Electromagnetic Scattering

Jake W. Liu

June 13, 2026

Outline

Chapter Overview

This chapter provides an overview of electromagnetic scattering, a fundamental phenomenon that occurs when a wave encounters an obstacle. We will begin by defining the concept of scattering cross-section, a key parameter used to quantify the strength and angular distribution of scattered fields. The chapter will then delve into the analytical solutions for scattering by canonical objects, starting with two-dimensional problems involving PEC cylinders, and then extending the analysis to three-dimensional scattering by dielectric spheres, commonly known as Mie theory.

Table of contents

Cross Sections

In the analysis of scattering problems, the concept of cross section is frequently employed to quantitatively characterize how an object interacts with and scatters electromagnetic waves in the far field. Essentially, the cross section measures the effective area that intercepts and re-radiates incident energy. In this chapter, various types of cross sections are defined. These quantities are fundamental for describing the strength and angular distribution of scattered fields, proving particularly useful when comparing the scattering behavior of different objects or materials.

A scattering problem typically involves determining the scattered field, $\vec{E}_s$, which is defined as the difference between the total field, $\vec{E}_t$, and the incident field, $\vec{E}_i$. The incident field is often represented as a plane wave propagating in the $\hat{i}$ direction:

$$\vec{E}_i = E_0 e^{-ik\hat{i}\cdot\vec{r}}. \tag{4.1.1}$$

Physically, this plane-wave representation approximates the far field produced by a radar antenna located at a large distance from the target. In this configuration, the wavefronts arriving at the target are effectively planar, and the observation (receiving) antenna is assumed to be in the target's far-field zone, where the scattered field can be described as a radiating spherical wave. The scattered field is then given by

$$\vec{E}_s = \vec{E}_t - \vec{E}_i. \tag{4.1.2}$$

Cross Sections (cont.)

Solving such a problem requires determining the scattered field subject to the matching conditions imposed on the total electric or magnetic field at the surface of the scatterer. In the far field, the scattered field takes the form:

$$\vec{E}_s = \begin{cases} E_0 \dfrac{e^{-ikr}}{r} \vec{f}(\hat{s}, \hat{i}) & \text{(3D)}, \\ E_0 \dfrac{e^{-ik\rho}}{\sqrt{\rho}} \vec{f}(\hat{s}, \hat{i}) & \text{(2D)}. \end{cases} \tag{4.1.3}$$

Here, $E_0 = |\vec{E}_0|$ and $\vec{f}(\hat{s}, \hat{i})$ is known as the scattering amplitude function, which describes the scattered wave in the $\hat{s}$ direction. From (1.5.7), the incident and scattered power flux densities are given by:

$$\vec{S}_i = \frac{1}{2}(\vec{E}_i \times \vec{H}_i^*) = \frac{|E_i|^2}{2\eta}\hat{i}, \quad \vec{S}_s = \frac{1}{2}(\vec{E}_s \times \vec{H}_s^*) = \frac{|E_s|^2}{2\eta}\hat{s}. \tag{4.1.4}$$

Cross Sections (cont.)

The differential scattering cross section is defined as:

$$\sigma_d(\hat{s},\hat{i}) = \begin{cases} \lim\limits_{r\to\infty} r^2 \dfrac{|\vec{S}_s|}{|\vec{S}_i|} & \text{(3D)} \\ \lim\limits_{\rho\to\infty} \rho \dfrac{|\vec{S}_s|}{|\vec{S}_i|} & \text{(2D)} \end{cases} = |\vec{f}(\hat{s},\hat{i})|^2. \tag{4.1.5}$$

The bistatic radar cross section (RCS) is then defined as:

$$\sigma_{bi}(\hat{s},\hat{i}) = \begin{cases} 4\pi\sigma_d(\hat{s},\hat{i}) & \text{(3D)}, \\ 2\pi\sigma_d(\hat{s},\hat{i}) & \text{(2D)}. \end{cases} \tag{4.1.6}$$

In two dimensions, σ_{bi} is also referred to as the echo width.
The bistatic RCS represents a hypothetical area that, if illuminated by the incident power density and scattering that power isotropically, would produce the same reflected power at the radar as the actual target. While the RCS is defined to be independent of the distance between the radar and the target, it strongly depends on various factors such as the incidence angle, observation angle, polarization, frequency, and the target's material and shape.

Cross Sections (cont.)

Specifically, the monostatic or backscattering RCS refers to the scenario where the radar transmitter and receiver are co-located, thereby measuring the power reflected directly back toward the source:

$$\sigma_{mono}(\hat{i}) = \sigma_{bi}(-\hat{i}, \hat{i}). \tag{4.1.7}$$

Finally, the total scattering cross section, which quantifies the amount of incident power scattered by an object in all directions, is defined as:

$$\sigma_{sca} = \begin{cases} \displaystyle\int_{4\pi} \sigma_d \, d\Omega = \int_{4\pi} r^2 \frac{|\vec{S}_s|}{|\vec{S}_i|} d\Omega & \text{(3D)}, \\ \displaystyle\int_{2\pi} \sigma_d \, d\phi = \int_{2\pi} \rho \frac{|\vec{S}_s|}{|\vec{S}_i|} d\phi & \text{(2D)}, \end{cases} \tag{4.1.8}$$

where $d\Omega$ represents the differential solid angle.

Table of contents

Cylindrical Waves

This section focuses on solving the wave and Helmholtz equations in cylindrical and spherical coordinates, along with their corresponding scattering phenomena. As noted in Section 33, in a source-free region, the scalar Helmholtz equation (1.4.16) applies to all three field components in Cartesian coordinates. However, in cylindrical coordinates, only the z-unit vector remains constant, meaning that (1.4.16) holds for the z-component alone. In spherical coordinates, none of the unit vectors are constant, necessitating additional analytical considerations, as discussed in Section 23.

Cylindrical Wave Solution

As previously mentioned, the vector Helmholtz equation can be simplified to a scalar equation for the z-component in cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0. \tag{4.2.1}$$

This equation can be solved using the method of separation of variables, by assuming a solution of the form:

$$\psi=B(\rho)\Phi(\phi)Z(z). \tag{4.2.2}$$

Substituting this into (4.2.1) yields:

$$\frac{1}{B\rho}\frac{d}{d\rho}\left(\rho\frac{dB}{d\rho}\right)+\frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}+k^2=0. \tag{4.2.3}$$

Since the variable z appears only in the third term of (4.2.3), we can separate it, leading to an equation similar to the rectangular case:

$$\frac{d^2Z}{dz^2}+k_z^2Z=0. \tag{4.2.4}$$

Cylindrical Wave Solution (cont.)

The elementary solution for Z is:

$$Z = e^{\pm i k_z z}. \tag{4.2.5}$$

Now, let

$$k_\rho^2 = k^2 - k_z^2. \tag{4.2.6}$$

Multiplying (4.2.3) by ρ^2, we obtain:

$$\frac{\rho}{B}\frac{d}{d\rho}\left(\rho\frac{dB}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + k_\rho^2\rho^2 = 0. \tag{4.2.7}$$

Similarly, we can separate the Φ term:

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0, \tag{4.2.8}$$

which has the elementary solution of:

$$\Phi = e^{\pm i\nu\phi}. \tag{4.2.9}$$

Cylindrical Wave Solution (cont.)

Here, $\nu = n$ is an integer constant, assuming Φ is periodic over 2π. Consequently, (4.2.7) simplifies to:

$$\frac{d^2B}{d\rho^2} + \frac{1}{\rho}\frac{dB}{d\rho} + \left(k_\rho^2 - \frac{\nu^2}{\rho^2}\right)B = 0. \tag{4.2.10}$$

This is known as the Bessel equation of order ν.
The first solution to (4.2.10) that remains finite at $\rho = 0$ is known as the Bessel function of the first kind:

$$J_\nu(k_\rho\rho) = \sum_{m=0}^{\infty} \frac{(-1)^m (k_\rho\rho/2)^{\nu+2m}}{m!\,\Gamma(m+\nu+1)}. \tag{4.2.11}$$

When ν is a nonnegative integer, $\Gamma(m+\nu+1) = (m+\nu)!$, recovering the factorial form. For non-integer values of ν, a second independent solution, $J_{-\nu}(k_\rho\rho)$, exists but is infinite at $\rho = 0$. If ν is an integer, denoted as n, then:

$$J_{-n}(k_\rho\rho) = (-1)^n J_n(k_\rho\rho). \tag{4.2.12}$$

Cylindrical Wave Solution (cont.)

A second independent solution, called the Neumann function, is constructed as:

$$Y_n(k_\rho \rho) = \lim_{\nu \to n} \frac{J_\nu(k_\rho \rho)\cos(\nu\pi) - J_{-\nu}(k_\rho \rho)}{\sin(\nu\pi)}. \tag{4.2.13}$$

The Hankel functions of the first and second kind are combinations of the Bessel and Neumann functions:

$$H_n^{(1)}(k_\rho \rho) = J_n(k_\rho \rho) + iY_n(k_\rho \rho), \tag{4.2.14}$$

$$H_n^{(2)}(k_\rho \rho) = J_n(k_\rho \rho) - iY_n(k_\rho \rho). \tag{4.2.15}$$

For large arguments ($k_\rho \rho \gg 1$), the Hankel functions can be asymptotically approximated as:

$$H_n^{(1)}(k_\rho \rho) \approx \sqrt{\frac{2}{\pi k_\rho \rho}} e^{i(k_\rho \rho - n\pi/2 - \pi/4)}, \tag{4.2.16}$$

$$H_n^{(2)}(k_\rho \rho) \approx \sqrt{\frac{2}{\pi k_\rho \rho}} e^{-i(k_\rho \rho - n\pi/2 - \pi/4)}. \tag{4.2.17}$$

These asymptotics, with $e^{i\omega t}$ time convention, represent incoming and outgoing traveling cylindrical waves, respectively.

Cylindrical Wave Transformation

Consider a z-polarized plane wave propagating in the x-direction:

$$\vec{E} = \hat{z}E_0 e^{-ikx} = \hat{z}E_0 e^{-ik\rho\cos\phi}. \tag{4.2.18}$$

We can expand this exponential term into an infinite sum of cylindrical waves:

$$e^{-ik\rho\cos\phi} = \sum_{n=-\infty}^{\infty} a_n J_n(k\rho) e^{in\phi}. \tag{4.2.19}$$

To determine the coefficients a_n, we utilize the following identities:

$$\int_0^{2\pi} e^{-i(k\rho\cos\phi + m\phi)} d\phi = 2\pi i^{-m} J_m(k\rho), \tag{4.2.20}$$

$$\int_0^{2\pi} e^{i(m-n)\phi} d\phi = 2\pi\delta_{mn}. \tag{4.2.21}$$

Multiplying (4.2.19) by $e^{-im\phi}$ and integrating over ϕ from 0 to 2π, we find that:

$$a_m = i^{-m}. \tag{4.2.22}$$

Cylindrical Wave Transformation (cont.)

Thus, we obtain the following expansion:

$$E_z = E_0 e^{-ikx} = E_0 e^{-ik\rho\cos\phi} = E_0 \sum_{n=-\infty}^{\infty} i^{-n} J_n(k\rho) e^{in\phi}. \tag{4.2.23}$$

This process is known as the cylindrical wave transformation, which expands a plane wave into a sum of cylindrical waves.

Table of contents

Scattering from PEC Cylinders

This section examines the scattering of a plane wave by a PEC cylinder of radius a, aligned along the z-axis. Two distinct incident polarizations are typically considered with respect to the cylinder's axis: E-polarization (transverse magnetic to z, or TM) and H-polarization (transverse electric to z, or TE). We first analyze the TM case and then obtain the TE result from the dual boundary condition.

Consider a plane wave propagating in the x-direction. Using the expansion from (4.2.23), the incident field can be expressed as:

$$\vec{E}_i = E_0 e^{-ikx}\hat{z} = \hat{z}E_0 \sum_{n=-\infty}^{\infty} i^{-n} J_n(k\rho) e^{in\phi}. \tag{4.3.1}$$

We assume the scattered field takes the form:

$$\vec{E}_s = \hat{z}E_0 \sum_{n=-\infty}^{\infty} i^{-n} a_n H_n^{(2)}(k\rho) e^{in\phi}. \tag{4.3.2}$$

Scattering from PEC Cylinders (cont.)

The Hankel function of the second kind, $H_n^{(2)}(k\rho)$, is used here to represent outgoing propagating waves. Applying the matching condition that the total tangential electric field E_z^t must be zero at the cylinder surface ($\rho = a$), we have:

$$E_z^t = E_0 \sum_{n=-\infty}^{\infty} i^{-n} \left[J_n(ka) + a_n H_n^{(2)}(ka) \right] e^{in\phi} = 0. \tag{4.3.3}$$

From this, the unknown coefficient a_n is determined as:

$$a_n = -\frac{J_n(ka)}{H_n^{(2)}(ka)}. \tag{4.3.4}$$

Substituting this back, the scattered field is:

$$\vec{E}_s = -\hat{z} E_0 \sum_{n=-\infty}^{\infty} i^{-n} \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) e^{in\phi}. \tag{4.3.5}$$

And the total field is:

$$\vec{E}_t = \hat{z} E_0 \sum_{n=-\infty}^{\infty} i^{-n} \left[J_n(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right] e^{in\phi}. \tag{4.3.6}$$

Scattering from PEC Cylinders (cont.)

From (4.2.17), the far-scattered field can be approximated as:

$$\vec{E}_s \approx -\hat{z}E_0\sqrt{\frac{2i}{\pi k\rho}}e^{-ik\rho}\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{in\phi}. \tag{4.3.7}$$

And from (4.1.6), we can compute the echo width:

$$\sigma_{bi} = \frac{4}{k}\left|\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{in\phi}\right|^2. \tag{4.3.8}$$

This result is often normalized by the wavelength λ as:

$$\sigma_{bi}/\lambda = \frac{2}{\pi}\left|\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{in\phi}\right|^2. \tag{4.3.9}$$

For TE polarization the longitudinal scalar is H_z rather than E_z:

$$\vec{H}_i = \hat{z}H_0\sum_{n=-\infty}^{\infty}i^{-n}J_n(k\rho)e^{in\phi}, \qquad \vec{H}_s = \hat{z}H_0\sum_{n=-\infty}^{\infty}i^{-n}b_nH_n^{(2)}(k\rho)e^{in\phi}. \tag{4.3.10}$$

Scattering from PEC Cylinders (cont.)

For a z-directed magnetic scalar in a homogeneous exterior region,

$$E_\phi = \frac{i\omega\mu}{k^2}\frac{\partial H_z}{\partial \rho}.$$

The PEC condition $E^t_\phi(\rho = a) = 0$ therefore gives the Neumann condition

$$\sum_{n=-\infty}^{\infty} i^{-n}\left[J_n'(ka) + b_n H_n^{(2)\prime}(ka)\right] e^{in\phi} = 0,$$

where the common factor kH_0 has been removed. Orthogonality of $e^{in\phi}$ then gives

$$b_n = -\frac{J_n'(ka)}{H_n^{(2)\prime}(ka)}. \tag{4.3.11}$$

Thus the TE echo-width expression follows from the TM expression by replacing $J_n(ka)/H_n^{(2)}(ka)$ with $J_n'(ka)/H_n^{(2)\prime}(ka)$.

Table of contents

Spherical Waves

Unlike cylindrical waves, in spherical coordinates none of the unit vectors remain constant, which leads to coupled terms when directly evaluating the vector Helmholtz equation. Special considerations are therefore required in this section.

Spherical Wave Solution

The application of the vector Helmholtz equation in spherical coordinates is not straightforward, as none of the unit vectors are constant throughout space. As a result, the vector Helmholtz equation cannot be directly reduced to separate scalar Helmholtz equations for each component. To illustrate this, consider (3.1.4) $\vec{\nabla} \times \vec{\nabla} \times \vec{A} - k^2\vec{A} = \mu\vec{J} - i\omega\mu\epsilon\vec{\nabla}\varphi$ in spherical coordinates. By assuming that $\vec{A}$ and $\vec{J}$ have only radial component, that is

$$\vec{A} = A_r\hat{r}, \quad \vec{J} = J_r\hat{r}. \tag{4.4.1}$$

Then, the r, θ, and ϕ components of (3.1.4) are given by

$$\left[\frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 + k^2\right] A_r = i\omega\mu\epsilon\partial_r\varphi - \mu J_r, \tag{4.4.2}$$

$$-\frac{1}{r}\partial_r\partial_\theta A_r = \frac{i\omega\mu\epsilon}{r}\partial_\theta\varphi, \tag{4.4.3}$$

$$-\frac{1}{r\sin\theta}\partial_r\partial_\phi A_r = \frac{i\omega\mu\epsilon}{r\sin\theta}\partial_\phi\varphi. \tag{4.4.4}$$

Spherical Wave Solution (cont.)

We notice that the Lorenz gauge condition (3.1.8) cannot be used to simplify the (4.4.3) and (4.4.4); instead, we choose

$$\partial_r A_r + i\omega\mu\epsilon\varphi = 0. \tag{4.4.5}$$

By doing so, (4.4.3) and (4.4.4) is satisfied and (4.4.2) is rewritten as

$$\left[\partial_r^2 + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2 \sin^2\theta}\partial_\phi^2 + k^2\right] A_r = -\mu J_r. \tag{4.4.6}$$

Let us introduce the Debye potential:

$$\pi_e = \frac{A_r}{i\omega\mu\epsilon r}. \tag{4.4.7}$$

Then (4.4.6) is transformed into the scalar Helmholtz equation in spherical coordinates with $\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2$:

$$\left(\nabla^2 + k^2\right)\pi_e = -\frac{J_r}{i\omega\epsilon r}. \tag{4.4.8}$$

Spherical Wave Solution (cont.)

By duality, we also have

$$\left(\nabla^2 + k^2\right)\pi_m = -\frac{M_r}{i\omega\mu r}. \tag{4.4.9}$$

Once the solutions of (4.4.8) and (4.4.9) are found, the fields can be found by plugging (4.4.5), (4.4.7) into (3.1.3), (3.1.1) and applying the duality transform:

$$\vec{E} = \vec{\nabla}\partial_r\Pi_e + k^2\vec{\Pi}_e - i\omega\mu\vec{\nabla}\times\vec{\Pi}_m, \tag{4.4.10}$$

$$\vec{H} = \vec{\nabla}\partial_r\Pi_m + k^2\vec{\Pi}_m + i\omega\epsilon\vec{\nabla}\times\vec{\Pi}_e. \tag{4.4.11}$$

where $\vec{\Pi}_e = r\pi_e\hat{r} = \Pi_e\hat{r}$ and $\vec{\Pi}_m = r\pi_m\hat{r} = \Pi_m\hat{r}$.
In terms of spherical components, the electric field is expressed as:

$$E_r = \left(\partial_r^2 + k^2\right)\Pi_e, \tag{4.4.12a}$$

$$E_\theta = \frac{1}{r}\partial_r\partial_\theta\Pi_e - \frac{i\omega\mu}{r\sin\theta}\partial_\phi\Pi_m, \tag{4.4.12b}$$

$$E_\phi = \frac{1}{r\sin\theta}\partial_r\partial_\phi\Pi_e + \frac{i\omega\mu}{r}\partial_\theta\Pi_m, \tag{4.4.12c}$$

Spherical Wave Solution (cont.)

and the magnetic field is expressed as:

$$H_r = \left(\partial_r^2 + k^2\right) \Pi_m, \tag{4.4.13a}$$

$$H_\theta = \frac{1}{r}\partial_r\partial_\theta\Pi_m + \frac{i\omega\epsilon}{r\sin\theta}\partial_\phi\Pi_e, \tag{4.4.13b}$$

$$H_\phi = \frac{1}{r\sin\theta}\partial_r\partial_\phi\Pi_m - \frac{i\omega\epsilon}{r}\partial_\theta\Pi_e. \tag{4.4.13c}$$

We considered here only the radial electric and magnetic current sources, where π_e and π_m are directly related to J_r and M_r. For sources with θ and ϕ-components, the relations become more complex. Nonetheless, the general electromagnetic field in spherical coordinates can still be fully described by the two scalar functions defined in (4.4.8)–(4.4.9). Away from the sources, simply set the right-hand side of these equations to zero. Notice that when $\Pi_m = 0$ and only Π_e exists, we have a field TM to r. Similarly, when $\Pi_e = 0$ and only Π_m exists, we have a field TE to r.

Now consider the scalar Helmholtz equation in spherical coordinates

$$\frac{1}{r^2}\partial_r(r^2\partial_r\psi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi + k^2\psi = 0. \tag{4.4.14}$$

Spherical Wave Solution (cont.)

Using separation of variables, let

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi). \tag{4.4.15}$$

Plug into (4.4.14), multiply $r^2 \sin^2\theta$ and divide it by ψ, we get

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + (kr\sin\theta)^2 = 0. \tag{4.4.16}$$

Similar to the cylindrical case, for the Φ term, we have

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0, \quad \Phi = e^{\pm i\nu\phi}, \tag{4.4.17}$$

where ν is a constant. Plug into (4.4.16) and divide it by $\sin^2\theta$, we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{\nu^2}{\sin^2\theta} + (kr)^2 = 0. \tag{4.4.18}$$

Spherical Wave Solution (cont.)

Now, let

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{\nu^2}{\sin^2\theta} = -\mu^2, \tag{4.4.19}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + (kr)^2 = \mu^2. \tag{4.4.20}$$

Since Φ is periodic over 2π, we let $\nu = m$ an integer value, and from (4.4.19), we get

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\mu^2 - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{4.4.21}$$

Let us consider two cases: the solution is azimuthal symmetric or non-azimuthal symmetric. If it is azimuthal symmetric, we have $m = 0$ and (4.4.21) is reduced to

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \mu^2\Theta = 0. \tag{4.4.22}$$

Spherical Wave Solution (cont.)

To ensure that all solutions of the equation remain finite at $\cos\theta = \pm 1$, the parameter μ^2 must take the specific form $\mu^2 = n(n+1)$, where n is a non-negative integer. Thus, we arrive at the following equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\Theta = 0, \tag{4.4.23}$$

which is known as the Legendre equation, and their solutions are known as the Legendre polynomials:

$$P_n(\cos\theta) = \frac{1}{2^n n!}\left(\frac{d}{d\cos\theta}\right)^n \left(\cos^2\theta - 1\right)^n. \tag{4.4.24}$$

The solutions form a complete set in the interval $-1 \leq \cos\theta \leq 1$. The orthogonality relation is

$$\int_{-1}^{1} P_n(\cos\theta)P_{n'}(\cos\theta)d\cos\theta = \frac{2}{2n+1}\delta_{nn'}. \tag{4.4.25}$$

Hence, if we have a function that lies in $0 \leq \theta \leq \pi$ $(-1 \leq \cos\theta \leq 1)$, we can expand it as

$$f(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta), \quad 0 \leq \theta \leq \pi, \tag{4.4.26}$$

Spherical Wave Solution (cont.)

where

$$a_n = \frac{2n+1}{2} \int_0^\pi f(\theta) P_n(\cos\theta) \sin\theta d\theta. \tag{4.4.27}$$

On the other hand, if the solution is not azimuthally symmetric, we obtain the associated Legendre equation

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta} \right] \Theta = 0. \tag{4.4.28}$$

The solutions to (4.4.28) are the associated Legendre functions of the first kind $P_n^m(\cos\theta)$ and second kind $Q_n^m(\cos\theta)$. Here, m is an integer that satisfies the periodic condition over ϕ, ensuring that the total field remains single-valued in azimuthal angle, that is,

$$e^{im(\phi+2\pi)} = e^{im\phi}.$$

All solutions are singular at $\cos\theta = \pm 1$ except for $P_n^m(\cos\theta)$ with integer n and m. For positive integers m and n, these functions are related to the ordinary Legendre polynomials by

$$P_n^m(\cos\theta) = \sin^m\theta \, P_n^{(m)}(\cos\theta), \quad m \leq n, \tag{4.4.29}$$

Spherical Wave Solution (cont.)

where $P_n^{(m)}$ denotes the m^{th} derivative of the n^{th}-order Legendre polynomial with respect to $\cos\theta$.
Now let us consider (4.4.20). Substituting $\mu^2 = n(n+1)$, and let

$$R = R_n(kr) = \sqrt{\frac{\pi}{2kr}} B_{n+1/2}(kr) = \sqrt{\frac{\pi}{2kr}} B, \tag{4.4.30}$$

then (4.4.20) becomes

$$\frac{d^2 B}{dr^2} + \frac{1}{r}\frac{dB}{dr} + \left[k^2 - \frac{(n+1/2)^2}{r^2}\right] B = 0, \tag{4.4.31}$$

which is the Bessel equation and $B_{n+1/2}$ is the Bessel function of order $n+1/2$. (4.4.30) is known as the spherical Bessel function. We often use lowercase letters to denote the spherical forms of the Bessel and Hankel functions. For example, the spherical Bessel and Hankel functions are defined as

$$j_n(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{n+1/2}(kr), \quad h_n^{(2)}(kr) = \sqrt{\frac{\pi}{2kr}}\, H_{n+1/2}^{(2)}(kr).$$

Spherical Wave Transformation

Similar to (4.2.18), consider an x-polarized plane wave propergating in z-direction:

$$\vec{E} = \hat{x} E_0 e^{-ikz} = \hat{x} E_0 e^{-ikr\cos\theta}. \tag{4.4.32}$$

Let us expand the exponential term into an infinite series of spherical waves. Since (4.4.32) is azimuthally symmetric, we set $m = 0$. Furthermore, because the solution must remain finite at the origin, we have

$$e^{-ikr\cos\theta} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\theta). \tag{4.4.33}$$

To find a_n, let us use the orthogonality relation (4.4.25):

$$a_n j_n(kr) = \frac{2n+1}{2} \int_{-1}^{1} e^{-ikr\cos\theta} P_n(\cos\theta) d\cos\theta. \tag{4.4.34}$$

Using the following identity

$$\int_{-1}^{1} e^{-ikr\cos\theta} P_n(\cos\theta) d\cos\theta = 2i^{-n} j_n(kr), \tag{4.4.35}$$

Spherical Wave Transformation (cont.)

we have

$$a_n = (2n+1)i^{-n}. \tag{4.4.36}$$

Thus, we have the following expansion

$$\begin{aligned} E_x &= E_0 e^{-ikz} = E_0 e^{-ikr\cos\theta} \\ &= E_0 \sum_{n=0}^{\infty} (2n+1) i^{-n} j_n(kr) P_n(\cos\theta). \end{aligned} \tag{4.4.37}$$

This is called the spherical wave transformation which expands a plane wave to a sum of spherical waves.

Table of contents

Scattering from Dielectric Spheres

The exact solution for the scattering of a plane electromagnetic wave by an isotropic, homogeneous dielectric sphere of arbitrary size is commonly known as Mie theory. Consider a dieletric sphere with radius a and permitivity / permeability equal to ϵ_d/μ_d placed in a medium with permitivity / permeability equal to ϵ/μ. Let the incident electric field be an x-polarized plane wave propergating in z-direction. The radial component of incident electric field can be expressed as

$$\begin{aligned} E_r^i &= E_0 e^{-ikz}\hat{x}\cdot\hat{r} = E_0 \sin\theta\cos\phi e^{-ikr\cos\theta} \\ &= \frac{1}{ikr}E_0\cos\phi\partial_\theta e^{-ikr\cos\theta}. \end{aligned} \tag{4.5.1}$$

Using (4.4.37), the radial component of incident electric field can be expressed as

$$\begin{aligned} E_r^i &= \frac{E_0\cos\phi}{ikr}\partial_\theta \sum_{n=0}^{\infty}(2n+1)i^{-n}j_n(kr)P_n(\cos\theta) \\ &= \frac{iE_0\cos\phi}{(kr)^2}\sum_{n=1}^{\infty}(2n+1)i^{-n}\hat{j}_n(kr)P_n^1(\cos\theta), \end{aligned} \tag{4.5.2}$$

Scattering from Dielectric Spheres (cont.)

where

$$-\partial_\theta P_n(\cos\theta) = P_n^1(\cos\theta), \tag{4.5.3}$$

$$\hat{j}_n(kr) = krj_n(kr). \tag{4.5.4}$$

To find the Debye potential of the incident field, (4.4.13a) must be satisfied:

$$E_r^i = \left(\partial_r^2 + k^2\right)\Pi_e^i = \left(\partial_r^2 + k^2\right)\left(r\pi_e^i\right). \tag{4.5.5}$$

Let us expand the Debye potential in terms of spherical harmonics

$$\pi_e^i = \sum_{n=0}^{\infty}\sum_{m=0}^{n} j_n(kr)P_n^m(\cos\theta)(A_{mn}\cos m\phi + B_{mn}\sin m\phi). \tag{4.5.6}$$

Substituting (4.5.6) into (4.5.5), we get

$$E_r^i = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{n(n+1)}{kr^2}\hat{j}_n(kr)P_n^m(\cos\theta)$$
$$(A_{mn}\cos m\phi + B_{mn}\sin m\phi). \tag{4.5.7}$$

Scattering from Dielectric Spheres (cont.)

Here, the standard identity is used for the derivation

$$\left(\frac{d^2}{dr^2}+k^2\right)[rj_n(kr)]=\frac{n(n+1)}{r}j_n(kr), \tag{4.5.8}$$

which came directly from (4.4.20). Comparing (4.5.7) and (4.5.2), we get

$$\begin{cases} A_{mn}=0, \quad m\neq 1, \\ A_{1n}=E_0(-i)^{n-1}\dfrac{2n+1}{kn(n+1)}, \end{cases} \tag{4.5.9}$$

$$B_{mn}=0. \tag{4.5.10}$$

Thus, we have

$$\pi_e^i=\frac{E_0\cos\phi}{k^2r}\sum_{n=1}^{\infty}(-i)^{n-1}\frac{2n+1}{n(n+1)}\hat{j}_n(kr)P_n^1(\cos\theta). \tag{4.5.11}$$

Scattering from Dielectric Spheres (cont.)

Following a similar process, we can obtain

$$\pi_m^i = \frac{E_0 \sin\phi}{\eta k^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} \hat{j}_n(kr) P_n^1(\cos\theta). \tag{4.5.12}$$

To find the scattered field inside and outside the sphere, let us first specify the matching conditions to be applied at $r = a$:

$$E_\theta = E_\theta^d, \quad E_\phi = E_\phi^d, \tag{4.5.13}$$

$$H_\theta = H_\theta^d, \quad H_\phi = H_\phi^d, \tag{4.5.14}$$

where the superscript d denotes the field inside the sphere. From (4.4.12) and (4.4.13), the matching conditions involve both $\pi_e = \pi_e^i + \pi_e^s$ and $\pi_m = \pi_m^i + \pi_m^s$, making them coupled. To simplify the analysis, it is convenient to decouple them by deriving matching conditions for π_e and π_m individually. To achieve this, we consider a linear combination of E_θ and E_ϕ in such a way that all terms involving π_m cancel out: $\partial_\theta(\sin\theta E_\theta) + \partial_\phi E_\phi = [\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin\theta}\partial_\phi^2]\frac{1}{r}\partial_r(r\pi_e)$. The complementary curl-type

Scattering from Dielectric Spheres (cont.)

combination of H_θ, H_ϕ isolates $\epsilon\pi_e$; by duality, the same two combinations isolate $\frac{1}{r}\partial_r(r\pi_m)$ and $\mu\pi_m$. At $r = a$, we have:

$$\frac{1}{r}\partial_r(r\pi_e) = \frac{1}{r}\partial_r\left(r\pi_e^d\right), \quad \epsilon\pi_e = \epsilon_d\pi_e^d, \tag{4.5.15}$$

$$\frac{1}{r}\partial_r(r\pi_m) = \frac{1}{r}\partial_r\left(r\pi_m^d\right), \quad \mu\pi_m = \mu_d\pi_m^d. \tag{4.5.16}$$

The matching conditions ensure that each Debye potential function outside the sphere couples exclusively to its corresponding Debye potential inside the sphere. As a result, if the incident field involves terms with $\cos\phi$ dependence, both the scattered and internal fields will exhibit the same $\cos\phi$ dependence. Likewise, all terms involving $\sin\phi$ will retain their $\sin\phi$ dependence. Thus, for the scattered field potential, we let

$$\pi_e^s = \frac{-E_0\cos\phi}{k^2 r}\sum_{n=1}^{\infty}(-i)^{n-1}\frac{2n+1}{n(n+1)}a_n\hat{h}_n^{(2)}(kr)P_n^1(\cos\theta), \tag{4.5.17}$$

$$\pi_m^s = \frac{-E_0\sin\phi}{\eta k^2 r}\sum_{n=1}^{\infty}(-i)^{n-1}\frac{2n+1}{n(n+1)}b_n\hat{h}_n^{(2)}(kr)P_n^1(\cos\theta), \tag{4.5.18}$$

Scattering from Dielectric Spheres (cont.)

where $\hat{h}_n^{(2)}(kr)$ is used to meet the radiation condition, and the coefficients a_n and b_n are to be determined. The total potential outside the sphere is thus written as

$$\pi_e = \frac{E_0 \cos\phi}{k^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} \left[\hat{j}_n(kr) - a_n \hat{h}_n^{(2)}(kr)\right] P_n^1(\cos\theta), \tag{4.5.19}$$

and

$$\pi_m = \frac{E_0 \sin\phi}{\eta k^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} \left[\hat{j}_n(kr) - b_n \hat{h}_n^{(2)}(kr)\right] P_n^1(\cos\theta). \tag{4.5.20}$$

For the Debye potential inside the sphere, we let

$$\pi_e^d = \frac{E_0 \cos\phi}{k_d^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} c_n \hat{j}_n(k_d r) P_n^1(\cos\theta), \tag{4.5.21}$$

$$\pi_m^d = \frac{E_0 \sin\phi}{\eta_d k_d^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} d_n \hat{j}_n(k_d r) P_n^1(\cos\theta). \tag{4.5.22}$$

Scattering from Dielectric Spheres (cont.)

Applying the matching conditions (4.5.15) and (4.5.16) to the above equations, the common angular factor for each n cancels. For the electric potential the two scalar conditions become

$$\frac{1}{k}\left[\hat{j}_n'(ka) - a_n \hat{h}_n^{(2)\prime}(ka)\right] = \frac{c_n}{k_d}\hat{j}_n'(k_d a),$$
$$\frac{\epsilon}{k^2}\left[\hat{j}_n(ka) - a_n \hat{h}_n^{(2)}(ka)\right] = \frac{\epsilon_d c_n}{k_d^2}\hat{j}_n(k_d a).$$

Since $k^2 = \omega^2\mu\epsilon$ and $k_d^2 = \omega^2\mu_d\epsilon_d$, the second equation is equivalently weighted by $1/\mu$ and $1/\mu_d$. Eliminating c_n gives a_n; the magnetic system follows by $\epsilon \leftrightarrow \mu$, giving b_n, d_n. Thus

$$a_n = \frac{\sqrt{\epsilon\mu_d}\,\hat{j}_n(ka)\hat{j}_n'(k_d a) - \sqrt{\epsilon_d\mu}\,\hat{j}_n'(ka)\hat{j}_n(k_d a)}{\sqrt{\epsilon\mu_d}\,\hat{h}_n^{(2)}(ka)\hat{j}_n'(k_d a) - \sqrt{\epsilon_d\mu}\,\hat{h}_n^{(2)\prime}(ka)\hat{j}_n(k_d a)}, \tag{4.5.23}$$

$$b_n = \frac{\sqrt{\epsilon_d\mu}\,\hat{j}_n(ka)\hat{j}_n'(k_d a) - \sqrt{\epsilon\mu_d}\,\hat{j}_n'(ka)\hat{j}_n(k_d a)}{\sqrt{\epsilon_d\mu}\,\hat{h}_n^{(2)}(ka)\hat{j}_n'(k_d a) - \sqrt{\epsilon\mu_d}\,\hat{h}_n^{(2)\prime}(ka)\hat{j}_n(k_d a)}, \tag{4.5.24}$$

$$c_n = \frac{i\mu_d\sqrt{\epsilon_d/\mu}}{\sqrt{\epsilon\mu_d}\,\hat{h}_n^{(2)}(ka)\hat{j}_n'(k_d a) - \sqrt{\epsilon_d\mu}\,\hat{h}_n^{(2)\prime}(ka)\hat{j}_n(k_d a)}, \tag{4.5.25}$$

Scattering from Dielectric Spheres (cont.)

$$d_n = \frac{i\mu_d\sqrt{\epsilon_d/\mu}}{\sqrt{\epsilon_d\mu}\,\hat{h}_n^{(2)}(ka)\hat{j}_n'(k_d a) - \sqrt{\epsilon\mu_d}\,\hat{h}_n^{(2)\prime}(ka)\hat{j}_n(k_d a)}. \tag{4.5.26}$$

Let us consider the case in far field. From (4.2.17), when $kr \to \infty$,

$$\hat{h}_n^{(2)}(kr) = \sqrt{\frac{\pi kr}{2}} H_{n+1/2}^{(2)}(kr) \approx i^{n+1}e^{-ikr}, \tag{4.5.27}$$

we get

$$\Pi_e^s = r\pi_e^s \approx e^{-ikr}\frac{E_0\cos\phi}{k^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}a_n P_n^1(\cos\theta), \tag{4.5.28}$$

$$\Pi_m^s = r\pi_m^s \approx e^{-ikr}\frac{E_0\sin\phi}{\eta k^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}b_n P_n^1(\cos\theta). \tag{4.5.29}$$

Noting that in the far field

$$\partial_r\Pi_e^s \approx -ik\Pi_e^s, \quad \partial_r\Pi_m^s \approx -ik\Pi_m^s, \tag{4.5.30}$$

Scattering from Dielectric Spheres (cont.)

we can thus express

$$E_\theta \approx f_\theta(\theta, \phi) \frac{e^{-ikr}}{r}, \tag{4.5.31}$$

$$E_\phi \approx f_\phi(\theta, \phi) \frac{e^{-ikr}}{r}, \tag{4.5.32}$$

$$f_\theta(\theta, \phi) = -i \cos\phi \, \frac{S_2(\theta)}{k}, \tag{4.5.33}$$

$$f_\phi(\theta, \phi) = i \sin\phi \, \frac{S_1(\theta)}{k}, \tag{4.5.34}$$

$$S_1(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n \pi_n(\cos\theta) + b_n \tau_n(\cos\theta)], \tag{4.5.35}$$

$$S_2(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n \tau_n(\cos\theta) + b_n \pi_n(\cos\theta)], \tag{4.5.36}$$

with

$$\pi_n(\cos\theta) = \frac{P_n^1(\cos\theta)}{\sin\theta}, \quad \tau_n(\cos\theta) = \frac{d}{d\theta} P_n^1(\cos\theta). \tag{4.5.37}$$

Scattering from Dielectric Spheres (cont.)

By (4.1.5), the differential cross section is

$$\sigma_d(\theta,\phi) = \left|\vec{f}(\theta,\phi)\right|^2 = \left|\cos\phi\frac{S_2(\theta)}{k}\right|^2 + \left|\sin\phi\frac{S_1(\theta)}{k}\right|^2, \tag{4.5.38}$$

and by (4.1.8), the scattering cross section is

$$\sigma_{sca} = \int_{4\pi}\sigma_d(\theta,\phi)d\Omega = \frac{\pi}{k^2}\int_0^\pi \left(|S_2(\theta)|^2 + |S_1(\theta)|^2\right)\sin\theta d\theta. \tag{4.5.39}$$

Computing the following:

$$\begin{aligned} |S_1(\theta)|^2 = &\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{2n'+1}{n'(n'+1)} \\ &\times\left[a_n\pi_n a_{n'}^*\pi_{n'} + b_n\tau_n b_{n'}^*\tau_{n'} + a_n\pi_n b_{n'}^*\tau_{n'} + b_n\tau_n a_{n'}^*\pi_{n'}\right], \end{aligned} \tag{4.5.40}$$

$$\begin{aligned} |S_2(\theta)|^2 = &\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{2n'+1}{n'(n'+1)} \\ &\times\left[a_n\tau_n a_{n'}^*\tau_{n'} + b_n\pi_n b_{n'}^*\pi_{n'} + a_n\tau_n b_{n'}^*\pi_{n'} + b_n\pi_n a_{n'}^*\tau_{n'}\right], \end{aligned} \tag{4.5.41}$$

Scattering from Dielectric Spheres (cont.)

and using the orthogonal property:

$$\int_0^\pi (\pi_n \pi_{n'} + \tau_n \tau_{n'}) \sin\theta d\theta = \begin{cases} 0, & \text{if } n \neq n' \\ \dfrac{2n^2(n+1)^2}{2n+1}, & \text{if } n = n' \end{cases} \quad (4.5.42)$$

we get

$$\sigma_{sca} = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1) \left(|a_n|^2 + |b_n|^2 \right). \quad (4.5.43)$$

For very small spheres, where $k_d a \ll 1$, only the $n = 1$ term contributes significantly to the scattered field, leading to the Rayleigh scattering approximation. The needed leading forms are

$$\hat{j}_1(x) \approx \frac{x^2}{3}, \quad \hat{j}_1'(x) \approx \frac{2x}{3}, \quad \hat{h}_1^{(2)}(x) \approx \frac{i}{x}, \quad \hat{h}_1^{(2)\prime}(x) \approx -\frac{i}{x^2}.$$

In this regime, the scattering coefficients reduce to

$$a_1 = \frac{2i}{3}(ka)^3 \frac{\epsilon_d - \epsilon}{\epsilon_d + 2\epsilon}, \quad b_1 = \frac{2i}{3}(ka)^3 \frac{\mu_d - \mu}{\mu_d + 2\mu}. \quad (4.5.44)$$

Scattering from Dielectric Spheres (cont.)

These expressions indicate that the scattered field strength depends on the particle size through $(ka)^3$. In the Rayleigh limit, the scattering cross section varies as ω^4, meaning shorter wavelengths are scattered more efficiently. This explains why the sky appears blue due to stronger scattering of blue light by air molecules, why sunlight becomes reddish at sunset after blue components are scattered out over a long atmospheric path, and why the lunar sky remains black in the absence of an atmosphere.

Scattering from Dielectric Spheres (cont.)

The PEC sphere follows from the dielectric Mie solution by taking $\epsilon_d \to -i\infty$, which enforces $\hat{r} \times \vec{E} = 0$ at $r = a$. In (4.5.23)–(4.5.24), the terms proportional to $\sqrt{\epsilon_d \mu}$ dominate and their common factor cancels, so

$$a_n = \frac{\hat{j}_n'(ka)}{\hat{h}_n^{(2)\prime}(ka)}, \tag{4.5.45}$$

$$b_n = \frac{\hat{j}_n(ka)}{\hat{h}_n^{(2)}(ka)}, \tag{4.5.46}$$

$$c_n = 0, \tag{4.5.47}$$

$$d_n = 0. \tag{4.5.48}$$

The internal-field coefficients vanish. Thus the PEC solution is the conducting limit of the dielectric formulation. The monostatic RCS appears in Fig. 1.

Scattering from Dielectric Spheres (cont.)

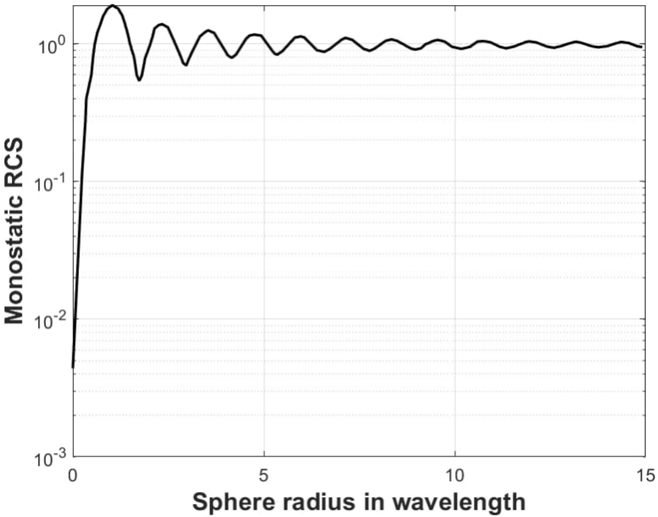

Figure: Monostatic RCS of a PEC sphere.

Scattering from Dielectric Spheres (cont.)

Further Reading

For a concise and accessible overview of special functions, the reader is encouraged to consult Novak and Fox Kyle A Novak and Fox Laura J. *Special Functions of Mathematical Physics: A Tourist's Guidebook*. CreateSpace Independent Publishing Platform, 2019, p. 172. ISBN: 9781985069695. A more comprehensive mathematical treatment can be found in Arfken, Weber, and Harris' *Mathematical Methods for Physicists* George B Arfken, Hans J Weber, and Frank E Harris. *Mathematical Methods for Physicists: a Comprehensive Guide*. Academic press, 2011, which provides detailed derivations and properties of Bessel, Legendre, and other special functions widely used in physics and engineering. The *NIST Digital Library of Mathematical Functions (DLMF)* *NIST Digital Library of Mathematical Functions*. https://dlmf.nist.gov/, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. URL: https://dlmf.nist.gov/ serves as an authoritative online reference for the definitions, identities, and asymptotic expansions of special functions. Harrington's text Roger F Harrington. *Time-Harmonic Electromagnetic Fields*. IEEE Press, 2001 includes dedicated chapters on cylindrical and spherical wave solutions. In this work, we adopt the same selection of canonical examples—conducting cylinders and dielectric spheres—as in Zangwill's treatment Andrew Zangwill. *Modern Electrodynamics*. Cambridge University Press, 2013, since these configurations serve as

Further Reading (cont.)

representative models for physical objects such as wires and raindrops. Ishimaru's classic Akira Ishimaru. *Electromagnetic Wave Propagation, Radiation and Scattering*. 2nd. Wiley, 2017 provides a rigorous formulation of Mie theory, which forms the principal reference for this chapter. For a broader discussion of electromagnetic scattering phenomena, the reader is referred to Osipov and Tretyakov's *Modern Electromagnetic Scattering Theory with Applications* Andrey V Osipov and Sergei A Tretyakov. *Modern Electromagnetic Scattering Theory with Applications*. John Wiley & Sons, 2017. In the context of light scattering by small particles, classical treatments include Van de Hulst's *Light Scattering by Small Particles* Hendrik Christoffel Hulst and Hendrik C van de Hulst. *Light Scattering by Small Particles*. Courier Corporation, 1981, Bohren and Huffman's *Absorption and Scattering of Light by Small Particles* Craig F Bohren and Donald R Huffman. *Absorption and Scattering of Light by Small Particles*. John Wiley & Sons, 2008, and Kerker's *Scattering of Light and Other Electromagnetic Radiation* Milton Kerker. *The Scattering of Light and Other Electromagnetic Radiation: Physical Chemistry: A Series of Monographs*. Vol. 16. Academic Press, 2013.

Problems

1. From (4.3.1) and (4.3.6) , plot $|\vec{E}_t/\vec{E}_i|$ with $k\rho$ for $ka = 1$.
2. From (4.3.5) and (4.3.6), find the solution of the scattered and total magnetic field.
3. From Problem 2, find the induced surface current.
4. Derive the case of TE scattering from a PEC cylinder.
5. Complete the derivation to obtain (4.5.23)–(4.5.26).
6. Verify (4.5.44).