Relativistic Electrodynamics

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Outline

1. Special Relativity and Lorentz Transformation
2. Lorentz Transformation of Maxwell Equations

Chapter Overview

This chapter bridges classical electromagnetism and the relativistic framework, demonstrating that Maxwell equations are inherently consistent with Einstein's special relativity. We begin by reviewing the foundational postulates and deriving the Lorentz transformation for spacetime coordinates. The corresponding transformation laws for the electric and magnetic fields are then developed, illustrating how these quantities are perceived by observers in relative motion. These ideas form the basis for many modern technologies—such as Global Positioning System (GPS) timing corrections, particle-accelerator beam dynamics, and relativistic charged-particle radiation—highlighting the practical importance of relativity in electromagnetic applications.

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1. Special Relativity and Lorentz Transformation

2. Lorentz Transformation of Maxwell Equations

Special Relativity and Lorentz Transformation

The Lorentz transformation describes how space and time coordinates transform between inertial frames moving at constant velocity relative to each other. Its form is determined by two fundamental postulates that underlie the theory of special relativity:

(P1) The laws of physics are the same in all inertial frames (principle of relativity).

(P2) The speed of light in vacuum, c, is the same for all inertial observers, independent of the motion of the source or the observer.

Postulate (P1) requires that transformations between inertial frames preserve the form of physical laws. In particular, it implies that the transformation must be *linear*, since non-linear mappings would distort inertial trajectories, thereby violating Newton's first law. Furthermore, (P1) enforces *spatial isotropy* and *homogeneity of space and time*, precluding any preferred direction or location in space-time. These constraints collectively restrict the transformation to be linear and symmetric in form.
Postulate (P2) imposes an additional and crucial constraint: among all permissible linear transformations, only those that preserve the constancy of the speed of light in all directions and in all inertial frames are valid. This requirement excludes Galilean transformations (which presumes the concept of absolute spacetime) and uniquely leads to the Lorentz transformation. The invariance of the light speed in vacuum necessitates

a mixing of space and time coordinates between frames, revealing their fundamentally intertwined nature in relativistic physics.

Consider two inertial frames, S and S', where S' moves with a constant velocity $\vec{v}$ relative to S. For simplicity, we let the origins of the two frames coincide at $t = t' = 0$. Suppose a light pulse is emitted from the origin at time $t = 0$. The wavefront expands radially at the speed of light c in all directions. Denoting by $\hat{n}$ the unit vector specifying the direction of propagation, the spatial and temporal coordinates of the wavefront satisfy

$$\vec{r} = ct\hat{n} \implies \vec{r}\cdot\vec{r} - c^2t^2 = 0. \tag{5.1.1}$$

Since the speed of light is the same in all inertial frames (P2), this relation must hold when the same light pulse is described in any inertial frame, and thus

$$\vec{r}\,'\cdot\vec{r}\,' - c^2t'^2 = \vec{r}\cdot\vec{r} - c^2t^2. \tag{5.1.2}$$

Let us define the spacetime interval as

$$s^2 = \vec{r}\cdot\vec{r} - c^2t^2. \tag{5.1.3}$$

Special Relativity and Lorentz Transformation (cont.)

Extending to the spacetime intervals of the same event in different inertial frames, the most general relation between the spacetime intervals in S and S' may be written as

$$s'^2 = \kappa(v) \left(s^2\right)^{\tau(v)}, \tag{5.1.4}$$

where $\kappa(v)$ and $\tau(v)$ are functions of the relative speed v between the two frames. Since by (P1) it is required that the transformation is linear, and this requirement is satisfied only when the exponent satisfies $\tau(v) = 1$, so that the transformation reduces to

$$s'^2 = \kappa(v)\, s^2.$$

Because the two inertial frames S and S' are on equal footing, exchanging their roles must leave the relation unchanged:

$$s^2 = \kappa(v)\, s'^2.$$

Combining the two expressions shows that $\kappa(v)^2 = 1$, so $\kappa(v) = \pm 1$. The negative sign, however, is incompatible with causality. Also notice that when the transformation reduces continuously to $v = 0$ the two frames coincide, so we have $s'^2 = s^2$ and hence $\kappa(0) = 1$. Therefore we must choose

$$\kappa(v) = 1.$$

Hence the intervals in the two frames satisfy

$$s'^2 = s^2. \tag{5.1.5}$$

Thus, we conclude that the spacetime interval (5.1.3) is invariant under transformations between inertial frames.

The structure of the invariant interval suggests a powerful analogy. A standard rotation in a two-dimensional Euclidean plane preserves the distance squared, $d^2 = x^2 + y^2$. Such a rotation is described by a linear transformation parametrized by an angle θ, which works precisely because of the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$.

The Lorentz transformation, by contrast, must preserve the spacetime interval $s^2 = \vec{r} \cdot \vec{r} - c^2 t^2$. For simplicity, let us first consider a boost along the x-axis, where the

Special Relativity and Lorentz Transformation (cont.)

invariant is $s^2 = x^2 - (ct)^2$. By analogy, the transformation must be a hyperbolic rotation parametrized by a quantity ϕ, often called the rapidity:

$$x' = x \cosh\phi - ct \sinh\phi, \tag{5.1.6}$$

$$ct' = -x \sinh\phi + ct \cosh\phi. \tag{5.1.7}$$

This form guarantees the invariance of $x^2 - (ct)^2$ because of the fundamental hyperbolic identity $\cosh^2\phi - \sinh^2\phi = 1$.

Now, we can connect the abstract parameter ϕ to the physical velocity v. The origin of the S' frame is defined by $x' = 0$. From (5.1.6), this gives:

$$0 = x \cosh\phi - ct \sinh\phi \implies \frac{x}{t} = c\frac{\sinh\phi}{\cosh\phi} = c \tanh\phi. \tag{5.1.8}$$

Since $v = x/t$ is the velocity of the S' frame relative to S, we find the crucial link between rapidity and velocity:

$$\beta = \frac{v}{c} = \tanh\phi. \tag{5.1.9}$$

Special Relativity and Lorentz Transformation (cont.)

With this connection, we can express the hyperbolic functions in terms of the dimensionless velocity β. From the identity $\cosh^2\phi - \sinh^2\phi = 1$, we can derive the expressions for $\cosh\phi$ and $\sinh\phi$ in terms of β.

$$\cosh\phi = \frac{1}{\sqrt{1-\tanh^2\phi}} = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma, \tag{5.1.10}$$

$$\sinh\phi = \tanh\phi\cosh\phi = \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}}. \tag{5.1.11}$$

The quantity γ is the celebrated Lorentz factor. Substituting these back into (5.1.6) and (5.1.7) yields the transformation for a boost along the x-axis:

$$x' = \gamma\left(x - \beta ct\right), \tag{5.1.12a}$$

$$ct' = \gamma\left(ct - \beta x\right). \tag{5.1.12b}$$

To generalize the Lorentz transformation to an arbitrary boost direction $\vec{\beta} = \vec{v}/c$, we first examine the behavior of coordinates *perpendicular* to the motion. Consider two identical rulers, one at rest in frame S and the other at rest in frame S', both oriented perpendicular to the relative velocity. When the origins of the two frames coincide, the rulers lie exactly on top of each other along their entire length, so they occupy the same spatial segment at that instant. If an observer in S were to see the perpendicular ruler in S' as shorter, then by the principle of relativity an observer in S' would have to make the same judgment about the ruler in S. But such mutual shortening is impossible because the rulers coincide point-by-point at the moment of comparison; any difference in length would create a mismatch at the endpoints. Hence, lengths perpendicular to the motion cannot change, and the corresponding coordinate components remain invariant under the boost.

In contrast, the case of rulers aligned *parallel* to the motion is fundamentally different. To measure the length of a moving ruler in frame S, the observer in S must record the positions of its two endpoints *simultaneously* at time t. However, the events corresponding to taking simultaneous endpoint measurements in S are *not* simultaneous in S', which is shown by the following argument.

Returning to the case of a boost along the x-axis for simplicity, consider two events whose space–time coordinates in the frame S are (t_1, x_1) and (t_2, x_2), with spatial and temporal separations

$$\Delta x = x_2 - x_1, \qquad \Delta t = t_2 - t_1.$$

Special Relativity and Lorentz Transformation (cont.)

Using (5.1.12), the corresponding spatial and temporal separations in frame S' are

$$\Delta x' = \gamma \left(\Delta x - \beta c \Delta t \right), \tag{5.1.13a}$$

$$c\Delta t' = \gamma \left(c\Delta t - \beta \Delta x \right). \tag{5.1.13b}$$

Alternatively, since S moves with a constant velocity $-v$ relative to S', we have

$$\Delta x = \gamma \left(\Delta x' + \beta c \Delta t' \right), \tag{5.1.14a}$$

$$c\Delta t = \gamma \left(c\Delta t' + \beta \Delta x' \right). \tag{5.1.14b}$$

Suppose now that an observer in S measures two events to be simultaneous but occurring at different locations, i.e.

$$\Delta x \neq 0, \qquad \Delta t = 0.$$

Substituting these conditions into the second equation of (5.1.13) yields

$$c\Delta t' = -\gamma\beta\, \Delta x \neq 0 \qquad (\Delta t = 0). \tag{5.1.15}$$

Therefore, two events that are simultaneous in frame S but spatially separated are *not* simultaneous when observed from frame S'. By the same reasoning, events that are simultaneous in S' at different spatial locations will not be simultaneous in frame S. Thus, unlike the perpendicular case, the two rulers do not share a point-by-point coincidence of their endpoints at a single moment in both frames. This lack of simultaneous spatial overlap removes the symmetry argument used earlier. Following the standard procedure used in length measurements, from (5.1.14), an observer in S will find that the distance between the two endpoints of the moving ruler (measured at the same time t, i.e. $\Delta t = 0$) is

$$\Delta x = \gamma \left(\Delta x' + \beta c \Delta t'\right) = \gamma \, \Delta x' \, (1 - \beta^2) = \Delta x' \sqrt{1 - \beta^2}, \tag{5.1.16}$$

which is shorter than its proper length $\Delta x'$. Conversely, an observer in S' applying the same simultaneity condition $\Delta t' = 0$ will conclude that a ruler moving with respect to S' is also contracted along the direction of motion. Therefore, each inertial frame observes the other's parallel ruler as shorter, and this mutual contraction is fully consistent with the relativity of simultaneity. Only the component of length parallel to the relative motion undergoes Lorentz contraction, while perpendicular components remain unchanged.

Special Relativity and Lorentz Transformation (cont.)

This crucial insight simplifies the problem immensely. Since $\vec{r}\,'_\perp = \vec{r}_\perp$, the entire spacetime-mixing effect of the transformation must occur within the two-dimensional spacetime plane defined by the time axis and the spatial direction parallel to the boost. This reduces the problem for the time and parallel spatial coordinates to the 1+1 dimensional case we have already solved. This leads to the following transformation rules:

1. The component of $\vec{r}$ perpendicular to the velocity, $\vec{r}_\perp$, is unchanged: $\vec{r}\,'_\perp = \vec{r}_\perp$.
2. The time coordinate and the parallel spatial component transform according to the one-dimensional rules. The time transformation generalizes by replacing βx with the scalar projection $\vec{\beta} \cdot \vec{r}$, where $\vec{\beta} = \beta\hat{\beta}$.

Special Relativity and Lorentz Transformation (cont.)

Applying these rules, the time coordinate transformation becomes:

$$ct' = \gamma \left(ct - \vec{\beta} \cdot \vec{r}\right). \tag{5.1.17}$$

For the spatial coordinates, we have $\vec{r}\,' = \vec{r}\,'_{\parallel} + \vec{r}\,'_{\perp}$. With $\vec{r}\,'_{\perp} = \vec{r}_{\perp}$ and $\vec{r}\,'_{\parallel} = \gamma(\vec{r}_{\parallel} - \vec{\beta} ct)$, we can write:

$$\begin{aligned} \vec{r}\,' &= \vec{r}_{\perp} + \gamma\,(\vec{r}_{\parallel} - \vec{\beta} ct) \\ &= (\vec{r} - \vec{r}_{\parallel}) + \gamma\vec{r}_{\parallel} - \gamma\vec{\beta} ct \\ &= \vec{r} + (\gamma - 1)\vec{r}_{\parallel} - \gamma\vec{\beta} ct. \end{aligned} \tag{5.1.18}$$

Using the definition $\vec{r}_{\parallel} = (\vec{\beta} \cdot \vec{r})\vec{\beta}/\beta^2$, we get the full spatial transformation:

$$\vec{r}\,' = \vec{r} + (\gamma - 1)\frac{\vec{\beta} \cdot \vec{r}}{\beta^2}\vec{\beta} - \gamma\vec{\beta} ct. \tag{5.1.19}$$

This expression is often written more compactly using dyadic notation. If we define the symmetric tensor

$$\bar{\bar{\alpha}} = \bar{\bar{I}} + (\gamma - 1)\frac{\vec{\beta}\vec{\beta}}{\beta^2}, \tag{5.1.20}$$

Special Relativity and Lorentz Transformation (cont.)

where $\bar{\bar{I}}$ is the identity tensor and $\vec{\beta}\vec{\beta}$ is the outer product, the spatial transformation is:

$$\vec{r}\,' = \bar{\bar{\alpha}} \cdot \vec{r} - \gamma\vec{\beta}ct. \tag{5.1.21}$$

(5.1.17) and (5.1.21) form the complete, general Lorentz transformation for a boost in any direction $\vec{v}$.

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2. Lorentz Transformation of Maxwell Equations

Lorentz Transformation of Maxwell Equations

Transformation formulas for electromagnetic field vectors are direct consequences of the Lorentz transformation for space and time. From the Lorentz transformation given in the previous section, the following transformation formulas can be derived:

$$\begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} = \gamma \begin{bmatrix} \bar{\bar{\alpha}}^{-1} & \bar{\bar{\beta}} \\ -\bar{\bar{\beta}} & \bar{\bar{\alpha}}^{-1} \end{bmatrix} \cdot \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix} \tag{5.2.1}$$

and

$$\begin{bmatrix} c\vec{D}' \\ \vec{H}' \end{bmatrix} = \gamma \begin{bmatrix} \bar{\bar{\alpha}}^{-1} & \bar{\bar{\beta}} \\ -\bar{\bar{\beta}} & \bar{\bar{\alpha}}^{-1} \end{bmatrix} \cdot \begin{bmatrix} c\vec{D} \\ \vec{H} \end{bmatrix}. \tag{5.2.2}$$

Here, $\bar{\bar{\alpha}}^{-1}$ represents the inverse of $\bar{\bar{\alpha}}$, which is given by:

$$\bar{\bar{\alpha}}^{-1} = \bar{\bar{I}} + \left(\frac{1}{\gamma} - 1\right) \frac{\vec{\beta}\vec{\beta}}{\beta^2} = \bar{\bar{\alpha}} - \gamma\vec{\beta}\vec{\beta}. \tag{5.2.3}$$

The 3×3 matrix $\bar{\bar{\beta}}$ has the explicit matrix form:

$$\bar{\bar{\beta}} = \begin{bmatrix} 0 & -\beta_z & \beta_y \\ \beta_z & 0 & -\beta_x \\ -\beta_y & \beta_x & 0 \end{bmatrix}. \tag{5.2.4}$$

Lorentz Transformation of Maxwell Equations (cont.)

It can be found that for any vector $\vec{u}$, $\bar{\bar{\beta}} \cdot \vec{u} \equiv \vec{\beta} \times \vec{u}$. From the operator identity

$$\bar{\bar{\beta}}^2 \cdot \vec{u} = \bar{\bar{\beta}} \cdot (\bar{\bar{\beta}} \cdot \vec{u}) = \vec{\beta} \times (\vec{\beta} \times \vec{u}) = \vec{\beta}\vec{\beta} \cdot \vec{u} - \beta^2 \vec{u},$$

it follows that:

$$\bar{\bar{\beta}}^2 = \vec{\beta}\vec{\beta} - \beta^2 \bar{\bar{I}}. \tag{5.2.5}$$

While both $\bar{\bar{\alpha}}$ and $\bar{\bar{\alpha}}^{-1}$ are symmetric, $\bar{\bar{\beta}}$ is skew-symmetric.
To prove this, we first deduce the Lorentz transformation for space-time derivatives.
This derivation utilizes the chain rule in differentiation:

$$\partial_{ct} = \left[\partial_{ct}(ct')\right] \partial_{ct'} + \left[\partial_{ct} x_i'\right] \partial_{x_i'}, \tag{5.2.6}$$

$$\partial_{x_i} = \left[\partial_{x_i}(ct')\right] \partial_{ct'} + \left[\partial_{x_i} x_j'\right] \partial_{x_j'}. \tag{5.2.7}$$

Substituting the Lorentz transformation and acknowledging the symmetrical nature of $\bar{\bar{\alpha}}$, we obtain:

$$\partial_{ct} = \gamma \partial_{ct'} - \gamma \vec{\beta} \cdot \vec{\nabla}', \tag{5.2.8}$$

$$\vec{\nabla} = \bar{\bar{\alpha}} \cdot \vec{\nabla}' - \gamma \vec{\beta} \partial_{ct'}. \tag{5.2.9}$$

Lorentz Transformation of Maxwell Equations (cont.)

To derive the transformation laws for all field vectors, we substitute (5.2.8) and (5.2.9) into the Maxwell equations in the S frame, requiring them to maintain the same form in the S' frame.
Let us first incorporate (5.2.8) and (5.2.9) into Gauss' magnetic field law and Faraday's law:

$$\left(\bar{\bar{\alpha}} \cdot \vec{\nabla}' - \gamma\vec{\beta}\partial_{ct'}\right) \cdot c\vec{B} = 0, \tag{5.2.10}$$

$$\left(\bar{\bar{\alpha}} \cdot \vec{\nabla}' - \gamma\vec{\beta}\partial_{ct'}\right) \times \vec{E} + \gamma\left(\partial_{ct'} - \vec{\beta} \cdot \vec{\nabla}'\right) c\vec{B} = 0. \tag{5.2.11}$$

To determine the transformation laws for $\vec{E}'$ and $c\vec{B}'$, our objective is to express (5.2.10) and (5.2.11) in the forms:

$$\vec{\nabla}' \cdot c\vec{B}' = 0, \tag{5.2.12}$$

$$\vec{\nabla}' \times \vec{E}' + \partial_{ct'} c\vec{B}' = 0. \tag{5.2.13}$$

Consider γ(5.2.10)$+\gamma\vec{\beta}\cdot$(5.2.11):

$$\gamma\left[\left(\bar{\bar{\alpha}} \cdot \vec{\nabla}'\right) \cdot c\vec{B} + \vec{\beta} \cdot \left(\bar{\bar{\alpha}} \cdot \vec{\nabla}'\right) \times \vec{E} - \gamma(\vec{\beta} \cdot \vec{\nabla}')\left(\vec{\beta} \cdot c\vec{B}\right)\right] = 0.$$

Lorentz Transformation of Maxwell Equations (cont.)

Using

$$\vec{\beta}\cdot[(\bar{\bar{\alpha}}\cdot\vec{\nabla}')\times\vec{E}] = \vec{\beta}\cdot\left[\left(\vec{\nabla}'+(\gamma-1)\frac{\vec{\beta}\cdot\vec{\nabla}'}{\beta^2}\vec{\beta}\right)\times\vec{E}\right]$$
$$= \vec{\beta}\cdot(\vec{\nabla}'\times\vec{E}) = -\vec{\nabla}'\cdot(\vec{\beta}\times\vec{E}),$$

where the term proportional to $\vec{\beta}\times\vec{E}$ is perpendicular to $\vec{\beta}$, and using (5.2.3), we can show that:

$$\vec{\nabla}'\cdot[\gamma\bar{\bar{\alpha}}^{-1}\cdot c\vec{B} - \gamma\vec{\beta}\times\vec{E}] = 0. \tag{5.2.14}$$

Similarly, consider $\bar{\bar{\alpha}}\cdot$(5.2.11)$+\gamma\vec{\beta}$(5.2.10):

$$\bar{\bar{\alpha}}\cdot\left(\bar{\bar{\alpha}}\cdot\vec{\nabla}'\right)\times\vec{E} - \gamma\bar{\bar{\alpha}}\cdot\left(\vec{\beta}\cdot\vec{\nabla}'\right)c\vec{B} + \gamma\vec{\beta}\left(\bar{\bar{\alpha}}\cdot\vec{\nabla}'\right)c\vec{B}$$
$$+\partial_{ct'}\left[-\gamma\bar{\bar{\alpha}}\cdot\left(\vec{\beta}\times\vec{E}\right)+\gamma\left(\bar{\bar{\alpha}}\cdot c\vec{B}\right)-\gamma^2\vec{\beta}\vec{\beta}\cdot c\vec{B}\right] = 0.$$

Applying the identity $\bar{\bar{\alpha}}\cdot[(\bar{\bar{\alpha}}\cdot\vec{\nabla}')\times\vec{E}] = \vec{\nabla}'\times(\gamma\bar{\bar{\alpha}}^{-1}\cdot\vec{E})$ and the identity $\vec{\beta}[(\bar{\bar{\alpha}}\cdot\vec{\nabla}')c\vec{B}] - \bar{\bar{\alpha}}\cdot(\vec{\beta}\cdot\vec{\nabla}')c\vec{B} = \vec{\beta}(\vec{\nabla}'\cdot c\vec{B}) - (\vec{\beta}\times\vec{\nabla}')c\vec{B} = \vec{\nabla}'\times(\vec{\beta}\times c\vec{B})$, we proceed term by term. The first identity follows from the cofactor rule

$$(\bar{\bar{\alpha}}\cdot\vec{a})\times(\bar{\bar{\alpha}}\cdot\vec{b}) = (\det\bar{\bar{\alpha}})\bar{\bar{\alpha}}^{-1}\cdot(\vec{a}\times\vec{b}), \qquad \det\bar{\bar{\alpha}} = \gamma,$$

Lorentz Transformation of Maxwell Equations (cont.)

so, writing $\vec{E} = \bar{\bar{\alpha}} \cdot (\bar{\bar{\alpha}}^{-1} \cdot \vec{E})$,

$$\bar{\bar{\alpha}} \cdot [(\bar{\bar{\alpha}} \cdot \vec{\nabla}') \times \vec{E}] = \gamma \, \vec{\nabla}' \times (\bar{\bar{\alpha}}^{-1} \cdot \vec{E}) = \vec{\nabla}' \times (\gamma \bar{\bar{\alpha}}^{-1} \cdot \vec{E}).$$

The two remaining spatial terms contain the common factor γ and become

$$\gamma \vec{\beta}[(\bar{\bar{\alpha}} \cdot \vec{\nabla}') c\vec{B}] - \gamma \bar{\bar{\alpha}} \cdot (\vec{\beta} \cdot \vec{\nabla}') c\vec{B} = \vec{\nabla}' \times (\gamma \vec{\beta} \times c\vec{B}).$$

The time-derivative bracket collapses by $\bar{\bar{\alpha}}^{-1} = \bar{\bar{\alpha}} - \gamma \vec{\beta}\vec{\beta}$ and by the fact that $\bar{\bar{\alpha}} \cdot (\vec{\beta} \times \vec{E}) = \vec{\beta} \times \vec{E}$:

$$\begin{aligned} & -\gamma \bar{\bar{\alpha}} \cdot (\vec{\beta} \times \vec{E}) + \gamma \bar{\bar{\alpha}} \cdot c\vec{B} - \gamma^2 \vec{\beta}\vec{\beta} \cdot c\vec{B} \\ & \qquad = -\gamma \vec{\beta} \times \vec{E} + \gamma(\bar{\bar{\alpha}} - \gamma \vec{\beta}\vec{\beta}) \cdot c\vec{B} = \gamma \bar{\bar{\alpha}}^{-1} \cdot c\vec{B} - \gamma \vec{\beta} \times \vec{E}. \end{aligned}$$

Substitution gives:

$$\vec{\nabla}' \times \left(\gamma \bar{\bar{\alpha}}^{-1} \cdot \vec{E} + \gamma \vec{\beta} \times c\vec{B}\right) + \partial_{ct'} \left(\gamma \bar{\bar{\alpha}}^{-1} \cdot c\vec{B} - \gamma \vec{\beta} \times \vec{E}\right) = 0. \tag{5.2.15}$$

Lorentz Transformation of Maxwell Equations (cont.)

By comparing (5.2.12), (5.2.13), (5.2.14), and (5.2.15), we derive the transformation formulas for $\vec{E}$ and $\vec{B}$:

$$\vec{E}' = \gamma\bar{\bar{\alpha}}^{-1} \cdot \vec{E} + \gamma\vec{\beta} \times c\vec{B}, \tag{5.2.16}$$

$$c\vec{B}' = \gamma\bar{\bar{\alpha}}^{-1} \cdot c\vec{B} - \gamma\vec{\beta} \times \vec{E}. \tag{5.2.17}$$

We can now express the Lorentz transformation formulas for $\vec{E}$ and $c\vec{B}$ as in (5.2.1). For $c\vec{D}$ and $\vec{H}$, start instead with Gauss' electric law and Ampere's law in frame S,

$$\vec{\nabla} \cdot c\vec{D} = c\rho, \qquad \vec{\nabla} \times \vec{H} - \partial_{ct}(c\vec{D}) = \vec{J}.$$

After substituting (5.2.8) and (5.2.9), these become

$$(\bar{\bar{\alpha}} \cdot \vec{\nabla}' - \gamma\vec{\beta}\partial_{ct'}) \cdot c\vec{D} = c\rho,$$

$$(\bar{\bar{\alpha}} \cdot \vec{\nabla}' - \gamma\vec{\beta}\partial_{ct'}) \times \vec{H} - \gamma(\partial_{ct'} - \vec{\beta} \cdot \vec{\nabla}')c\vec{D} = \vec{J}.$$

Now form γ times the first equation minus $\gamma\vec{\beta}\cdot$ the second. The time-derivative terms cancel, and the same scalar-triple-product identity used above gives

$$\vec{\nabla}' \cdot \left(\gamma\bar{\bar{\alpha}}^{-1} \cdot c\vec{D} + \gamma\vec{\beta} \times \vec{H}\right) = \gamma(c\rho - \vec{\beta} \cdot \vec{J}).$$

Lorentz Transformation of Maxwell Equations (cont.)

Likewise, $\bar{\bar{\alpha}}\cdot$ the second equation minus $\gamma\vec{\beta}$ times the first gives

$$\vec{\nabla}' \times \left(\gamma\bar{\bar{\alpha}}^{-1}\cdot\vec{H} - \gamma\vec{\beta}\times c\vec{D}\right) - \partial_{ct'}\left(\gamma\bar{\bar{\alpha}}^{-1}\cdot c\vec{D} + \gamma\vec{\beta}\times\vec{H}\right)$$
$$= \bar{\bar{\alpha}}\cdot\vec{J} - \gamma\vec{\beta}c\rho.$$

These are exactly $\vec{\nabla}'\cdot c\vec{D}' = c\rho'$ and $\vec{\nabla}'\times\vec{H}' - \partial_{ct'}(c\vec{D}') = \vec{J}'$ when

$$c\rho' = \gamma(c\rho - \vec{\beta}\cdot\vec{J}), \qquad \vec{J}' = \bar{\bar{\alpha}}\cdot\vec{J} - \gamma\vec{\beta}c\rho,$$

and $c\vec{D}', \vec{H}'$ are identified as in (5.2.2).
Consider the charge conservation equation. Transforming from frame S to S' yields:

$$\left(\bar{\bar{\alpha}}\cdot\vec{\nabla}' - \gamma\vec{\beta}\partial_{ct'}\right)\cdot\vec{J} + \gamma\left(\partial_{ct'} - \vec{\beta}\cdot\vec{\nabla}'\right)c\rho = 0. \tag{5.2.18}$$

To put this into the form $\vec{\nabla}'\cdot\vec{J}' + \partial_{ct'}(c\rho') = 0$, collect the $\vec{\nabla}'$ terms and the $\partial_{ct'}$ terms separately. Since $\bar{\bar{\alpha}}$ is symmetric and $\vec{\beta}$ is constant,

$$(\bar{\bar{\alpha}}\cdot\vec{\nabla}')\cdot\vec{J} - \gamma(\vec{\beta}\cdot\vec{\nabla}')c\rho = \vec{\nabla}'\cdot\left(\bar{\bar{\alpha}}\cdot\vec{J} - \gamma\vec{\beta}c\rho\right),$$
$$-\gamma\partial_{ct'}(\vec{\beta}\cdot\vec{J}) + \gamma\partial_{ct'}(c\rho) = \partial_{ct'}\left[\gamma(c\rho - \vec{\beta}\cdot\vec{J})\right].$$

Lorentz Transformation of Maxwell Equations (cont.)

Thus (5.2.18) becomes

$$\vec{\nabla}' \cdot \left(\bar{\bar{\alpha}} \cdot \vec{J} - \gamma\vec{\beta}c\rho\right) + \partial_{ct'}\left[\gamma(c\rho - \vec{\beta}\cdot\vec{J})\right] = 0.$$

Consequently, the charge conservation equation is Lorentz-covariant if:

$$c\rho' = \gamma\left(c\rho - \vec{\beta}\cdot\vec{J}\right), \tag{5.2.19}$$

$$\vec{J}' = \bar{\bar{\alpha}} \cdot \vec{J} - \gamma\vec{\beta}c\rho. \tag{5.2.20}$$

It is important to note that a charge distribution stationary in frame S will inevitably generate a current in frame S'. Furthermore, as indicated by the above equations, a uniform current element in S also gives rise to a charge distribution in S'.
For the transformation of $\vec{E}$ and $c\vec{B}$ as described in (5.2.1), we decompose the field vectors into components parallel and perpendicular to the velocity $\vec{v}$. It is noteworthy that the field components parallel to the velocity remain invariant:

$$\vec{E}'_{||} = \vec{E}_{||}, \tag{5.2.21}$$

$$\vec{B}'_{||} = \vec{B}_{||}, \tag{5.2.22}$$

$$\vec{D}'_{||} = \vec{D}_{||}, \tag{5.2.23}$$

Lorentz Transformation of Maxwell Equations (cont.)

$$\vec{H}'_{||} = \vec{H}_{||}. \tag{5.2.24}$$

Conversely, the perpendicular components transform as follows:

$$\vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \vec{\beta} \times c\vec{B}_{\perp}\right), \tag{5.2.25}$$

$$c\vec{B}'_{\perp} = \gamma\left(c\vec{B}_{\perp} - \vec{\beta} \times \vec{E}_{\perp}\right), \tag{5.2.26}$$

$$c\vec{D}'_{\perp} = \gamma\left(c\vec{D}_{\perp} + \vec{\beta} \times \vec{H}_{\perp}\right), \tag{5.2.27}$$

$$\vec{H}'_{\perp} = \gamma\left(\vec{H}_{\perp} - \vec{\beta} \times c\vec{D}_{\perp}\right). \tag{5.2.28}$$

This behavior contrasts with the transformation of space coordinates, where the perpendicular components are typically left unchanged.
According to the Lorentz transformation (5.2.25), a pure magnetic field $\vec{B}$ in frame S induces an electric field $\vec{E}'$ in frame S'. This implies that a voltage is generated in a moving conductor when its velocity vector has a component perpendicular to the magnetic field lines. Conversely, according to the Lorentz transformation (5.2.26), a pure electric field $\vec{E}$ in frame S is observed as a magnetic field from a moving frame. Thus, a stationary electron, when viewed from a moving frame, will exhibit a magnetic field.

Lorentz Transformation of Maxwell Equations (cont.)

Now, let us find relations that are invariant under the Lorentz transformation. We first designate the 6×6 matrix appearing in (5.2.2) and (5.2.1) as $\bar{\bar{\Lambda}}$:

$$\bar{\bar{\Lambda}}(\vec{\beta}) = \gamma \begin{bmatrix} \bar{\bar{\alpha}}^{-1} & \bar{\bar{\beta}} \\ -\bar{\bar{\beta}} & \bar{\bar{\alpha}}^{-1} \end{bmatrix}. \tag{5.2.29}$$

The inverse transformation is determined by the inverse of $\bar{\bar{\Lambda}}(\vec{\beta})$. It can be verified that:

$$\bar{\bar{\Lambda}}^{-1}(\vec{\beta}) = \bar{\bar{\Lambda}}(-\vec{\beta}) = \gamma \begin{bmatrix} \bar{\bar{\alpha}}^{-1} & -\bar{\bar{\beta}} \\ \bar{\bar{\beta}} & \bar{\bar{\alpha}}^{-1} \end{bmatrix}. \tag{5.2.30}$$

From a physical perspective, inverting a pure Lorentz transformation is equivalent to reversing the direction of the velocity.

We further investigate some properties of the $\bar{\bar{\Lambda}}$ matrix. Given that $\bar{\bar{\alpha}}$ is symmetric and $\bar{\bar{\beta}}$ is skew-symmetric, we find that:

$$\bar{\bar{\Lambda}}^T = \gamma \begin{bmatrix} (\bar{\bar{\alpha}}^{-1})^T & (-\bar{\bar{\beta}})^T \\ \bar{\bar{\beta}}^T & (\bar{\bar{\alpha}}^{-1})^T \end{bmatrix} = \bar{\bar{\Lambda}} \tag{5.2.31}$$

Lorentz Transformation of Maxwell Equations (cont.)

where the superscript T denotes the transpose of the matrix. This confirms that $\bar{\bar{\Lambda}}$ is a symmetric 6×6 matrix.
Additionally, we can demonstrate the following identities:

$$\bar{\bar{\Lambda}}^T \cdot \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{0}} \\ \bar{\bar{0}} & -\bar{\bar{I}} \end{bmatrix} \cdot \bar{\bar{\Lambda}} = \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{0}} \\ \bar{\bar{0}} & -\bar{\bar{I}} \end{bmatrix}, \tag{5.2.32}$$

$$\bar{\bar{\Lambda}}^T \cdot \begin{bmatrix} \bar{\bar{0}} & \bar{\bar{I}} \\ \bar{\bar{I}} & \bar{\bar{0}} \end{bmatrix} \cdot \bar{\bar{\Lambda}} = \begin{bmatrix} \bar{\bar{0}} & \bar{\bar{I}} \\ \bar{\bar{I}} & \bar{\bar{0}} \end{bmatrix}. \tag{5.2.33}$$

(5.2.32) can be employed to identify relations that remain invariant under the Lorentz transformation. Applying the Lorentz transformation to the entity of intensity as defined in (5.2.1), we have:

$$\begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix}^T \cdot \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{0}} \\ \bar{\bar{0}} & -\bar{\bar{I}} \end{bmatrix} \cdot \begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} = \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix}^T \cdot \bar{\bar{\Lambda}}^T \cdot \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{0}} \\ \bar{\bar{0}} & -\bar{\bar{I}} \end{bmatrix} \cdot \bar{\bar{\Lambda}} \cdot \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix}. \tag{5.2.34}$$

Lorentz Transformation of Maxwell Equations (cont.)

Considering (5.2.31), equation (5.2.34) simplifies to:

$$|\vec{E}'|^2 - |c\vec{B}'|^2 = |\vec{E}|^2 - |c\vec{B}|^2. \tag{5.2.35}$$

This result clearly shows that the relative velocity between observers S and S' does not appear in (5.2.35). Therefore, the difference between the squared magnitude of $\vec{E}$ and the squared magnitude of $c\vec{B}$ is a numerical constant, independent of motion. Any quantity that remains unchanged under a Lorentz transformation is termed a Lorentz invariant.

Another Lorentz invariant can be derived using (5.2.1) and (5.2.33):

$$\begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix}^T \cdot \begin{bmatrix} \bar{\bar{0}} & \bar{\bar{I}} \\ \bar{\bar{I}} & \bar{\bar{0}} \end{bmatrix} \cdot \begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} = \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix}^T \cdot \bar{\bar{\Lambda}}^T \cdot \begin{bmatrix} \bar{\bar{0}} & \bar{\bar{I}} \\ \bar{\bar{I}} & \bar{\bar{0}} \end{bmatrix} \cdot \bar{\bar{\Lambda}} \cdot \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix}. \tag{5.2.36}$$

This yields another Lorentz-invariant quantity:

$$\vec{E}' \cdot c\vec{B}' = \vec{E} \cdot c\vec{B}. \tag{5.2.37}$$

Further Reading

The discussion of Lorentz transformation of Maxwell equations largely follows Kong's dyadic formulation Jin Au Kong. *Theory of Electromagnetic Waves*. New York, Niley, 1975; Jin Au Kong. *Electromagnetic Wave Theory*. 2nd. Cambridge, MA: EMW Publishing, 2008, which aligns with the notation used in Chapter 3. Advanced tools such as tensor notation and four-vectors are omitted due to the scope of this course; fuller treatments of relativistic electrodynamics may be found in Zangwill Andrew Zangwill. *Modern Electrodynamics*. Cambridge University Press, 2013 and in Jackson's classic text John David Jackson. *Classical Electrodynamics*. 3rd. John Wiley & Sons, 2012, which presents elegant derivations in Gaussian units. For a broader physical perspective on radiation—especially emission from accelerated charges and its connection to relativistic theory—Smith's pedagogical exposition Glenn S Smith. *An Introduction to Classical Electromagnetic Radiation*. Cambridge University Press, 1997 is recommended.

Problems

1. Consider the one-dimensional Lorentz transformation, show explicitly that $-(ct')^2 + (x')^2 + (y')^2 + (z')^2 = -(ct)^2 + x^2 + y^2 + z^2$.
2. Show (5.2.3) by direct multiplication.
3. Perform the complementary derivation for the transformation laws of $\vec{D}$ and $\vec{H}$ starting by substituting the Lorentz transformation for space-time derivatives, (5.2.8) and (5.2.9), into Ampere's Law ($\vec{\nabla} \times \vec{H} - \partial_{ct}(c\vec{D}) = \vec{J}$) and Gauss's Electric Field Law ($\vec{\nabla} \cdot c\vec{D} = c\rho$) in the S frame.
4. Demonstrate that similar invariant quantities can be formed from $\vec{D}$ and $\vec{H}$ fields. Specifically, show that:

$$\vec{D}' \cdot \vec{H}' = \vec{D} \cdot \vec{H},$$

$$|c\vec{D}'|^2 - |\vec{H}'|^2 = |c\vec{D}|^2 - |\vec{H}|^2.$$

Bessel Functions of the First Kind

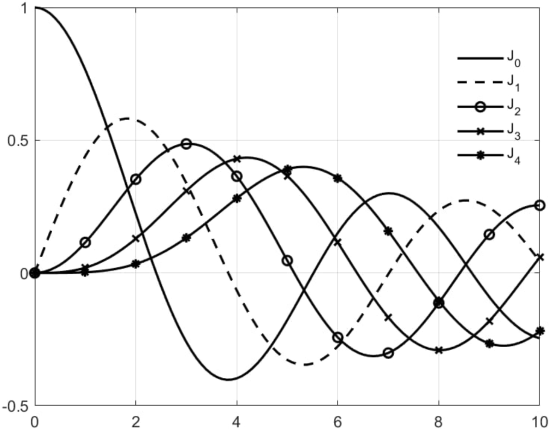

Figure: Bessel functions of the first kind for $\nu = 0, 1, 2, 3, 4$.

Bessel Functions of the Second Kind

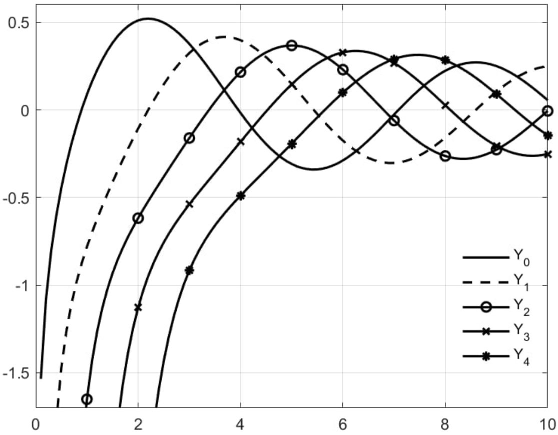

Figure: Bessel functions of the second kind for $\nu = 0, 1, 2, 3, 4$.

Legendre Polynomials

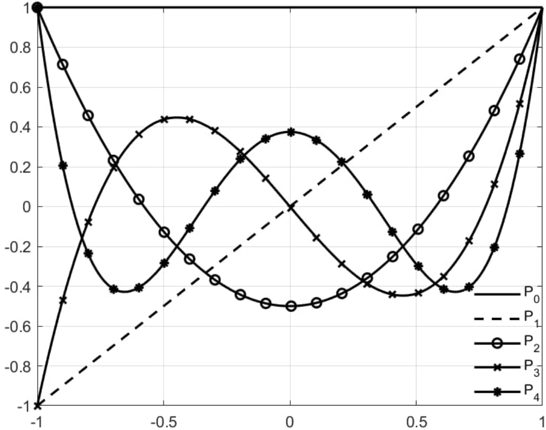

Figure: Legendre polynomials for $n = 0, 1, 2, 3, 4, 5$.