

$$\begin{aligned}\frac{d}{dx}(y'\,L_{y'}) &= y''\,L_{y'} + y'\,\frac{dL_{y'}}{dx} \\ &= y''\,L_{y'} + y'\,L_y,\end{aligned} \tag{7.9}$$

where the last equality uses $dL_{y'}/dx = L_y$ from the EL equation (7.3). Subtract (7.9) from (7.8):

$$\frac{d}{dx}\big(L - y'L_{y'}\big) = \big(L_y y' + L_{y'} y''\big) - \big(y'' L_{y'} + y' L_y\big) = 0.$$

Hence $L - y'L_{y'}$ is constant along the extremal. □

Remark 7.6 (Cyclic coordinates). *A related first integral exists whenever L does not depend on y (i.e. $L_y = 0$; y is called a* cyclic *or* ignorable *coordinate). Then EL reduces to $(d/dx)L_{y'} = 0$, so $L_{y'}$ is conserved. This is the variational form of momentum conservation associated with a translation symmetry.*

7.4 Worked examples

We now apply the Euler–Lagrange equation and the Beltrami identity to the classical examples.

Example 7.7 (Shortest path in the plane). *Among all C^2 curves $y : [a,b] \to \mathbb{R}$ joining (a, y_a) to (b, y_b), find the shortest one. The arclength functional is*

$$\mathcal{L}[y] = \int_a^b \sqrt{1 + y'(x)^2}\, dx,$$

so $L(x,y,p) = \sqrt{1+p^2}$, independent of both x and y. Compute:

$$L_y = 0, \qquad L_{y'} = \frac{y'}{\sqrt{1+y'^2}}.$$

The EL equation (7.3) *reads*

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0,$$

so $y'/\sqrt{1+y'^2}$ is a constant, call it $k \in (-1,1)$. The function

$$s \longmapsto \frac{s}{\sqrt{1+s^2}}$$

is strictly increasing on $\mathbb{R}$, so this already forces y' itself to be constant. Solving explicitly,

$$\frac{y'}{\sqrt{1+y'^2}} = k \implies \frac{y'^2}{1+y'^2} = k^2 \implies y'^2 = k^2(1+y'^2) \implies y'^2(1-k^2) = k^2,$$

and since y' has the same sign as k,

$$y' = \frac{k}{\sqrt{1-k^2}} =: m.$$

Here m is a real constant. Integrating $y' = m$,

$$y(x) = mx + c,$$

with m, c fixed by the two boundary conditions $y(a) = y_a$, $y(b) = y_b$: solving,

$$m = \frac{y_b - y_a}{b-a}, \qquad c = y_a - ma.$$

The extremal is the straight line segment. It is also the unique global minimizer: every other path has length at least the Euclidean distance between the endpoints, with equality only for the straight segment.

Example 7.8 (Brachistochrone)**.** Johann Bernoulli, 1696. *Find the curve $y = y(x)$ joining $(0,0)$ to (x_1, y_1), with $y_1 > 0$, along which a bead slides from rest under gravity in the shortest time. We take the y-axis downward, so larger y means lower height.*

Setting up the functional. *Conservation of energy at depth y (measuring downward positive) gives $\frac{1}{2}v^2 = gy$, so the speed is $v = \sqrt{2gy}$. The arclength element along the curve is $ds = \sqrt{1+y'^2}\,dx$, so the transit time is*

$$T[y] = \int_0^{x_1} \frac{ds}{v} = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \frac{1}{\sqrt{2g}}\int_0^{x_1} \sqrt{\frac{1+y'^2}{y}}\,dx.$$

The constant $1/\sqrt{2g}$ does not affect the extremal, so take $L = \sqrt{(1+p^2)/y}$. The integrand is singular at the release point $y = 0$, so we should not treat that endpoint as regular. Apply Euler–Lagrange and Beltrami only on interior subintervals where $y \geq \varepsilon > 0$. On each such interval L is C^2, so the first integral below holds wherever $y > 0$. We impose $(0,0)$ after solving the interior ODE, and then check that the resulting extremal meets the origin with a vertical tangent. Since $L_x = 0$, Beltrami (7.7) applies on those interior intervals.

Computing the Beltrami integrand. *We need $L_{y'}$:*

$$L_{y'} = \frac{1}{\sqrt{y}} \cdot \frac{y'}{\sqrt{1+y'^2}} = \frac{y'}{\sqrt{y(1+y'^2)}}.$$

Then

$$\begin{aligned} L - y'L_{y'} &= \sqrt{\frac{1+y'^2}{y}} - y' \cdot \frac{y'}{\sqrt{y(1+y'^2)}} \\ &= \frac{1+y'^2}{\sqrt{y(1+y'^2)}} - \frac{y'^2}{\sqrt{y(1+y'^2)}} \\ &= \frac{(1+y'^2)-y'^2}{\sqrt{y(1+y'^2)}} = \frac{1}{\sqrt{y(1+y'^2)}} =: \frac{1}{\sqrt{2R}}, \end{aligned}$$

where the name $1/\sqrt{2R}$, with $R > 0$, is chosen to match the usual rolling-circle parametrization of a cycloid. Squaring,

$$y(1+y'^2) = 2R. \tag{7.10}$$

This is the implicit first-order ODE of the brachistochrone.

Parametric cycloid solution. *We verify that the cycloid*

$$x(\theta) = R(\theta - \sin\theta), \qquad y(\theta) = R(1-\cos\theta), \qquad \theta \in [0, \theta_1], \tag{7.11}$$

satisfies (7.10)*. Differentiate:*

$$\frac{dx}{d\theta} = R(1-\cos\theta), \qquad \frac{dy}{d\theta} = R\sin\theta,$$

so

$$y'(x) = \frac{dy/d\theta}{dx/d\theta} = \frac{R\sin\theta}{R(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}.$$

Then

$$\begin{aligned}1+y'^2 &= 1+\frac{\sin^2\theta}{(1-\cos\theta)^2}\\ &= \frac{(1-\cos\theta)^2+\sin^2\theta}{(1-\cos\theta)^2}\\ &= \frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{(1-\cos\theta)^2}\\ &= \frac{2-2\cos\theta}{(1-\cos\theta)^2} = \frac{2(1-\cos\theta)}{(1-\cos\theta)^2} = \frac{2}{1-\cos\theta}.\end{aligned}$$

Thus

$$y(1+y'^2) \;=\; R(1-\cos\theta)\cdot\frac{2}{1-\cos\theta} \;=\; 2R,$$

which is exactly (7.10)*. Hence* (7.11) *is a solution. The boundary condition* $x(0)=y(0)=0$ *is automatic at* $\theta=0$*; the endpoint* (x_1,y_1) *fixes* R *and* θ_1 *through* $x(\theta_1)=x_1$*,* $y(\theta_1)=y_1$*.*

What happens at the singular endpoint. *Expand near* $\theta=0$*:*

$$x(\theta)=R\left(\frac{\theta^3}{6}+O(\theta^5)\right), \qquad y(\theta)=R\left(\frac{\theta^2}{2}+O(\theta^4)\right).$$

Therefore

$$y'(x)=\frac{\sin\theta}{1-\cos\theta}\sim\frac{2}{\theta}\to\infty \qquad (\theta\downarrow 0),$$

so the curve meets the origin with a cusp and vertical tangent. But the travel-time element is still regular:

$$ds=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=2R\sin(\theta/2)\,d\theta,$$

while

$$v=\sqrt{2gy}=\sqrt{2gR(1-\cos\theta)}=2\sqrt{gR}\sin(\theta/2).$$

Hence

$$\frac{ds}{v}=\sqrt{\frac{R}{g}}\,d\theta,$$

which stays finite as $\theta\downarrow 0$*. Thus the singular endpoint is integrable. The cycloid is the limit of the regularized interior problem, not a contradiction to it.*

Remark 7.9. *The calculation identifies the cycloid as the variational extremal and shows that the singular release point is compatible with the first-integral derivation. Proving that it is the* global *time-minimizer needs a separate comparison argument.*

Example 7.10 (Catenoid: stationary surface of revolution for the area functional)**.** *Two parallel coaxial circles of radii* $y_a, y_b > 0$ *lie in the planes* $x=a$ *and* $x=b$*. Rotate a curve* $y=y(x)$ *about the* x*-axis to join them. We seek the Euler–Lagrange extremals of the lateral area functional. The area is*

$$A[y] \;=\; 2\pi\int_a^b y\sqrt{1+y'^2}\,dx,$$

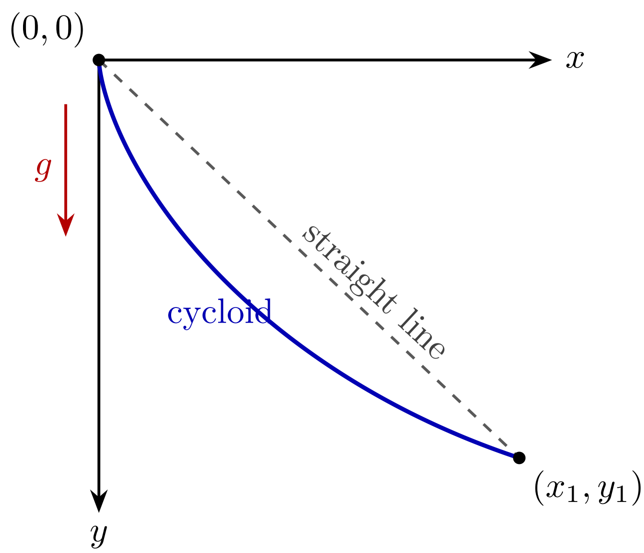

Figure 14: The brachistochrone extremal is a cycloid. The y axis points downward (gravity direction). The curve leaves the origin vertically, which matches the integrable endpoint singularity discussed in Example 7.8.

so up to the constant 2π the Lagrangian is $L = y\sqrt{1+p^2}$. Since $L_x = 0$, apply Beltrami:

$$L_{y'} = y \cdot \frac{y'}{\sqrt{1+y'^2}},$$

$$L - y'L_{y'} = y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} = \frac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = \frac{y}{\sqrt{1+y'^2}} = c,$$

where $c > 0$ is the constant of integration (distinct from the interval endpoint a). Thus $y^2 = c^2(1+y'^2)$, i.e. $y'^2 = (y/c)^2 - 1$. Separating,

$$dx = \frac{c\,dy}{\sqrt{y^2 - c^2}}.$$

Substitute $y = c\cosh u$, $dy = c\sinh u\,du$, $\sqrt{y^2-c^2} = c\sinh u$:

$$dx = \frac{c \cdot c\sinh u\,du}{c\sinh u} = c\,du,$$

so $x - x_0 = cu$ and

$$y(x) = c\cosh\!\left(\frac{x-x_0}{c}\right). \tag{7.12}$$

The surface generated is the catenoid*, so every smooth rotational extremal is a catenoid. For generic boundary data,* (7.12) *may admit zero, one, or two solutions for (c, x_0). The EL computation identifies stationary surfaces; deciding which one minimizes area is a separate stability question.*

Example 7.11 (Surface geodesics)**.** *Parametrize the surface by $(r(z)\cos\phi, r(z)\sin\phi, z)$, with $r \in C^2$, $r > 0$. The induced line element is*

$$ds^2 = (1 + r'(z)^2)\,dz^2 + r(z)^2\,d\phi^2.$$

Writing $\phi = \phi(z)$, the arclength functional is

$$\mathcal{L}[\phi] = \int_{z_0}^{z_1} \sqrt{1 + r'^2 + r^2\phi'^2}\,dz,$$

with $L = \sqrt{1+r'^2+r^2\phi'^2}$. *Since* L *does not depend on* ϕ *(it is a cyclic coordinate), Remark 7.6 gives* $L_{\phi'} =$ *const:*

$$L_{\phi'} \;=\; \frac{r^2\phi'}{\sqrt{1+r'^2+r^2\phi'^2}} \;=\; C. \tag{7.13}$$

Let ψ *be the angle between the geodesic and a meridian (a* $\phi =$ *const curve). Along the geodesic,* $ds^2 = (1+r'^2)dz^2 + r^2 d\phi^2$, *and* $\sin\psi = r\,d\phi/ds$. *Using* (7.13) *together with* $ds/dz = L$, *we get*

$$r(z)\sin\psi(z) \;=\; C, \tag{7.14}$$

Clairaut's relation. *Geometrically, the product of distance to the rotation axis and sine of the angle with a meridian is conserved along a geodesic.*

Solve for ϕ' *from* (7.13)*:*

$$r^4\phi'^2 \;=\; C^2(1+r'^2+r^2\phi'^2) \;\Rightarrow\; \phi'^2(r^4-C^2r^2) \;=\; C^2(1+r'^2),$$

$$\phi'(z) \;=\; \frac{C\sqrt{1+r'^2}}{r\sqrt{r^2-C^2}},$$

so the geodesic is obtained by quadrature:

$$\phi(z)-\phi(z_0) \;=\; \int_{z_0}^{z} \frac{C\sqrt{1+r'(\zeta)^2}}{r(\zeta)\sqrt{r(\zeta)^2-C^2}}\,d\zeta.$$

On the round sphere of radius R, *parametrize by colatitude* θ*:* $r(\theta) = R\sin\theta$, $z(\theta) = R\cos\theta$. *Then* $ds^2 = R^2d\theta^2 + R^2\sin^2\theta\,d\phi^2$, *and Clairaut's relation becomes* $\sin\theta\,\sin\psi = C$. *Since*

$$\sin\psi = \frac{\sin\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}},$$

where now $\phi' = d\phi/d\theta$, *we get*

$$\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}} = C \quad\Longrightarrow\quad \phi' = \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}}.$$

For $C = 0$ *this gives* $\phi' = 0$, *a meridian, which is already a great circle. For* $C \neq 0$, *set* $u = \cot\theta$ *in the last integral. Since* $\sin^2\theta = 1/(1+u^2)$ *and* $d\theta = -du/(1+u^2)$,

$$d\phi = -\frac{C\,du}{\sqrt{1-C^2-C^2u^2}}.$$

Let $B = \sqrt{1-C^2}$. *Then*

$$d\phi = -\frac{C\,du}{\sqrt{B^2-C^2u^2}} \quad\Longrightarrow\quad \phi_0 - \phi = \arcsin\!\left(\frac{Cu}{B}\right),$$

after absorbing the integration constant into ϕ_0. *Therefore* $u = (B/C)\sin(\phi_0-\phi)$, *so, since* $u = \cot\theta$,

$$\cot\theta = A\sin(\phi_0-\phi)$$

with $A = B/C$. *To see the plane explicitly, write a point on the unit sphere as*

$$x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta.$$

Multiplying $\cot\theta = A\sin(\phi_0-\phi)$ *by* $\sin\theta$ *gives*

$$z = A\sin\theta(\sin\phi_0\cos\phi - \cos\phi_0\sin\phi) = A(\sin\phi_0\,x - \cos\phi_0\,y).$$

This is a linear equation through the origin, hence a plane through the origin intersected with the sphere. Its intersection with the sphere is a great circle. Problem 7.4 asks for the same result directly from the sphere Lagrangian.

Remark 7.12 (General geodesic form). *For a Riemannian metric $g_{\mu\nu}(x)$, meaning the coordinate-dependent matrix that measures squared lengths $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, extremizing $\int \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,d\tau$ yields the* geodesic equation *after choosing $\tau = s$ to be arclength:*

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = 0, \qquad \Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}\,g^{\mu\nu}\big(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\big). \tag{7.15}$$

Here $g^{\mu\nu}$ is the matrix inverse of $g_{\mu\nu}$, with summation over repeated indices. The quantities $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols*; they encode how the coordinate basis changes from point to point. For a non-arclength parameter, the EL equation for the length functional has an extra term from reparametrization freedom. The equivalent* energy *Lagrangian $\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ gives* (7.15) *directly when τ is an affine parameter, meaning a parameter for which the geodesic equation has no extra term proportional to the velocity. The surface-of-revolution example is the special case with one cyclic coordinate ϕ; the conservation law $L_{\phi'} =$ const is Clairaut's relation.*

Example 7.13 (Fermat's principle and Snell's law). *Fermat's principle says that light rays in a medium with refractive index $n(\mathbf{r})$ extremize the optical path length*

$$\mathcal{L} = \int n(\mathbf{r})\,ds.$$

Consider a piecewise-constant medium with index n_1 in $y > 0$ and n_2 in $y < 0$, a ray going from (x_A, y_A) with $y_A > 0$ to (x_B, y_B) with $y_B < 0$. In each homogeneous half-space the path is straight (shortest-path example), so the ray consists of two segments meeting at $(x_0, 0)$. The optical path length is

$$\mathcal{L}(x_0) = n_1\sqrt{(x_0 - x_A)^2 + y_A^2} + n_2\sqrt{(x_B - x_0)^2 + y_B^2}.$$

Stationarity, $d\mathcal{L}/dx_0 = 0$:

$$n_1 \cdot \frac{x_0 - x_A}{\sqrt{(x_0 - x_A)^2 + y_A^2}} = n_2 \cdot \frac{x_B - x_0}{\sqrt{(x_B - x_0)^2 + y_B^2}}.$$

The two quotients are $\sin\theta_1$ and $\sin\theta_2$, the angles of the two segments with the surface normal (the y-axis). Hence Snell's law*:*

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{7.16}$$

More generally, for smooth $n = n(y)$, write $L(y, y') = n(y)\sqrt{1 + y'^2}$, independent of x. Beltrami gives

$$L - y'L_{y'} = \frac{n(y)}{\sqrt{1 + y'^2}} = \text{const}.$$

Since $1/\sqrt{1 + y'^2} = \cos\alpha$ where α is the angle the ray makes with the x-axis, and $\cos\alpha = \sin\theta$ with θ the angle to the y-normal, this reads $n(y)\sin\theta(y) =$ const, the continuum Snell's law.

7.5 Multiple dependent and independent variables

n dependent variables, one independent. For $L(x, y_1, \ldots, y_n, y_1', \ldots, y_n')$ and

$$\mathcal{L}[y_1, \ldots, y_n] = \int_a^b L\,dx,$$

the stationarity condition $\delta\mathcal{L} = 0$ under independent variations η_i (all vanishing at endpoints) yields the system

$$\frac{\partial L}{\partial y_i} - \frac{d}{dx}\frac{\partial L}{\partial y_i'} = 0, \qquad i = 1, \ldots, n. \tag{7.17}$$

The proof repeats the derivation of (7.5) for each η_i separately, then uses the fundamental lemma.

Field theory. For a field $\phi(x^\mu)$ on a region $\Omega \subset \mathbb{R}^{1+d}$ of spacetime, a *Lagrangian density* is the quantity integrated over spacetime to form the action. With density $\mathcal{L}(\phi, \partial_\mu \phi)$ and action

$$S[\phi] = \int_\Omega \mathcal{L}(\phi, \partial_\mu \phi)\, d^{1+d}x,$$

varying $\phi \mapsto \phi + \varepsilon\eta$ with η vanishing on $\partial\Omega$ yields (summation over repeated Greek indices)

$$\begin{aligned}\delta S &= \int_\Omega \left[\frac{\partial \mathcal{L}}{\partial \phi}\eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu \eta\right] d^{1+d}x \\ &= \int_\Omega \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\eta\, d^{1+d}x + \int_{\partial\Omega} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\eta\, n_\mu\, dS,\end{aligned}$$

where n_μ is the outward unit normal to the boundary. The divergence theorem gives the boundary term. It vanishes because $\eta|_{\partial\Omega} = 0$, and the multivariable fundamental lemma gives the field Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0. \tag{7.18}$$

7.6 Constraints: isoperimetric and holonomic

An isoperimetric constraint fixes an integral quantity, such as length. A holonomic constraint is pointwise and depends only on the variables, such as $g(x, y_1, \ldots, y_n) = 0$; a non-holonomic constraint usually involves velocities such as y_i'.

Theorem 7.14 (Isoperimetric problem: Lagrange multiplier). *Let $\mathcal{L}[y] = \int_a^b L\, dx$ and $\mathcal{G}[y] = \int_a^b G\, dx$, with $L, G \in C^2$. Suppose $y \in \mathcal{A}$ extremizes $\mathcal{L}$ subject to the constraint $\mathcal{G}[y] = C$. Also suppose y is not a critical point of $\mathcal{G}$ alone. Equivalently, there is at least one admissible perturbation that changes $\mathcal{G}$ to first order; this is what* nondegenerate constraint *means here. Then there exists $\lambda \in \mathbb{R}$ such that y is an unconstrained EL-extremal of*

$$\int_a^b \big(L - \lambda G\big)\, dx. \tag{7.19}$$

Proof. Reduce the constrained problem to one parameter using the implicit function theorem. This theorem says that if one partial derivative of an equation is nonzero, then locally we can solve that equation for the corresponding variable.

Step 1: choose one variation that moves the constraint. Since y is not a critical point of $\mathcal{G}$ alone, there exists an admissible perturbation η_2 such that

$$\delta\mathcal{G}[y; \eta_2] \neq 0. \tag{7.20}$$

Let η_1 be any other admissible perturbation, and consider

$$y_{\varepsilon_1,\varepsilon_2} := y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2.$$

Define

$$F(\varepsilon_1, \varepsilon_2) := \mathcal{L}[y_{\varepsilon_1,\varepsilon_2}], \qquad H(\varepsilon_1, \varepsilon_2) := \mathcal{G}[y_{\varepsilon_1,\varepsilon_2}].$$

Both F and H are C^1 near $(0,0)$, and

$$\frac{\partial H}{\partial \varepsilon_2}(0,0) = \delta\mathcal{G}[y; \eta_2] \neq 0$$

by (7.20).

Step 2: solve the constraint locally. Since $\partial H/\partial \varepsilon_2(0,0) \neq 0$, the implicit function theorem gives a C^1 function ϕ defined for $|\varepsilon_1|$ small such that

$$\phi(0) = 0, \qquad H(\varepsilon_1, \phi(\varepsilon_1)) = C.$$

Then

$$\tilde{y}_{\varepsilon_1} := y + \varepsilon_1 \eta_1 + \phi(\varepsilon_1)\eta_2$$

satisfies the constraint $\mathcal{G}[\tilde{y}_{\varepsilon_1}] = C$.

Step 3: differentiate the constrained family. Since y extremizes $\mathcal{L}$ among constrained curves,

$$f(\varepsilon_1) := \mathcal{L}[\tilde{y}_{\varepsilon_1}] = F(\varepsilon_1, \phi(\varepsilon_1))$$

has an extremum at $\varepsilon_1 = 0$. Hence

$$0 = f'(0) = \delta\mathcal{L}[y;\eta_1] + \phi'(0)\,\delta\mathcal{L}[y;\eta_2]. \tag{7.21}$$

Differentiate the identity $H(\varepsilon_1, \phi(\varepsilon_1)) = C$ at $\varepsilon_1 = 0$:

$$0 = \delta\mathcal{G}[y;\eta_1] + \phi'(0)\,\delta\mathcal{G}[y;\eta_2]. \tag{7.22}$$

Since $\delta\mathcal{G}[y;\eta_2] \neq 0$, solve (7.22) for

$$\phi'(0) = -\frac{\delta\mathcal{G}[y;\eta_1]}{\delta\mathcal{G}[y;\eta_2]}.$$

Substitute this into (7.21):

$$\delta\mathcal{L}[y;\eta_1] = \frac{\delta\mathcal{L}[y;\eta_2]}{\delta\mathcal{G}[y;\eta_2]}\,\delta\mathcal{G}[y;\eta_1].$$

The ratio on the right depends only on the fixed choice of η_2, not on the arbitrary perturbation η_1. Define

$$\lambda := \frac{\delta\mathcal{L}[y;\eta_2]}{\delta\mathcal{G}[y;\eta_2]}.$$

Then the previous identity becomes

$$\delta\mathcal{L}[y;\eta_1] = \lambda\,\delta\mathcal{G}[y;\eta_1]$$

for the arbitrary admissible perturbation η_1.

Step 4: read off the Euler–Lagrange equation. Since η_1 is arbitrary,

$$\delta\big(\mathcal{L} - \lambda\mathcal{G}\big)[y;\eta_1] = 0$$

for every admissible variation. Theorem 7.3, applied to $\tilde{L} = L - \lambda G$, gives the claim. □

Example 7.15 (Catenary: the hanging chain)**.** *A flexible chain of fixed length ℓ and uniform linear mass density ρ hangs between two fixed points in a uniform gravitational field g. Up to an additive constant, its gravitational potential energy is*

$$U[y] \;=\; \rho g \int_a^b y\sqrt{1+y'^2}\,dx,$$

since $dm = \rho\sqrt{1+y'^2}\,dx$ *and height is* y *measured upward. The constraint is fixed length:*

$$\mathcal{G}[y] \;=\; \int_a^b \sqrt{1+y'^2}\,dx \;=\; \ell.$$

By Theorem 7.14, there is a multiplier $\lambda \in \mathbb{R}$ *such that* y *is EL-stationary for* $\tilde{L} = (\rho g y - \lambda)\sqrt{1+y'^2}$. *Since* $\tilde{L}$ *has no explicit* x*-dependence, apply Beltrami (7.7). Compute*

$$\tilde{L}_{y'} \;=\; (\rho g y - \lambda)\cdot \frac{y'}{\sqrt{1+y'^2}},$$

so

$$\begin{aligned}\tilde{L} - y'\tilde{L}_{y'} &= (\rho g y - \lambda)\sqrt{1+y'^2} - (\rho g y - \lambda)\frac{y'^2}{\sqrt{1+y'^2}}\\ &= (\rho g y - \lambda)\cdot \frac{(1+y'^2)-y'^2}{\sqrt{1+y'^2}}\\ &= \frac{\rho g y - \lambda}{\sqrt{1+y'^2}} \;=:\; \rho g\, a,\end{aligned}$$

naming the constant on the right for convenience. Write $y_0 := \lambda/(\rho g)$*; then*

$$\frac{y-y_0}{\sqrt{1+y'^2}} \;=\; a \implies (y-y_0)^2 \;=\; a^2(1+y'^2) \implies y'^2 \;=\; \frac{(y-y_0)^2}{a^2} - 1.$$

Separate variables: $dx \;=\; a\,dy/\sqrt{(y-y_0)^2-a^2}$. *Substitute* $y - y_0 \;=\; a\cosh u$, *so* $dy \;=\; a\sinh u\,du$ *and* $\sqrt{(y-y_0)^2-a^2} = a\sinh u$. *Then*

$$dx \;=\; a\,\frac{a\sinh u\,du}{a\sinh u} \;=\; a\,du,$$

so $x = au + x_0$*, i.e.* $u = (x-x_0)/a$. *Therefore*

$$y(x) \;=\; y_0 + a\cosh\!\left(\frac{x-x_0}{a}\right), \tag{7.23}$$

the catenary. *The constants* y_0*,* x_0*, and* a *are determined by the two endpoints and the length constraint* $\int \sqrt{1+y'^2}dx = \ell$.

Example 7.16 (Isoperimetric problem: circle as the stationary curve for fixed perimeter)**.** *Among all simple closed planar curves of perimeter* P*, traversed once counterclockwise, find the one enclosing the largest area. Here* simple *means the curve has no self-intersections except that its endpoint returns to its starting point. Parametrize by arclength* $s \in [0,P]$, $\big(x(s), y(s)\big)$*, with* $x(0) = x(P)$*,* $y(0) = y(P)$*, and* $x'^2 + y'^2 = 1$. *By Green's theorem,*

$$A \;=\; \tfrac{1}{2}\oint (x\,dy - y\,dx) \;=\; \tfrac{1}{2}\int_0^P (xy' - yx')\,ds.$$

Introduce a multiplier function $\mu(s)$ *for the pointwise constraint and consider*

$$\tilde{L}(x,y,x',y',s) \;=\; \tfrac{1}{2}(xy'-yx') - \mu(s)(x'^2+y'^2-1).$$

EL in x*:* $\tilde{L}_x = \frac{1}{2}y'$, $\tilde{L}_{x'} = -\frac{1}{2}y - 2\mu x'$*, so*

$$\tfrac{1}{2}y' - \frac{d}{ds}(-\tfrac{1}{2}y - 2\mu x') \;=\; \tfrac{1}{2}y' + \tfrac{1}{2}y' + 2(\mu x')' \;=\; 0 \implies (\mu x')' \;=\; -\tfrac{1}{2}y'.$$

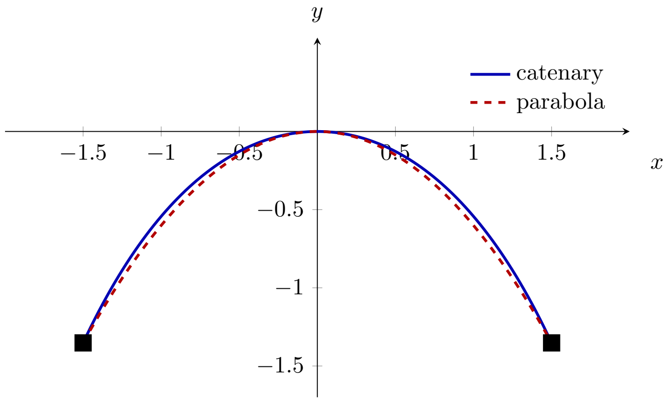

Figure 15: A hanging chain under uniform gravity takes the shape of a catenary $y = a\cosh(x/a)$, not a parabola. A matching parabola with the same endpoints and sag is only a close approximation.

EL in y: $\tilde{L}_y = -\frac{1}{2}x'$, $\tilde{L}_{y'} = \frac{1}{2}x - 2\mu y'$, so

$$-\tfrac{1}{2}x' - \frac{d}{ds}(\tfrac{1}{2}x - 2\mu y') \;=\; -\tfrac{1}{2}x' - \tfrac{1}{2}x' + 2(\mu y')' \;=\; 0 \implies (\mu y')' \;=\; \tfrac{1}{2}x'.$$

Differentiate $x'^2 + y'^2 = 1$: $x'x'' + y'y'' = 0$. Now multiply $(\mu x')' = -\frac{1}{2}y'$ by x' and $(\mu y')' = \frac{1}{2}x'$ by y', then add:

$$x'(\mu x')' + y'(\mu y')' \;=\; -\tfrac{1}{2}x'y' + \tfrac{1}{2}x'y' \;=\; 0,$$

i.e. $(\mu(x'^2 + y'^2))' - \mu(x'x'' + y'y'') = \mu' \cdot 1 - 0 = \mu' = 0$. Thus $\mu = $ const, and we can set $R := 2\mu$. The system becomes

$$Rx'' \;=\; -y', \qquad Ry'' \;=\; x'.$$

Because $x'^2 + y'^2 = 1$, there is an angle $\theta(s)$ such that

$$x'(s) = \cos\theta(s), \qquad y'(s) = \sin\theta(s).$$

Differentiate these identities:

$$x'' = -\theta' \sin\theta, \qquad y'' = \theta' \cos\theta.$$

Substitute into $Rx'' = -y'$ and $Ry'' = x'$:

$$-R\theta' \sin\theta = -\sin\theta, \qquad R\theta' \cos\theta = \cos\theta.$$

Wherever $\sin\theta$ or $\cos\theta$ is nonzero, these equations give $\theta' = 1/R$; by continuity the same identity holds for every s. Hence

$$\theta(s) = \frac{s}{R} + \theta_0$$

for a constant θ_0, and therefore

$$x'(s) = \cos\!\left(\frac{s}{R} + \theta_0\right), \qquad y'(s) = \sin\!\left(\frac{s}{R} + \theta_0\right).$$

Integrating,

$$x(s) \;=\; x_0 + R\sin\!\left(\frac{s}{R} + \theta_0\right), \qquad y(s) \;=\; y_0 - R\cos\!\left(\frac{s}{R} + \theta_0\right),$$

which is a circle of radius R. Closure $x(P) = x(0)$ and $y(P) = y(0)$ gives

$$\sin\left(\frac{P}{R} + \theta_0\right) = \sin\theta_0, \qquad \cos\left(\frac{P}{R} + \theta_0\right) = \cos\theta_0,$$

so $P/R = 2\pi m$ for some integer m. Since the curve is simple and traversed once, $m = 1$ and $R = P/(2\pi)$. The stationary curve is a circle, with area $A = \pi R^2 = P^2/(4\pi)$. The global isoperimetric inequality $4\pi A \le P^2$ requires a separate comparison argument.

Holonomic constraints. If the curves must satisfy $g(x, y_1, \dots, y_n) = 0$ pointwise, introduce a multiplier *function* $\mu(x)$ and extremize $\int (L - \mu(x) g)\, dx$. Treating y_i and μ as independent, the EL equations are

$$\frac{\partial L}{\partial y_i} - \frac{d}{dx}\frac{\partial L}{\partial y_i'} = \mu(x)\frac{\partial g}{\partial y_i}, \qquad g(x, y_1, \dots, y_n) = 0. \tag{7.24}$$

An integral constraint $\int g\, dx = \text{const}$ uses one multiplier constant λ. A pointwise constraint $g = 0$ uses a multiplier function $\mu(x)$, one scalar at each x.

Worked example (particle on a sphere). A point mass in a gravitational potential $V(\mathbf{r})$ constrained to the sphere $g(\mathbf{r}) = x^2 + y^2 + z^2 - R^2 = 0$ has $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(\mathbf{r})$, and (7.24) (reading $x \to t$, $y_i \to \mathbf{r}$) gives

$$m\ddot{\mathbf{r}} = -\nabla V - \mu(t)\nabla g = -\nabla V - 2\mu(t)\mathbf{r}.$$

The multiplier is, up to a factor 2, the normal force coefficient. It is determined by differentiating $g = 0$ twice in time and solving for μ.

For non-holonomic velocity constraints $\sum a_i(x, y) y_i' + b(x, y) = 0$, the multiplier method is not automatic. Vakonomic mechanics constrains the variations; d'Alembert–Lagrange mechanics constrains only the trajectories and gives the physical rolling equations. Since the two can differ, we do not use non-holonomic constraints in the sequel.

7.7 Second variation and the Jacobi criterion

The first variation finds stationary curves. The second variation tests whether a stationary curve is a minimum:

$$\begin{aligned}\delta^2\mathcal{L}[y;\eta] &= \frac{d^2}{d\varepsilon^2}\mathcal{L}[y + \varepsilon\eta]\bigg|_{\varepsilon=0} \\ &= \int_a^b \left[L_{yy}\eta^2 + 2L_{yy'}\eta\eta' + L_{y'y'}\eta'^2\right] dx,\end{aligned} \tag{7.25}$$

where the coefficients are evaluated on the extremal y. Integrate the middle term by parts, using $\eta(a) = \eta(b) = 0$:

$$\int_a^b 2L_{yy'}\eta\eta'\, dx = \int_a^b L_{yy'}(\eta^2)'\, dx = \left[L_{yy'}\eta^2\right]_a^b - \int_a^b \frac{dL_{yy'}}{dx}\eta^2\, dx = -\int_a^b \frac{dL_{yy'}}{dx}\eta^2\, dx.$$

Collect terms with $P(x) := L_{y'y'}$, $Q(x) := L_{yy} - (d/dx)L_{yy'}$:

$$\delta^2\mathcal{L}[y;\eta] = \int_a^b \left[P\eta'^2 + Q\eta^2\right] dx. \tag{7.26}$$

- **Legendre condition (necessary):** If y is a minimizer, then $P(x) = L_{y'y'} \geq 0$ on $[a, b]$. If $P(x_0) < 0$ at some interior point, continuity gives an interval $J \subset (a, b)$ containing x_0, with closure still inside (a, b), and a number $p_0 > 0$ such that $P(x) \leq -p_0$ on J. Choose a nonzero cutoff $\chi \in C_c^2(J)$, meaning χ is C^2 and vanishes outside J, and set

$$\eta_\varepsilon(x) \;=\; \varepsilon\,\chi(x)\sin\Big(\frac{x - x_0}{\varepsilon}\Big).$$

  This perturbation satisfies the endpoint conditions because it is supported inside J. It is small in amplitude, $\eta_\varepsilon = O(\varepsilon)$, but its derivative has an order-one oscillating part:

$$\eta_\varepsilon'(x) = \chi(x)\cos\Big(\frac{x - x_0}{\varepsilon}\Big) + \varepsilon\chi'(x)\sin\Big(\frac{x - x_0}{\varepsilon}\Big).$$

  Hence

$$\int_a^b P\,\eta_\varepsilon'^2\,dx = \int_J P(x)\chi(x)^2\cos^2\Big(\frac{x - x_0}{\varepsilon}\Big)\,dx + O(\varepsilon).$$

  The fast factor $\cos^2((x - x_0)/\varepsilon)$ averages to $1/2$: over many tiny periods, the oscillatory part has mean zero, while the slowly varying factor $P(x)\chi(x)^2$ barely changes across one period. Therefore, for all sufficiently small ε, the main term is bounded above by a negative constant. Meanwhile the $Q\eta_\varepsilon^2$ term in (7.26) is $O(\varepsilon^2)$ (and, in the unintegrated form (7.25), the mixed term is only $O(\varepsilon)$). Thus $\delta^2\mathcal{L}[y; \eta_\varepsilon] < 0$ for small ε, contradicting minimality.

- **Jacobi accessory equation.** The Euler–Lagrange equation of (7.26) (with η as the unknown) is the linear Sturm–Liouville ODE

$$-\frac{d}{dx}\big(P\,\eta'\big) + Q\,\eta \;=\; 0. \tag{7.27}$$

  A nonzero solution η with $\eta(a) = 0$ and $\eta(c) = 0$ for some $c \in (a, b]$ makes c a *conjugate point*. Under the strengthened Legendre condition $P > 0$, an interior conjugate point rules out weak local minimality (minimality among sufficiently nearby curves). Conversely, if there is no conjugate point in $(a, b]$, the second-variation quadratic form is positive definite, which is the standard Jacobi sufficient test for strict weak local minimality. A conjugate point at the endpoint leaves a zero direction and rules out strictness. Strong local minimality requires the additional Weierstrass condition; the sphere example below shows the second-variation mechanism concretely.

Example 7.17 (Geodesics on the sphere: conjugate point at the antipode)**.** *By rotational symmetry, take the reference geodesic to be the equator of the unit sphere. Use coordinates* (s, u) *near the equator:*

$$X(s, u) = (\cos u\cos s,\ \cos u\sin s,\ \sin u),$$

where s is longitude and u is signed latitude. The equator is $u = 0$.

Differentiate:

$$X_s = (-\cos u\sin s,\ \cos u\cos s,\ 0), \qquad X_u = (-\sin u\cos s,\ -\sin u\sin s,\ \cos u).$$

Hence

$$|X_s|^2 = \cos^2 u, \qquad |X_u|^2 = 1, \qquad X_s \cdot X_u = 0,$$

so the metric in these coordinates is

$$d\ell^2 = du^2 + \cos^2 u\,ds^2. \tag{7.28}$$

Fix an equatorial arc $s \in [0, s_1]$. A nearby curve with the same endpoints can be written as $u = \varepsilon\eta(s)$, with $\eta(0) = \eta(s_1) = 0$. By (7.28), its arclength is

$$L[\varepsilon\eta] = \int_0^{s_1} \sqrt{\varepsilon^2\eta'(s)^2 + \cos^2(\varepsilon\eta(s))}\, ds. \tag{7.29}$$

Use the Taylor expansion

$$\cos^2(\varepsilon\eta) = 1 - \varepsilon^2\eta^2 + O(\varepsilon^4)$$

uniformly in s, so the quantity under the square root is

$$1 + \varepsilon^2(\eta'^2 - \eta^2) + O(\varepsilon^4).$$

Applying $\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$ with $x = \varepsilon^2(\eta'^2 - \eta^2) + O(\varepsilon^4)$ gives

$$L[\varepsilon\eta] = s_1 + \frac{\varepsilon^2}{2}\int_0^{s_1} \left(\eta'^2 - \eta^2\right) ds + O(\varepsilon^4). \tag{7.30}$$

The quadratic form for the second variation is

$$Q[\eta] = \int_0^{s_1} \left(\eta'^2 - \eta^2\right) ds,$$

and the corresponding Jacobi equation is

$$\eta''(s) + \eta(s) = 0. \tag{7.31}$$

Case $s_1 < \pi$. *The Wirtinger inequality on $[0, s_1]$ for functions with $\eta(0) = \eta(s_1) = 0$ says*

$$\int_0^{s_1} \eta'^2\, ds \geq \left(\frac{\pi}{s_1}\right)^2 \int_0^{s_1} \eta^2\, ds.$$

This is the one-dimensional Poincaré inequality. The constant is sharp and is achieved by $\eta(s) = \sin(\pi s/s_1)$, the lowest-frequency sine mode that vanishes at both endpoints. One-line derivation: *expand η in the orthonormal sine basis $e_k(s) = \sqrt{2/s_1}\sin(k\pi s/s_1)$, $k = 1, 2, \ldots$, of $L^2[0, s_1]$; if $\eta = \sum_k c_k e_k$ then $\eta'(s) = \sum_k c_k (k\pi/s_1)\sqrt{2/s_1}\cos(k\pi s/s_1)$ and by Parseval*

$$\int_0^{s_1} \eta'^2\, ds = \sum_{k\geq 1} c_k^2 \left(k\pi/s_1\right)^2 \geq (\pi/s_1)^2 \sum_{k\geq 1} c_k^2 = (\pi/s_1)^2 \int_0^{s_1} \eta^2\, ds,$$

using $k \geq 1$ in the inequality. Since $s_1 < \pi$, we have $(\pi/s_1)^2 > 1$, so for every nonzero η,

$$Q[\eta] \geq \left[\left(\frac{\pi}{s_1}\right)^2 - 1\right] \int_0^{s_1} \eta^2\, ds > 0.$$

So the equatorial arc is a strict local minimizer.

Case $s_1 = \pi$. *Equality in Wirtinger's inequality is attained by $\eta(s) = A\sin s$, so*

$$Q[A\sin s] = 0.$$

This Jacobi field solves (7.31) with $\eta(0) = \eta(\pi) = 0$. The first conjugate point is the antipode.

Case $s_1 > \pi$. *Choose the admissible variation*

$$\eta(s) = \sin\left(\frac{\pi s}{s_1}\right).$$

Then

$$\int_0^{s_1} \eta'^2\, ds = \left(\frac{\pi}{s_1}\right)^2 \int_0^{s_1} \eta^2\, ds,$$

so

$$Q[\eta] = \left[\left(\frac{\pi}{s_1}\right)^2 - 1\right] \int_0^{s_1} \eta^2\, ds < 0.$$

Equation (7.30) *therefore gives*

$$L[\varepsilon\eta] < s_1$$

for all sufficiently small nonzero ε*. A chosen great-circle arc of length greater than* π *is therefore not a local minimizer.*

A great-circle arc on S^2 *is locally minimizing exactly up to the antipode. The antipode is the first conjugate point.*

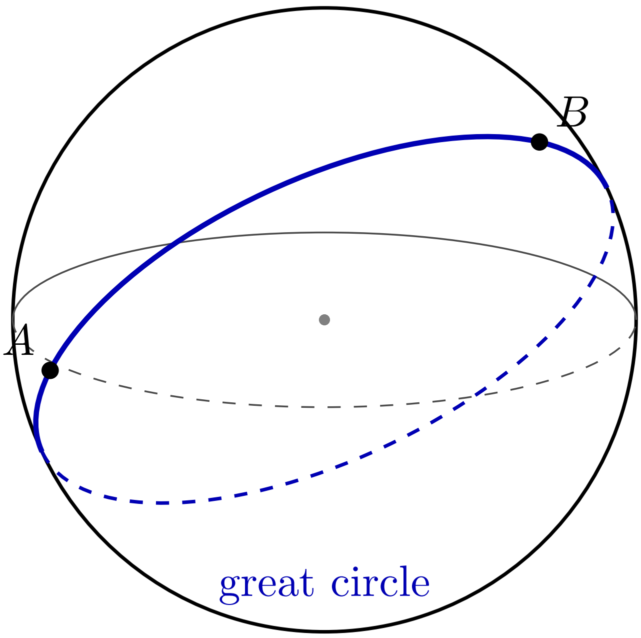

Figure 16: The shortest path between non-antipodal points A, B on the unit sphere is the shorter arc of a great circle. A chosen great-circle arc of length $s_1 < \pi$ is a strict local minimum; at $s_1 = \pi$ antipodal endpoints are minimizing but no longer unique; for $s_1 > \pi$ the chosen arc is not minimizing.

7.8 Noether's theorem

Noether's theorem says that every continuous symmetry of the action produces a conservation law.

7.8.1 Point-mechanics version

Let $L(t, q, \dot{q})$ be a Lagrangian for generalized coordinates $q = (q^1, \dots, q^n)$. Consider

$$t \mapsto t' = t + \varepsilon\, \tau(t, q), \qquad q^i \mapsto q'^i = q^i + \varepsilon\, \xi^i(t, q), \tag{7.32}$$

with ε small and τ, ξ^i smooth. Along a trajectory,

$$q'^i(t') = q^i(t) + \varepsilon\, \xi^i(t, q(t)). \tag{7.33}$$

Taylor expanding at fixed parameter value t and using (7.33),

$$q'^i(t) - q^i(t) \;=\; \varepsilon\big[\xi^i(t,q) - \dot q^i\,\tau(t,q)\big] + O(\varepsilon^2). \tag{7.34}$$

The combination in brackets is the *total variation* at fixed t:

$$\bar\delta q^i \;:=\; \xi^i - \dot q^i\,\tau. \tag{7.35}$$

The minus sign reflects that a time shift changes the value seen at fixed t by $-\dot q\,\tau$.

Repeated indices i are summed from 1 to n. Let $\dot\tau = d(\tau(t,q(t)))/dt$ denote the total derivative along the trajectory. At fixed time t, the curve change gives $L \mapsto L + \varepsilon\bar\delta L$. The time label also shifts by $\varepsilon\tau$, contributing $\varepsilon\tau\,\dot L$, and the time element changes by $dt' = (1+\varepsilon\dot\tau)dt$. Therefore, to first order in ε,

$$L'\,dt' = \big(L + \varepsilon\bar\delta L + \varepsilon\tau\dot L\big)(1+\varepsilon\dot\tau)\,dt = \left[L + \varepsilon\left(\bar\delta L + \frac{d(L\tau)}{dt}\right)\right]dt.$$

This explains the extra total derivative term in the action density:

The first-order change of the action density is

$$(L'\,dt' - L\,dt)/\varepsilon \;=\; \bar\delta L + \frac{d(L\tau)}{dt}, \tag{7.36}$$

where $\bar\delta L := L_{q^i}\bar\delta q^i + L_{\dot q^i}(d\bar\delta q^i/dt)$ is the variation at fixed t.

Theorem 7.18 (Noether, point-mechanics). *Suppose the action $S = \int L\,dt$ is* quasi-invariant *under* (7.32)*, meaning the Lagrangian changes only by a total derivative:*

$$\bar\delta L + \frac{d}{dt}(L\tau) \;=\; \frac{dF}{dt} \tag{7.37}$$

for some $F = F(t,q)$. Then along any Euler–Lagrange solution,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot q^i}\,\xi^i - \left(\frac{\partial L}{\partial \dot q^i}\dot q^i - L\right)\tau - F\right] \;=\; 0. \tag{7.38}$$

The bracketed quantity is the Noether charge.

Proof. Compute $\bar\delta L$ at fixed t:

$$\begin{aligned}\bar\delta L &= L_{q^i}\,\bar\delta q^i + L_{\dot q^i}\,\frac{d\bar\delta q^i}{dt}\\ &= \left[L_{q^i} - \frac{d}{dt}L_{\dot q^i}\right]\bar\delta q^i + \frac{d}{dt}\big[L_{\dot q^i}\,\bar\delta q^i\big]\\ &= \frac{d}{dt}\big[L_{\dot q^i}\,\bar\delta q^i\big],\end{aligned}$$

where the last step uses the EL equations (7.17). Substitute into quasi-invariance (7.37):

$$\frac{d}{dt}\big[L_{\dot q^i}\,(\xi^i - \dot q^i\tau)\big] + \frac{d}{dt}(L\tau) \;=\; \frac{dF}{dt}.$$

Move the total derivatives to one side:

$$\frac{d}{dt}\big[L_{\dot q^i}\,(\xi^i - \dot q^i\tau) + L\tau - F\big] \;=\; 0.$$

Combine the two τ-terms inside the bracket:

$$\frac{d}{dt}\big[L_{\dot q^i}\xi^i - (L_{\dot q^i}\dot q^i - L)\tau - F\big] \;=\; 0,$$

i.e. the Noether charge (7.38) has zero total time derivative. □

Example 7.19 (Time translation ⇒ energy). *If L does not depend explicitly on t, then $\tau = 1$, $\xi^i = 0$, $F = 0$ is a symmetry. The Noether charge* (7.38) *reduces to*

$$Q = -\left(L_{\dot q^i}\dot q^i - L\right) = -H,$$

where $H = p_i\dot q^i - L$ is the Hamiltonian. Since $H = -Q$, conservation of Q is equivalently conservation of energy. For $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V$, $H = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V$.

Example 7.20 (Space translation ⇒ momentum). *For $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(\mathbf{r})$ with V independent of x, the translation $x \mapsto x + \varepsilon$ ($\tau = 0$, $\xi = \hat{x}$, $F = 0$) leaves L invariant. The Noether charge is*

$$p_x = L_{\dot x} = m\dot x,$$

the x-component of linear momentum. Similarly for y, z if V is fully translation-invariant.

Example 7.21 (Rotations in 3D ⇒ angular momentum). *An infinitesimal rotation about an axis $\hat n$ acts by $\mathbf{r} \mapsto \mathbf{r} + \varepsilon(\hat n \times \mathbf{r}) + O(\varepsilon^2)$. Thus $\xi^i = \epsilon^i{}_{jk}\, n^j r^k$ and $\tau = 0$. For $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(|\mathbf{r}|)$, this is a symmetry with $F = 0$. The Noether charge is*

$$\begin{aligned} Q_{\hat n} &= L_{\dot r^i}\, \xi^i = m\dot{\mathbf{r}} \cdot (\hat n \times \mathbf{r}) \\ &= \hat n \cdot (\mathbf{r} \times m\dot{\mathbf{r}}) = \hat n \cdot \mathbf{L}, \end{aligned}$$

the component of angular momentum along $\hat n$. Since $\hat n$ is arbitrary, $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$ is conserved.

Example 7.22 (Galilean boost ⇒ center-of-mass motion). *Consider N particles with $L = \sum_a \frac{1}{2}m_a|\dot{\mathbf{r}}_a|^2 - V$, where V depends only on differences $\mathbf{r}_a - \mathbf{r}_b$. A Galilean boost along $\hat u$ sends $\mathbf{r}_a \mapsto \mathbf{r}_a + \varepsilon t\hat u$ ($\tau = 0$, $\xi_a = t\hat u$). Compute*

$$\bar\delta L = \sum_a m_a\, \dot{\mathbf{r}}_a \cdot \hat u = \frac{d}{dt}\left(\hat u \cdot \sum_a m_a\, \mathbf{r}_a\right),$$

since V is invariant under the common shift. Thus (7.37) *holds with $F = \hat u \cdot \sum_a m_a \mathbf{r}_a$. The Noether charge is*

$$\begin{aligned} Q &= \sum_a L_{\dot{\mathbf{r}}_a} \cdot (t\hat u) - F \\ &= \hat u \cdot \left[t\sum_a m_a\dot{\mathbf{r}}_a - \sum_a m_a\mathbf{r}_a \right] \\ &= M\,\hat u \cdot (t\mathbf{V}_{cm} - \mathbf{R}_{cm}), \end{aligned}$$

where $M = \sum m_a$, $M\mathbf{R}_{cm} = \sum m_a\mathbf{r}_a$, $M\mathbf{V}_{cm} = \sum m_a\dot{\mathbf{r}}_a$. Conservation of Q says the center of mass moves uniformly.

7.8.2 Field-theory version

For fields ϕ^A with Lagrangian density $\mathcal{L}(\phi^A, \partial_\mu\phi^A)$, suppose an infinitesimal change $\delta\phi^A = \xi^A$ has

$$\delta\mathcal{L} = \partial_\mu K^\mu \tag{7.39}$$

This quasi-invariance gives a conserved current. On solutions of the field EL equation (7.18),

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi^A}\xi^A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\partial_\mu\xi^A \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\right)\xi^A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\partial_\mu\xi^A \\ &= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\,\xi^A\right]. \end{aligned}$$

Equating with (7.39),

$$\partial_\mu j^\mu = 0, \qquad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\xi^A - K^\mu. \tag{7.40}$$

Integrating over space gives a conserved charge

$$Q = \int j^0\, d^d x, \qquad \frac{dQ}{dt} = 0$$

when boundary terms vanish.

Translation example and the stress tensor. The same formula includes spacetime translations if we use the active fixed-coordinate variation

$$\delta\phi^A = -a^\nu \partial_\nu \phi^A,$$

where a^ν is a constant translation vector. If $\mathcal{L}$ has no explicit dependence on the coordinates, then

$$\delta\mathcal{L} = -a^\nu \partial_\nu \mathcal{L} = \partial_\mu(-a^\mu \mathcal{L}).$$

Thus $K^\mu = -a^\mu \mathcal{L}$, and (7.40) gives

$$j^\mu = -a^\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \partial_\nu \phi^A - \delta^\mu_\nu \mathcal{L} \right].$$

Because the constants a^ν are arbitrary, the bracketed tensor is conserved:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \partial_\nu \phi^A - \delta^\mu_\nu \mathcal{L}, \qquad \partial_\mu T^\mu{}_\nu = 0. \tag{7.41}$$

Here δ^μ_ν is the Kronecker delta: 1 if $\mu = \nu$ and 0 otherwise. This is the canonical energy–momentum tensor. The minus sign in $j^\mu = -a^\nu T^\mu{}_\nu$ comes from the convention that a positive coordinate translation changes the field at a fixed coordinate by $-a^\nu \partial_\nu \phi^A$.

7.9 Hamilton's principle of stationary action

Theorem 7.23 (Hamilton's principle)**.** *For a particle system written in Cartesian coordinates with Lagrangian $L = T - V$ (kinetic minus potential energy), the physical trajectories are the extremals of the action*

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt$$

at fixed endpoints $q(t_0)$, $q(t_1)$. In Cartesian coordinates, the Euler–Lagrange equations are Newton's equations. Other coordinate systems give the same dynamics after coordinate change.

Derivation of Newton's law from the EL equation. For a single Cartesian particle in 3D with $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(\mathbf{r})$, compute component-wise for each r^i:

$$\frac{\partial L}{\partial r^i} = -\frac{\partial V}{\partial r^i}, \qquad \frac{\partial L}{\partial \dot{r}^i} = m\dot{r}^i, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}^i} = m\ddot{r}^i.$$

The EL equation (7.17) becomes

$$-\frac{\partial V}{\partial r^i} - m\ddot{r}^i = 0 \iff m\ddot{r}^i = -\frac{\partial V}{\partial r^i},$$

i.e. Newton's second law $m\ddot{\mathbf{r}} = -\nabla V = \mathbf{F}$. Conversely, Newton's law gives the EL equation by the same identities.

For several particles in Cartesian coordinates, the same computation applies componentwise to each coordinate of each particle. □

7.10 The Hamiltonian formulation

7.10.1 Legendre transform

The *Legendre transform* used here is the change from velocity variables $\dot{q}$ to momentum variables p. Define the canonical *momentum*; here canonical means the standard momentum coordinate paired with q^i:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}(t,q,\dot{q}).$$

Assume the Hessian $(L_{\dot{q}^i\dot{q}^j})$, the matrix of second partial derivatives with respect to the velocities, is invertible. Then the implicit function theorem, the standard theorem that lets one solve equations locally when the derivative matrix is invertible, solves for $\dot{q} = \dot{q}(t,q,p)$. Define the *Hamiltonian*

$$H(t,q,p) = p_i\dot{q}^i(t,q,p) - L\big(t,q,\dot{q}(t,q,p)\big). \tag{7.42}$$

Compute dH treating H as a function of (t,q,p):

$$\begin{aligned} dH &= \dot{q}^i\,dp_i + p_i\,d\dot{q}^i - \frac{\partial L}{\partial t}dt - \frac{\partial L}{\partial q^i}dq^i - \frac{\partial L}{\partial \dot{q}^i}d\dot{q}^i \\ &= \dot{q}^i\,dp_i + \left(p_i - \frac{\partial L}{\partial \dot{q}^i}\right)d\dot{q}^i - \frac{\partial L}{\partial t}\,dt - \frac{\partial L}{\partial q^i}\,dq^i \\ &= \dot{q}^i\,dp_i - \frac{\partial L}{\partial t}\,dt - \frac{\partial L}{\partial q^i}\,dq^i, \end{aligned} \tag{7.43}$$

using the definition of p_i to cancel the $d\dot{q}$ term. Thus H is naturally a function of (t,q,p).

Remark 7.24 (Geometric picture). *For fixed (t,q), view L as a function of the single variable $\dot{q}$ in one dimension. The construction*

$$H(p) = \sup_{\dot{q}}\big[p\,\dot{q} - L(\dot{q})\big]$$

is the standard convex Legendre transform from optimization. For each slope p, draw the line $\dot{q} \mapsto p\dot{q}$; the supremum is achieved at the $\dot{q}$ where the graph of L has slope p, namely where $L'(\dot{q}) = p$. The number $H(p)$ is then the (signed) vertical gap by which the line $p\dot{q}$ overshoots L at the matched point. When L is strictly convex in $\dot{q}$ (so $L_{\dot{q}\dot{q}} > 0$), every slope p is achieved exactly once, the supremum is a maximum, and the recipe $\dot{q} = \dot{q}(p)$ is single-valued — this is the invertibility assumption made above. The same picture explains why $L \longleftrightarrow H$ are reciprocal: applying the transform twice returns the original function (Problem 7.7). The multivariable version is the same picture component-wise.

7.10.2 Hamilton's equations

Reading partial derivatives from (7.43),

$$\frac{\partial H}{\partial p_i} = \dot{q}^i, \qquad \frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

On EL solutions, $\partial L/\partial q^i = (d/dt)(\partial L/\partial \dot{q}^i) = \dot{p}_i$, so the second identity reads $\partial H/\partial q^i = -\dot{p}_i$. Combining,

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{7.44}$$

Hamilton's equations are the first-order form of the Euler–Lagrange equations.

Example 7.25 (Cartesian particle). *For* $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(\mathbf{r})$*:* $\mathbf{p} = m\dot{\mathbf{r}}$*, so* $\dot{\mathbf{r}} = \mathbf{p}/m$*, and*

$$H = \mathbf{p}\cdot\dot{\mathbf{r}} - L = \frac{|\mathbf{p}|^2}{m} - \frac{|\mathbf{p}|^2}{2m} + V = \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{r}),$$

the total mechanical energy. Hamilton's equations read $\dot{\mathbf{r}} = \mathbf{p}/m$ *(definition of momentum) and* $\dot{\mathbf{p}} = -\nabla V$ *(Newton's second law).*

7.10.3 Poisson brackets

For functions $f(q,p,t)$, $g(q,p,t)$ on phase space, meaning the space whose coordinates are positions q and momenta p, define

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \right). \tag{7.45}$$

The bracket has four basic algebraic properties:

- It is *bilinear*, meaning linear in each input separately.
- It is *antisymmetric*: $\{f,g\} = -\{g,f\}$.
- It satisfies the Leibniz rule, so it differentiates products in each slot.
- It satisfies the Jacobi identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0.$$

The basic position and momentum variables have the canonical brackets

$$\{q^i,p_j\} = \delta^i_j, \qquad \{q^i,q^j\} = 0, \qquad \{p_i,p_j\} = 0.$$

Here *canonical* means these are the defining bracket relations for the chosen coordinates, and δ^i_j is the Kronecker delta: 1 if $i = j$ and 0 otherwise.

Hamilton's equations (7.44) take the compact form $\dot{q}^i = \{q^i,H\}$, $\dot{p}_i = \{p_i,H\}$, and more generally for any phase-space function

$$\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}.$$

A function f with $\partial f/\partial t = 0$ is conserved along a trajectory if $\{f,H\} = 0$ along that trajectory; if the bracket vanishes on phase space, f is conserved for all trajectories.

Quantum aside. In the standard passage from classical mechanics to quantum mechanics, Poisson brackets correspond to commutators via $\{f,g\} \mapsto (1/i\hbar)[\hat{f},\hat{g}]$, at leading order in $\hbar$.

7.11 Connections to complex analysis and special functions

This subsection closes the loop with Sections 2–6. None of it is needed to do calculus of variations, but it explains *why* a chapter on variations belongs in a book centred on complex analysis and the classical special functions.

7.11.1 Sturm–Liouville framing: every special-function ODE is an EL equation

The pattern. For each of the special-function families of Sections 4–6, the defining ODE is the Euler–Lagrange equation of a single quadratic functional of *Rayleigh* type

$$\mathcal{R}[y] = \frac{\displaystyle\int_a^b \big[p(x)\,y'(x)^2 + q(x)\,y(x)^2\big]\,dx}{\displaystyle\int_a^b w(x)\,y(x)^2\,dx}, \tag{7.46}$$

where $p(x) \geq 0$, $w(x) > 0$, and $q(x)$ are fixed real-valued weights on (a, b). The numerator and denominator are the same Sturm–Liouville structure we already met in Theorem 5.15 for Legendre and Theorem 4.19 for Bessel.

Fix the denominator to 1 by a Lagrange multiplier λ and consider the constrained variational problem

$$\text{extremize} \quad \int_a^b \big[p\,y'^2 + q\,y^2\big]\,dx \qquad \text{subject to} \qquad \int_a^b w\,y^2\,dx = 1. \tag{7.47}$$

The constrained Lagrangian is $L_\lambda(x, y, y') = p\,y'^2 + q\,y^2 - \lambda w\,y^2$. Compute the EL equation: $L_{\lambda,y'} = 2p\,y'$, so $(d/dx)L_{\lambda,y'} = 2(p\,y')'$, while $L_{\lambda,y} = 2q\,y - 2\lambda w\,y$. The EL equation (7.3) (Theorem 7.3) gives

$$-\frac{d}{dx}\big[p(x)\,y'(x)\big] + q(x)\,y(x) = \lambda\,w(x)\,y(x), \tag{7.48}$$

the *Sturm–Liouville eigenvalue problem*. The multiplier λ is the eigenvalue; the extremal y is the corresponding eigenfunction. The Rayleigh quotient (7.46) delivers the variational characterization of those eigenvalues: the n-th eigenvalue equals the minimum of $\mathcal{R}$ over the space of admissible y orthogonal (in the weight w) to the first $n-1$ eigenfunctions.

Translation to the catalog. Choosing (p, q, w, a, b) from the table reproduces the ODEs of the previous sections:

Family	$p(x)$	$q(x)$	$w(x)$	(a, b)	ODE label
Legendre P_ℓ	$1-x^2$	0	1	$(-1, 1)$	(5.40)
Assoc. Legendre P_ℓ^m	$1-x^2$	$m^2/(1-x^2)$	1	$(-1, 1)$	(5.66)
Bessel (ν fixed)	r	ν^2/r	r	$(0, a)$	(4.111)
Hermite H_n	e^{-x^2}	0	e^{-x^2}	$\mathbb{R}$	(6.8)
Laguerre $L_n^{(\alpha)}$	$x^{\alpha+1}e^{-x}$	0	$x^\alpha e^{-x}$	$(0, \infty)$	(6.27)
Chebyshev T_n	$\sqrt{1-x^2}$	0	$1/\sqrt{1-x^2}$	$(-1, 1)$	(6.39)

Worked check: Hermite from a variational principle. For Hermite, the Sturm–Liouville form of equation (6.8) is

$$-(e^{-x^2}y')' = \lambda\,e^{-x^2}y, \qquad \lambda = 2n.$$

Multiplying out the derivative and dividing by e^{-x^2} recovers $y'' - 2xy' + \lambda y = 0$, which is (6.8) at $\lambda = 2n$. So the Rayleigh quotient is

$$\mathcal{R}[y] = \frac{\int_{-\infty}^{\infty} y'(x)^2\,e^{-x^2}\,dx}{\int_{-\infty}^{\infty} y(x)^2\,e^{-x^2}\,dx}.$$

Its minimum over nonzero y vanishes ($y \equiv 1$ achieves the value 0), realized by $H_0 \equiv 1$ with eigenvalue $\lambda_0 = 0 = 2 \cdot 0$. Minimizing among y orthogonal to H_0 in the weight e^{-x^2} gives the next eigenvalue $\lambda_1 = 2$, realized by $H_1(x) = 2x$, and so on. Going back through the substitution $\psi = ye^{-x^2/2}$, a one-line calculation (integrate $-2xyy'e^{-x^2}$ by parts) gives

$$\int_{\mathbb{R}} \left[\tfrac{1}{2}\psi'^2 + \tfrac{1}{2}x^2\psi^2\right]dx = \tfrac{1}{2}\int_{\mathbb{R}} y'^2 e^{-x^2}\,dx + \tfrac{1}{2}\int_{\mathbb{R}} y^2 e^{-x^2}\,dx, \qquad \int_{\mathbb{R}} \psi^2\,dx = \int_{\mathbb{R}} y^2 e^{-x^2}\,dx,$$

so the Hermite Rayleigh quotient $\mathcal{R}[y]$ and the Schrödinger expectation value are related by $\langle \hat{H} \rangle_\psi = \frac{1}{2}(\mathcal{R}[y] + 1)$. Hence the quantum-oscillator energies $E_n = n + 1/2$ from Example 6.6 acquire the variational characterization

$$E_n = \min_{\substack{\psi \perp \psi_0, \dots, \psi_{n-1} \\ \|\psi\|_{L^2(\mathbb{R})} = 1}} \int_{\mathbb{R}} \left[\tfrac{1}{2}\psi'^2 + \tfrac{1}{2}x^2\psi^2\right]dx,$$

the unweighted L^2 Rayleigh quotient for the Schrödinger Hamiltonian. This is the standard *Rayleigh–Ritz* basis of numerical quantum mechanics.

Why orthogonality is automatic. The integration-by-parts identity used to prove orthogonality (Theorem 5.15 for Legendre, Proposition 5.18 for P_ℓ^m, Theorem 4.19 for Bessel) is exactly the symmetry $\langle Ly_1, y_2 \rangle_w = \langle y_1, Ly_2 \rangle_w$ of the Sturm–Liouville operator $L = -(d/dx)(p\,d/dx) + q$ with respect to the weighted inner product $\langle f, g \rangle_w = \int_a^b f\,g\,w\,dx$. Eigenfunctions for distinct eigenvalues are orthogonal in this inner product, and the same symmetry is what makes the boundary terms vanish in each of those theorems. So the Sturm–Liouville framing is not a renaming of the orthogonality theorems — it is the single principle behind all of them.

7.11.2 The brachistochrone transit time is a Beta integral

The brachistochrone solution in Example 7.8 identified the curve but did not compute the transit time. Doing it explicitly turns the result of Section 7 into a Beta value from Section 3.

Take the cycloid (7.11), $x(\theta) = R(\theta - \sin\theta)$, $y(\theta) = R(1 - \cos\theta)$. The transit time is

$$T = \int_0^{\theta_1} \frac{ds}{v}.$$

From the body of Example 7.8,

$$ds = 2R\sin(\theta/2)\,d\theta, \qquad v = 2\sqrt{gR}\sin(\theta/2),$$

so $ds/v = \sqrt{R/g}\,d\theta$ and

$$T(\theta_1) = \sqrt{\frac{R}{g}}\,\theta_1. \tag{7.49}$$

This is the closed form in the natural parameter. To exhibit it as a Beta integral, change variable to $u = y/(2R) \in [0,1]$ along the full half-cycloid $\theta_1 = \pi$, where $y_1 = R(1 - \cos\pi) = 2R$. From $y = 2R\sin^2(\theta/2)$,

$$\sin(\theta/2) = \sqrt{u}, \qquad \cos(\theta/2) = \sqrt{1-u}, \qquad dy = 2R\,du.$$

Along the cycloid $ds = 2R\sin(\theta/2)\,d\theta$ and $dy = R\sin\theta\,d\theta = 2R\sin(\theta/2)\cos(\theta/2)\,d\theta$, so $ds/dy = 1/\cos(\theta/2) = (1-u)^{-1/2}$. The transit time is $T = \int (ds/v)$ with $v = \sqrt{2gy} = 2\sqrt{gRu}$.

Therefore

$$T(\pi) = \int_0^{2R} \frac{ds/dy}{\sqrt{2gy}}\,dy = \int_0^1 \frac{(1-u)^{-1/2}}{2\sqrt{gRu}}\,2R\,du$$
$$= \sqrt{\frac{R}{g}} \int_0^1 u^{-1/2}(1-u)^{-1/2}\,du = \sqrt{\frac{R}{g}}\,B\big(\tfrac{1}{2},\tfrac{1}{2}\big) = \pi\sqrt{\frac{R}{g}}, \tag{7.50}$$

using $B(1/2,1/2) = \Gamma(1/2)^2/\Gamma(1) = \pi$ from the Beta–Gamma identity (Proposition 3.9) and $\Gamma(1/2) = \sqrt{\pi}$ (Corollary 3.14). The answer agrees with the direct evaluation (7.49) at $\theta_1 = \pi$. So the brachistochrone half-period *is* the Beta integral $B(1/2,1/2)$ in disguise; the appearance of π is the same π as in $\Gamma(1/2) = \sqrt{\pi}$. This is the only place in these notes where Sections 7 and 3 cross paths, and it does so cleanly.

7.11.3 Stationary action, steepest descent, and the WKB picture

Action as the phase of an oscillatory integral. In quantum mechanics, transition amplitudes are oscillatory integrals over paths,

$$K(q_b, t_b\,; q_a, t_a) \;=\; \int_{\gamma:q(t_a)=q_a}^{q(t_b)=q_b} \exp\Big(\frac{i}{\hbar}S[\gamma]\Big)\,\mathcal{D}\gamma, \qquad S[\gamma] = \int_{t_a}^{t_b} L\big(q,\dot{q},t\big)\,dt. \tag{7.51}$$

We will not need the path integral as a defined object; what we need is the variable-by-variable analogue. For each fixed finite-dimensional discretization, the integrand has the form $e^{iS/\hbar} \times$ slowly varying studied by stationary phase (Theorem 2.14). As $\hbar \to 0$, the integral is dominated by the points where the phase S is stationary, i.e. by the curves γ satisfying

$$\delta S[\gamma] \;=\; 0.$$

By Theorem 7.3, these are exactly the Euler–Lagrange solutions: the *classical* trajectories. The leading-order asymptotic value of (7.51) is therefore $e^{iS_{\mathrm{cl}}/\hbar}$ times a Gaussian fluctuation integral, with Gaussian width set by the second variation $\delta^2 S$ at the classical path. This is Feynman's path-integral derivation of classical mechanics in one paragraph, and it explicitly uses the same saddle-point machinery as Section 2.5.

WKB connection (filling out the sketch from Section 2.6.14). For the one-dimensional time-independent Schrödinger equation $-\hbar^2\psi''/2 + V\psi = E\psi$, write $\psi = e^{iW/\hbar}$ and expand $W = W_0 + \hbar W_1 + \cdots$. The leading-order eikonal equation $\frac{1}{2}(W_0')^2 + V = E$ has the explicit solution

$$W_0(x) \;=\; \pm \int^x \sqrt{2(E - V(x'))}\,dx',$$

which is exactly the abbreviated classical action $S_0 = \int p\,dq$ for the Hamiltonian $H = p^2/2 + V$ at energy E, with $p = \sqrt{2(E-V)}$ obtained from Section 7.10. So a complex-analytic identity (the eikonal expansion of the wave function) reproduces a classical-mechanics quantity (the abbreviated action). The connection formula across a turning point $E = V$ is exactly the Airy matching computed in Sections 2.6.7 and 2.6.8: outside the classically allowed region the WKB function decays like $e^{-2x^{3/2}/3}/(2\sqrt{\pi}x^{1/4})$ (matching (2.24)); inside it oscillates like $\cos(2x^{3/2}/3 - \pi/4)/(\sqrt{\pi}x^{1/4})$ (matching (2.31)). The same cubic Hankel contour that yielded $J_\nu(\nu) \sim 2^{1/3}/[3^{2/3}\Gamma(2/3)]\,\nu^{-1/3}$ in Theorem 4.14 reappears here as the Airy turning-point matching: the variational object (Hamilton's action) and the special-function object (Airy/Bessel) are coupled by the very same saddle-point asymptotics.

Hamilton–Jacobi as a method-of-characteristics. The Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0$$

is a nonlinear first-order PDE for $S(t, q)$. Its method of characteristics is exactly Hamilton's equations (7.44). For separable Hamiltonians one writes $S = S_1(q^1) + S_2(q^2) + \cdots$, and each separated piece is a one-dimensional Sturm–Liouville eigenfunction integral; quantizing the action $\oint p\,dq = 2\pi\hbar(n + \frac{1}{2})$ then recovers Bohr–Sommerfeld energies. The hydrogen radial Laguerre quantization of Example 6.12 and the harmonic-oscillator Hermite quantization of Example 6.6 fall out of this scheme.

7.11.4 Summary: one diagram

We can summarize the structural arrows in the notes as

$$\underbrace{\text{action } S[\gamma]}_{\S 7} \xrightarrow{\delta S = 0} \underbrace{\text{EL/Sturm–Liouville ODE}}_{\S 4\text{–}6} \xleftarrow{\text{coeffs}} \underbrace{\text{generating function / contour}}_{\S 1, 3}$$

and the asymptotics of all three boxes is supplied by the saddle-point methods of Section 2. The complex-analytic methods of §1–2 *construct* the special functions; the variational principle of §7 *characterizes* them as critical curves. The two viewpoints are dual, and a working physicist uses both.

Exercises

Problem 7.1. *Derive the EL equation for $\mathcal{L}[y] = \int_0^1 (y'^2 + y^2)\,dx$ with $y(0) = 0$, $y(1) = 1$, and solve for the extremal. Compute $\mathcal{L}$ on the extremal.*

Problem 7.2. *Find the rotational stationary surfaces connecting two parallel circles of equal radius y_0 in the planes $x = \pm L$. Using the catenoid formula (7.12), show that a catenoid solution exists if and only if y_0/L is at least a critical value, determine that value numerically, and explain why equality gives one catenoid while strict inequality gives two.*

Problem 7.3. *Verify that the parametric cycloid $(x, y) = (R(\theta - \sin\theta), R(1 - \cos\theta))$ also satisfies the original EL equation for the brachistochrone, not only the Beltrami first integral. Compute the transit time T as a function of R and θ_1.*

Problem 7.4. *Derive the geodesic equations on the round unit sphere starting from the Lagrangian $L = \sqrt{\dot\theta^2 + \sin^2\theta\,\dot\phi^2}$. Identify the Clairaut conserved quantity, integrate, and show the geodesics are great circles.*

Problem 7.5. *For the harmonic oscillator $L = \frac{1}{2}\dot q^2 - \frac{1}{2}\omega^2 q^2$, verify the phase-space rotation $q \mapsto q\cos\alpha + (p/\omega)\sin\alpha$, $p \mapsto p\cos\alpha - \omega q\sin\alpha$, where $p = \dot q$. Find the conserved generator and show it equals the total energy divided by ω.*

Problem 7.6. *For a relativistic free particle, $L = -mc^2\sqrt{1 - |\dot{\mathbf{r}}|^2/c^2}$. Derive the EL equation; show the relativistic momentum is $\mathbf{p} = m\dot{\mathbf{r}}/\sqrt{1 - |\dot{\mathbf{r}}|^2/c^2}$, and compute $H = \mathbf{p}\cdot\dot{\mathbf{r}} - L$. Show $H = \sqrt{|\mathbf{p}|^2c^2 + m^2c^4}$.*

Problem 7.7. *Prove that the Legendre transform is an involution, meaning that applying the dual construction twice returns the original function: if* $H(q,p) = \sup_{\dot q}\big[p\dot q - L(q,\dot q)\big]$ *and* L *is smooth, strictly convex in* $\dot q$ *(curves upward with no flat line segments), and superlinear in* $\dot q$ *(so* $L(q,v)/\|v\| \to \infty$ *as* $\|v\| \to \infty$*), then* $L(q,\dot q) = \sup_p\big[p\dot q - H(q,p)\big]$*. Here* sup *means least upper bound; under these hypotheses, the displayed suprema are attained maxima.*

Problem 7.8. *Derive Snell's law* (7.16) *directly from Fermat's principle with a continuously varying index* $n(y)$ *by checking the conservation of* $n \sin\theta$ *along a ray. Specialize to a medium with linear profile* $n(y) = n_0 + \alpha y$ *and solve for the ray trajectory.*

Problem 7.9. *(Kepler problem.) For* $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + GMm/|\mathbf{r}|$*, use Noether's theorem to obtain the three conserved components of angular momentum, and verify explicitly* $\{L_i, L_j\} = \epsilon_{ijk}L_k$ *using Poisson brackets. Here* ϵ_{ijk} *is the Levi–Civita symbol: it is* $+1$ *for cyclic permutations of* $(1,2,3)$*,* -1 *for reversed permutations, and* 0 *if any index repeats.*

Problem 7.10. *Compute the second variation of the arclength functional on the plane and show that the straight line is a strict minimum (no conjugate points). Contrast with the sphere (Ex. 7.17).*

Problem 7.11. *Show that the isoperimetric extremals on the sphere (closed curves of fixed length on the unit sphere that maximize enclosed area) are latitude circles. Hint: use spherical coordinates with area element* $dA = \sin\theta\, d\theta\, d\phi$ *and perimeter* $P = \int \sqrt{\dot\theta^2 + \sin^2\theta\, \dot\phi^2}\, dt$*.*

Problem 7.12. *(Scalar field theory.) For* $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \frac{1}{2}m^2\phi^2$*, derive the Klein–Gordon equation* $(\Box + m^2)\phi = 0$ *from* (7.18)*. Using translation invariance, compute the conserved energy-momentum (stress) tensor* $T^{\mu\nu}$*.*

Problem 7.13. *(Rotation in 2D.) Let* $L = \frac{1}{2}(\dot x^2 + \dot y^2) - V(\sqrt{x^2 + y^2})$*. Take the rotation* $(x,y) \mapsto (x - \varepsilon y, y + \varepsilon x)$*, verify it is a symmetry, and compute the Noether charge. Confirm it equals* $L_z = x\dot y - y\dot x$*.*

Problem 7.14. *For* $H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\omega^2(x^2 + y^2)$*, compute the Poisson brackets among* $A = \frac{1}{2}(p_x^2 - p_y^2) + \frac{1}{2}\omega^2(x^2 - y^2)$*,* $B = p_x p_y + \omega^2 xy$*,* $L_z = xp_y - yp_x$*, and identify the* $SU(2)$ *algebra structure.*

Problem 7.15. *Use the Beltrami identity to derive the shape of a light ray in a medium with* $n(y) = n_0/y$ *(upper half-plane) and show that rays are circular arcs with centers on the* x*-axis.*

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