Electromagnetic Radiation

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Outline

Chapter Overview

This chapter examines electromagnetic radiation—the generation and propagation of waves from time-varying sources. We begin by introducing the scalar and vector potentials used to compute radiated fields, followed by the application of Green functions to solve the inhomogeneous Helmholtz equations governing these potentials. The general free-space solution and the far-field approximation are then derived, together with the Stratton–Chu formulation. These tools are applied to the Hertzian dipole to analyze its radiation properties and field patterns, after which image theory is introduced. The chapter concludes with the analysis of radiation from an aperture.

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Vector and Scalar Potentials

In the absence of magnetic sources, Gauss' law for magnetism states that $\vec{\nabla} \cdot \vec{B} = 0$. A key vector identity states that the divergence of the curl of any vector field is always zero. This allows us to express the magnetic flux density $\vec{B}$ as the curl of a vector field $\vec{A}$, known as the magnetic vector potential:

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{3.1.1}$$

Substituting this into Faraday's law (1.4.5) with no magnetic sources ($\vec{M} = 0$), we obtain:

$$\vec{\nabla} \times \left(\vec{E} + i\omega\vec{A}\right) = 0. \tag{3.1.2}$$

Since the curl of the quantity in parentheses is zero, it can be expressed as the gradient of a scalar function. This allows us to define the electric field intensity in terms of $\vec{A}$ and an electric scalar potential φ:

$$\vec{E} = -\vec{\nabla}\varphi - i\omega\vec{A}. \tag{3.1.3}$$

Substituting the potential representations (3.1.1) and (3.1.3) into Ampere's law (1.4.6), in a simple medium we get a wave-like equation for $\vec{A}$:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu\vec{J} + \omega^2\mu\epsilon\vec{A} - i\omega\mu\epsilon\vec{\nabla}\varphi. \tag{3.1.4}$$

Vector and Scalar Potentials (cont.)

Similarly, substituting (3.1.3) into Gauss' law (1.4.7) yields an equation for φ:

$$\nabla^2\varphi + i\omega\vec{\nabla}\cdot\vec{A} = -\frac{\rho}{\epsilon}. \tag{3.1.5}$$

Using the vector identity $\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$, and rearranging terms, equations (3.1.4) and (3.1.5) can be rewritten as a pair of coupled wave equations:

$$\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + i\omega\mu\epsilon\varphi\right), \tag{3.1.6}$$

$$\nabla^2\varphi + k^2\varphi = -\rho/\epsilon - i\omega\left(\vec{\nabla}\cdot\vec{A} + i\omega\mu\epsilon\varphi\right), \tag{3.1.7}$$

where $k^2 = \omega^2\mu\epsilon$. So far, we have only specified the curl of $\vec{A}$. To uniquely define a vector field, we must also specify its divergence. We have the freedom to choose the divergence of $\vec{A}$ in a way that simplifies these equations. By choosing the following:

$$\vec{\nabla}\cdot\vec{A} + i\omega\mu\epsilon\varphi = 0, \tag{3.1.8}$$

which is known as the Lorenz gauge condition, the equations become decoupled:

$$\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J}, \tag{3.1.9}$$

Vector and Scalar Potentials (cont.)

$$\nabla^2 \varphi + k^2 \varphi = -\frac{\rho}{\epsilon} = \frac{1}{i\omega\epsilon} \vec{\nabla} \cdot \vec{J}. \tag{3.1.10}$$

Starting from the case when $\vec{\nabla} \cdot \vec{D} = 0$, a similar reasoning leads to the introduction of an electric vector potential, defined by $\vec{D} = -\vec{\nabla} \times \vec{F}$, which is applicable when there are no electric sources and only magnetic sources are present.

The Green Function Method

Both the scalar potential equation (3.1.10) and the Cartesian components of the vector potential equation (3.1.9) are forms of the inhomogeneous scalar Helmholtz equation. A powerful method for solving such equations is the Green function technique. The Green function G is the solution to the Helmholtz equation for a unit point source (a Dirac delta function):

$$\nabla^2 G + k^2 G = -\delta(\vec{r} - \vec{r}\,'). \tag{3.1.11}$$

The solution to this equation, representing the field at position $\vec{r}$ due to a point source located at $\vec{r}\,'$, is given by

$$G(\vec{r}; \vec{r}\,') = \frac{e^{-ikR}}{4\pi R}, \tag{3.1.12}$$

where $R = |\vec{r} - \vec{r}\,'|$ is the distance between the source and observation points. The negative sign in the exponential term ensures that the Green function represents an outward-radiating spherical wave. This form satisfies the *Sommerfeld radiation condition* at infinity, which requires that any radiating field behaves as an outgoing wave that decays with distance:

$$\lim_{R \to \infty} R \left(\partial_R G + ikG \right) = 0. \tag{3.1.13}$$

The Green Function Method (cont.)

If the opposite sign were used in the exponential term, the resulting solution would describe an incoming spherical wave converging toward the source, which is unphysical for radiation problems.
The delta function is defined such that:

$$\begin{aligned} \delta\left(\vec{r}-\vec{r}\,'\right) &= 0, \quad \vec{r} \neq \vec{r}\,', \\ \int_V \delta\left(\vec{r}-\vec{r}\,'\right)\, dv &= 1, \quad \vec{r}\,' \text{ in } V. \end{aligned} \tag{3.1.14}$$

The Green function G is the fundamental solution, or impulse response, of the Helmholtz equation. From (3.1.14), equation (3.1.11) can be re-expressed as the following set of conditions on G:

$$\begin{aligned} \nabla^2 G + k^2 G &= 0, \qquad \vec{r} \neq \vec{r}\,', \\ \int_V \left(\nabla^2 G + k^2 G\right) dv &= -1, \quad \vec{r}\,' \text{ in } V. \end{aligned} \tag{3.1.15}$$

The Green Function Method (cont.)

We can verify that our solution for G satisfies these conditions. First, for $\vec{r} \neq \vec{r}\,'$ (i.e., $R \neq 0$):

$$\begin{aligned} \nabla^2 G &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\frac{e^{-ikR}}{4\pi R}\right) \\ &= -k^2 \frac{e^{-ikR}}{4\pi R} = -k^2 G. \end{aligned} \tag{3.1.16}$$

Thus, $\nabla^2 G + k^2 G = 0$ for $R \neq 0$. To verify the second condition, we integrate over an infinitesimal spherical volume V_0 centered at $\vec{r}\,'$ with radius R_0:

$$\begin{aligned} \int_{V_0} \nabla^2 G dv &= \oint_{S_0} \vec{\nabla} G \cdot d\vec{s} = \oint_{S_0} (\partial_R G|_{R_0})\hat{R} \cdot d\vec{s} \\ &= -e^{-ikR_0}(1 + ikR_0) \end{aligned} \tag{3.1.17}$$

The Green Function Method (cont.)

and

$$
\begin{aligned}
\int_{V_0} k^2 G\, dV &= \int_{V_0} \frac{k^2 e^{-ikR}}{4\pi R} R^2 \sin\theta\, d\theta\, d\phi\, dR \\
&= k^2 \int_0^{R_0} R e^{-ikR}\, dR \\
&= \frac{k^2}{-ik} \int_0^{R_0} R\, d\left(e^{-ikR}\right) \\
&= ik\left[R_0 e^{-ikR_0} + \frac{1}{ik}\left(e^{-ikR_0} - 1\right)\right] \\
&= e^{-ikR_0}(ikR_0 + 1) - 1.
\end{aligned}
\tag{3.1.18}
$$

Adding these two results, we find that $\int_{V_0}(\nabla^2 G + k^2 G)dv = -1$, which confirms that our expression for the Green function is correct.

The General Radiation Solution

By multiplying the Green function equation (3.1.11) by $\mu\vec{J}(\vec{r}\,')$ and integrating over a volume containing all sources, we obtain:

$$\begin{aligned}
&\int_{V'} \mu\vec{J}(\vec{r}\,')\left[\left(\nabla^2 + k^2\right) G(\vec{r};\vec{r}\,')\right] dv' \\
&= \left(\nabla^2 + k^2\right) \int_{V'} \mu\vec{J}(\vec{r}\,')G(\vec{r};\vec{r}\,')dv' \\
&= -\int_{V'} \mu\vec{J}(\vec{r}\,')\delta(\vec{r}-\vec{r}\,')dv' = -\mu\vec{J}(\vec{r}).
\end{aligned} \tag{3.1.19}$$

By comparing this result with the wave equation for the vector potential (3.1.9), we can express the solution for $\vec{A}(\vec{r})$ as a convolution of the source distribution with the Green function:

$$\vec{A}(\vec{r}) = \int_{V'} \mu\vec{J}(\vec{r}\,')G(\vec{r};\vec{r}\,')dv'. \tag{3.1.20}$$

Similarly, from (3.1.11) and (3.1.10), the scalar potential is given by:

$$\varphi(\vec{r}) = \frac{-1}{i\omega\epsilon} \int_{V'} \vec{\nabla}' \cdot \vec{J}(\vec{r}\,')G(\vec{r};\vec{r}\,')dv'. \tag{3.1.21}$$

The General Radiation Solution (cont.)

Notice that the symbol $\vec{\nabla}$ acts on the observation coordinate $\vec{r}$, whereas $\vec{\nabla}'$ acts on the source coordinate $\vec{r}\,'$. Substituting these expressions for the potentials into (3.1.3) and using the relation $\omega\mu = k\eta$, we obtain the electric field in terms of the source distribution:

$$\vec{E}(\vec{r}) = -ik\eta \int_{V'} \left[\vec{J}(\vec{r}\,')G(\vec{r};\vec{r}\,') + \frac{1}{k^2}\left(\vec{\nabla}' \cdot \vec{J}(\vec{r}\,')\right)\vec{\nabla}G(\vec{r};\vec{r}\,')\right] dv'. \tag{3.1.22}$$

The magnetic field can be found by substituting (3.1.20) into (3.1.1):

$$\vec{H}(\vec{r}) = -\int_{V'} \vec{J}(\vec{r}\,') \times \vec{\nabla}G(\vec{r};\vec{r}\,')dv'. \tag{3.1.23}$$

It is convenient to define the following integral operators:

$$\mathfrak{L}\{\vec{X}\} = -ik \int_{V'} \left[\vec{X}G + \frac{1}{k^2}(\vec{\nabla}' \cdot \vec{X})\vec{\nabla}G \right] dv', \tag{3.1.24}$$

$$\mathfrak{K}\{\vec{X}\} = -\int_{V'} \vec{X} \times \vec{\nabla}G dv'. \tag{3.1.25}$$

The General Radiation Solution (cont.)

This expanded form makes the operator placement explicit: $\vec{\nabla}'$ differentiates the source, while $\vec{\nabla}$ differentiates G. Compact notations with G outside the bracket should be read this way. Using these operators, the electric and magnetic fields generated by electric and magnetic sources can be expressed in a compact and symmetric form. For electric sources:

$$\vec{E} = \eta\mathfrak{L}\{\vec{J}\}, \tag{3.1.26}$$

$$\vec{H} = \mathfrak{K}\{\vec{J}\}, \tag{3.1.27}$$

and by applying the duality transform, for magnetic sources:

$$\vec{E} = -\mathfrak{K}\{\vec{M}\}, \tag{3.1.28}$$

$$\vec{H} = \frac{1}{\eta}\mathfrak{L}\{\vec{M}\}. \tag{3.1.29}$$

By the principle of superposition, the total fields from both electric and magnetic sources are:

$$\vec{E} = \eta\mathfrak{L}\{\vec{J}\} - \mathfrak{K}\{\vec{M}\}, \tag{3.1.30}$$

The General Radiation Solution (cont.)

$$\vec{H} = \mathfrak{K}\{\vec{J}\} + \frac{1}{\eta}\mathfrak{L}\{\vec{M}\}. \tag{3.1.31}$$

It is worth noting that an alternative expression for the electric field can be derived by substituting (3.1.20) and the Lorenz gauge condition (3.1.8) into (3.1.3):

$$\vec{E}(\vec{r}) = -ik\eta \int_{V'} \left(1 + \frac{1}{k^2}\vec{\nabla}\vec{\nabla}\cdot\right) \left[\vec{J}(\vec{r}\,')G(\vec{r};\vec{r}\,')\right] dv'. \tag{3.1.32}$$

It is important to understand the distinction between the two expressions for the electric field, (3.1.22) and (3.1.32). In (3.1.22), $\vec{\nabla}'$ differentiates the source current and $\vec{\nabla}$ differentiates only the Green function. This results in a weaker singularity in the integrand. In contrast, in (3.1.32), both $\vec{\nabla}$ operators act on the field point $\vec{r}$, and therefore on the Green function, leading to a higher-order singularity. This form is typically more suitable for far-field calculations, where simplifying approximations can be made.

The Stratton-Chu Formulation

Based on the surface equivalence principle (Section 4), if all sources are enclosed within a closed surface S_0, we can replace them with a set of equivalent surface currents:

$$\begin{cases} \vec{J}_s = \hat{n} \times \vec{H}, \\ \vec{M}_s = -\hat{n} \times \vec{E}, \end{cases} \tag{3.1.33}$$

where $\hat{n}$ is the outward-pointing unit normal vector on the surface S_0. By placing these currents on the surface, we can set the fields inside S_0 to zero. Using the general radiation solution (3.1.30), the electric field outside S_0 can be expressed as:

$$\begin{aligned} \vec{E} &= \eta \mathfrak{L}\{\vec{J}_s\} - \mathfrak{K}\{\vec{M}_s\} \\ &= -ik\eta \oint_{S_0} \left[\vec{J}_s G + \frac{1}{k^2} \left(\vec{\nabla}' \cdot \vec{J}_s \right) \vec{\nabla} G \right] ds' + \oint_{S_0} \vec{M}_s \times \vec{\nabla} G ds'. \end{aligned} \tag{3.1.34}$$

From the continuity equation and the matching condition for the normal component of the electric field (1.2.8), we have:

$$\vec{\nabla}' \cdot \vec{J}_s = -i\omega\rho_s = -i\omega\epsilon \left(\hat{n} \cdot \vec{E} \right). \tag{3.1.35}$$

The Stratton-Chu Formulation (cont.)

Substituting (3.1.33) and (3.1.35) into (3.1.34) and using the property $\vec{\nabla}' G = -\vec{\nabla} G$, we arrive at the expression for the electric field:

$$\vec{E} = \oint_{S_0} \left[-ik\eta \left(\hat{n} \times \vec{H} \right) G + \left(\hat{n} \cdot \vec{E} \right) \vec{\nabla}' G + \left(\hat{n} \times \vec{E} \right) \times \vec{\nabla}' G \right] ds'. \tag{3.1.36}$$

By applying the duality transform, we can obtain the corresponding expression for the magnetic field:

$$\vec{H} = \oint_{S_0} \left[i\frac{k}{\eta} \left(\hat{n} \times \vec{E} \right) G + \left(\hat{n} \cdot \vec{H} \right) \vec{\nabla}' G + \left(\hat{n} \times \vec{H} \right) \times \vec{\nabla}' G \right] ds'. \tag{3.1.37}$$

This pair of equations constitutes the *Stratton–Chu formulation*, which provides a rigorous and general framework for solving radiation problems. Notably, the Stratton–Chu formulation explicitly involves the normal components of the electric and magnetic fields.

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The Far-Field Approximation

In the far-field region, the observation distance r is much larger than the source size $a = \max|\vec{r}\,'|$, and the observation point is many wavelengths away, $kr \gg 1$. These are independent requirements: $r \gg a$ controls the source-size expansion, while $kr \gg 1$ controls the radiation-zone $1/r$ approximation. The denominator of the Green function (3.1.12) can be approximated as $4\pi r$, and the phase term in the numerator can be expanded as

$$R = |\vec{r} - \vec{r}\,'| = \sqrt{r^2 - 2r(\hat{r}\cdot\vec{r}\,') + r'^2}$$
$$= r\left(1 - 2\frac{\hat{r}\cdot\vec{r}\,'}{r} + \frac{r'^2}{r^2}\right)^{1/2}.$$

Using the generalized binomial theorem with small argument x,

$$(1+x)^{1/2} \approx 1 + \tfrac{1}{2}x,$$

and retaining only the first-order term, we obtain

$$R \approx r - \hat{r}\cdot\vec{r}\,'.$$

The Far-Field Approximation (cont.)

The amplitude term $1/R$ may be approximated as $1/r$, since its correction terms are of order $1/(kr)$ and are negligible in the far zone. However, in the phase term e^{-ikR}, even small variations in R produce significant phase differences when multiplied by k. Hence, one additional term must be retained in the phase approximation:

$$e^{-ikR} \approx e^{-ikr(1-\hat{r}\cdot\vec{r}\,'/r)} \approx e^{-ikr}e^{ik\hat{r}\cdot\vec{r}\,'}. \tag{3.2.1}$$

This approximation preserves the correct angular dependence of the radiation pattern. Thus, in the far field, the Green function simplifies to

$$G(\vec{r};\vec{r}\,') \approx \frac{e^{-ikr}}{4\pi r}\,e^{ik\hat{r}\cdot\vec{r}\,'} = G_r(r)G_a(\theta,\phi), \tag{3.2.2}$$

where $G_r(r) = e^{-ikr}/(4\pi r)$ represents the radial dependence, and $G_a(\theta,\phi) = e^{ik\hat{r}\cdot\vec{r}\,'}$ represents the angular dependence.
Here $k = 2\pi/\lambda$ indicates that distances are effectively measured in wavelengths, and higher-order terms of order $1/(kr)$, $1/(kr)^2$, and beyond are neglected in the far-field

approximation. To apply this to the radiation integral (3.1.32), we first find the gradients of G_r and G_a:

$$\vec{\nabla} G_r = \hat{r}\partial_r \left(\frac{e^{-ikr}}{4\pi r}\right) = \hat{r}\left[-ikG_r + O\left(\frac{1}{r}\right)\right], \tag{3.2.3}$$

$$\vec{\nabla} G_a = \hat{\theta}\frac{1}{r}\partial_\theta \left(e^{ik\hat{r}\cdot\vec{r}\,'}\right) + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi \left(e^{ik\hat{r}\cdot\vec{r}\,'}\right) = O\left(\frac{1}{r}\right). \tag{3.2.4}$$

In the far field, the $1/r^2$ terms can be neglected, and the gradient of the Green function becomes:

$$\begin{aligned} \vec{\nabla} G &= G_a\vec{\nabla} G_r + G_r\vec{\nabla} G_a \\ &= -ikG\hat{r} + O\left(\frac{1}{r}\right) \approx -ikG\hat{r}. \end{aligned} \tag{3.2.5}$$

Then, the term $\vec{\nabla}\vec{\nabla}\cdot[\vec{J}(\vec{r}\,')G]$ can be approximated as:

$$\begin{aligned} \vec{\nabla}\vec{\nabla}\cdot\left[\vec{J}(\vec{r}\,')G\right] &= \vec{\nabla}\left[\vec{J}(\vec{r}\,')\cdot\vec{\nabla} G\right] \approx -ik\vec{\nabla}\left[\hat{r}\cdot\vec{J}(\vec{r}\,')G\right] \\ &= -ik\left\{\hat{r}\cdot\vec{J}(\vec{r}\,')\vec{\nabla} G + G\vec{\nabla}\left[\hat{r}\cdot\vec{J}(\vec{r}\,')\right]\right\}. \end{aligned} \tag{3.2.6}$$

The Far-Field Approximation (cont.)

To proceed with this derivation, we first note the following vector identity involving the unit dyadic $\bar{\bar{I}}$:

$$\begin{aligned}\vec{\nabla}\vec{r} &= (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)(\hat{x}x + \hat{y}y + \hat{z}z) \\ &= \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} = \bar{\bar{I}},\end{aligned} \tag{3.2.7}$$

where a dyadic is a tensor of rank two, which can be represented as a square matrix:

$$\begin{aligned}\bar{\bar{F}} &= F_{xx}\hat{x}\hat{x} + F_{yx}\hat{y}\hat{x} + F_{zx}\hat{z}\hat{x} \\ &\quad + F_{xy}\hat{x}\hat{y} + F_{yy}\hat{y}\hat{y} + F_{zy}\hat{z}\hat{y} \\ &\quad + F_{xz}\hat{x}\hat{z} + F_{yz}\hat{y}\hat{z} + F_{zz}\hat{z}\hat{z} \\ &= \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}.\end{aligned} \tag{3.2.8}$$

The juxtaposition of two vectors, such as $\bar{\bar{F}} = \vec{a}\vec{b}$, forms a dyadic product with components $F_{mn} = a_m b_n$. We have the following rules for dyadic calculations:

- $\vec{c} \cdot (\vec{a}\vec{b}) = (\vec{c} \cdot \vec{a})\vec{b}$

The Far-Field Approximation (cont.)

- $(\vec{a}\vec{b}) \cdot \vec{c} = \vec{a}(\vec{b} \cdot \vec{c})$
- $\vec{c} \times (\vec{a}\vec{b}) = (\vec{c} \times \vec{a})\vec{b}$
- $(\vec{a}\vec{b}) \times \vec{c} = \vec{a}(\vec{b} \times \vec{c})$

From the identity $\vec{\nabla}\vec{r} = \vec{\nabla}(r\hat{r}) = \vec{\nabla}(r)\hat{r} + r\vec{\nabla}(\hat{r}) = \hat{r}\hat{r} + r\vec{\nabla}(\hat{r}) = \bar{\bar{I}}$, we can derive:

$$\vec{\nabla}\hat{r} = \frac{\bar{\bar{I}} - \hat{r}\hat{r}}{r}. \tag{3.2.9}$$

The term $\vec{\nabla}[\hat{r} \cdot \vec{J}(\vec{r}\,')]$ in (3.2.6) can be simplified using the vector identity $\vec{\nabla}(\vec{a} \cdot \vec{b}) = \vec{a} \times (\vec{\nabla} \times \vec{b}) + \vec{b} \times (\vec{\nabla} \times \vec{a}) + (\vec{a} \cdot \vec{\nabla})\vec{b} + (\vec{b} \cdot \vec{\nabla})\vec{a}$:

$$\begin{aligned} \vec{\nabla}\left[\hat{r} \cdot \vec{J}(\vec{r}\,')\right] &= \left[\vec{J}(\vec{r}\,') \cdot \vec{\nabla}\right]\hat{r} = \vec{J}(\vec{r}\,') \cdot \vec{\nabla}\hat{r} \\ &= \frac{\vec{J} - J_r\hat{r}}{r} = \frac{J_\theta\hat{\theta} + J_\phi\hat{\phi}}{r} = O\left(\frac{1}{r}\right). \end{aligned} \tag{3.2.10}$$

The Far-Field Approximation (cont.)

Thus, the expression in (3.2.6) becomes:

$$\begin{aligned} \dots &= -ik\left\{\hat{r}\cdot\vec{J}(\vec{r}\,')\left[-ikG\hat{r} + O\left(\frac{1}{r}\right)\right] + G\times O\left(\frac{1}{r}\right)\right\} \\ &= -k^2\left[\vec{J}(\vec{r}\,')\cdot\hat{r}\right]\hat{r}G + O\left(\frac{1}{r}\right). \end{aligned} \tag{3.2.11}$$

Applying these approximations, the expression for the electric field in (3.1.32) simplifies to:

$$\begin{aligned} \vec{E} &\approx -ik\eta\int_{V'} G\left[\vec{J}(\vec{r}\,') - J_r\hat{r}\right]dv' \\ &= -ik\eta\frac{e^{-ikr}}{4\pi r}\int_{V'}\left(J_\theta\hat{\theta} + J_\phi\hat{\phi}\right)e^{ik\hat{r}\cdot\vec{r}\,'}dv' \\ &= -ik\eta\frac{e^{-ikr}}{4\pi r}\int_{V'}\left(\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}\right)\cdot\vec{J}(\vec{r}\,')e^{ik\hat{r}\cdot\vec{r}\,'}dv' \\ &= ik\eta\frac{e^{-ikr}}{4\pi r}\int_{V'}\hat{r}\times\left[\hat{r}\times\vec{J}(\vec{r}\,')\right]e^{ik\hat{r}\cdot\vec{r}\,'}dv'. \end{aligned} \tag{3.2.12}$$

The Far-Field Approximation (cont.)

From (3.1.23), the magnetic field in the far-field region is given by:

$$\vec{H} \approx -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}\left[\hat{r}\times\vec{J}(\vec{r}\,')\right]e^{ik\hat{r}\cdot\vec{r}\,'}dv' = \frac{1}{\eta}\hat{r}\times\vec{E}. \tag{3.2.13}$$

For the general case including both electric and magnetic sources, we can apply the duality principle to obtain the far-field expression for the electric field and the magnetic field:

$$\vec{E} \approx -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}\left[\eta\left(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi}\right)\cdot\vec{J}(\vec{r}\,')-\hat{r}\times\vec{M}(\vec{r}\,')\right]e^{ik\hat{r}\cdot\vec{r}\,'}dv', \tag{3.2.14}$$

$$\vec{H} \approx -ik\frac{e^{-ikr}}{4\pi r}\int_{V'}\left[\frac{1}{\eta}\left(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi}\right)\cdot\vec{M}(\vec{r}\,')+\hat{r}\times\vec{J}(\vec{r}\,')\right]e^{ik\hat{r}\cdot\vec{r}\,'}dv'. \tag{3.2.15}$$

It is interesting to note that by expressing the term $k\hat{r}$ in Cartesian coordinates, the far-field radiation pattern can be interpreted as the inverse Fourier transform (up to a constant factor) of the spatial distribution of the source currents.

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Radiation from a Hertzian Dipole

The Hertzian dipole is the simplest and most fundamental radiating element. It consists of an infinitesimal segment of current, dl, carrying a time-harmonic current $I = i\omega q$. When this dipole is oriented along the z-axis and located at the origin, the source distribution is described by $\vec{J}dv' = I\hat{z}dz'$. From the general radiation solution (3.1.32), the electric field is given by:

$$\vec{E}(\vec{r}) = -ik\eta Idl\left(1 + \frac{1}{k^2}\vec{\nabla}\vec{\nabla}\cdot\right)\hat{z}G. \tag{3.3.1}$$

Similarly, from (3.1.23), the magnetic field is:

$$\vec{H}(\vec{r}) = -Idl\,\hat{z}\times\vec{\nabla}G. \tag{3.3.2}$$

To express these fields in spherical coordinates, we compute:

$$\vec{\nabla}G = \vec{\nabla}\left(\frac{e^{-ikr}}{4\pi r}\right) = -\left(ik + \frac{1}{r}\right)G\hat{r}, \tag{3.3.3}$$

$$\vec{\nabla}\cdot(\hat{z}G) = \hat{z}\cdot\vec{\nabla}G = -\left(ik + \frac{1}{r}\right)G\cos\theta, \tag{3.3.4}$$

Radiation from a Hertzian Dipole (cont.)

$$\vec{\nabla}\left[\vec{\nabla}\cdot(\hat{z}G)\right] = G\left[\left(-k^2+\frac{2ik}{r}+\frac{2}{r^2}\right)\cos\theta\hat{r}+\frac{1}{r}\left(ik+\frac{1}{r}\right)\sin\theta\hat{\theta}\right], \quad (3.3.5)$$

$$\hat{z}\times\vec{\nabla}G = \left(\cos\theta\hat{r}-\sin\theta\hat{\theta}\right)\times\vec{\nabla}G = -\left(ik+\frac{1}{r}\right)G\sin\theta\hat{\phi}. \quad (3.3.6)$$

Using these, the electric and magnetic fields become:

$$\vec{E} = \frac{\eta Idl}{r}\left(1+\frac{1}{ikr}\right)2\cos\theta G\hat{r} + ik\eta Idl\left(1+\frac{1}{ikr}-\frac{1}{k^2r^2}\right)\sin\theta G\hat{\theta}, \quad (3.3.7)$$

$$\vec{H} = ikIdl\left(1+\frac{1}{ikr}\right)\sin\theta G\hat{\phi}. \quad (3.3.8)$$

These expressions include terms that scale as $1/r$, $1/r^2$, and $1/r^3$, corresponding to radiative, inductive, and electrostatic components, respectively. This decomposition enables classification of the field into near-field ($kr \ll 1$) and far-field ($kr \gg 1$) regions.

Radiation from a Hertzian Dipole (cont.)

In the near-field, the higher-order terms dominate. Using the approximation $e^{-ikr} \approx 1$, the fields reduce to:

$$\vec{E} \approx -i\frac{\eta Idl}{4\pi kr^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}), \tag{3.3.9}$$

$$\vec{H} \approx \frac{Idl}{4\pi r^2}\sin\theta\hat{\phi}. \tag{3.3.10}$$

In the far-field, only the $1/r$ terms survive, and the fields simplify to:

$$\vec{E} \approx ik\eta Idl\sin\theta G\hat{\theta}, \tag{3.3.11}$$

$$\vec{H} \approx ikIdl\sin\theta G\hat{\phi}. \tag{3.3.12}$$

In this region, the fields are purely transverse and orthogonal, with power radiating outward. The time-averaged Poynting vector defined in (1.5.6) gives the radiated power density :

$$\vec{S}_{\text{avg}} = \frac{1}{2}\Re\left\{\vec{E}\times\vec{H}^*\right\} \approx \hat{r}\frac{1}{2}\eta(kIdl)^2\sin^2\theta|G|^2. \tag{3.3.13}$$

Radiation from a Hertzian Dipole (cont.)

To compute the total radiated power, we integrate the average Poynting vector over a spherical surface:

$$P_{\text{rad}} = \int_0^{2\pi} \int_0^{\pi} \vec{S}_{\text{avg}} \cdot \hat{r}\, r^2 \sin\theta \, d\theta \, d\phi = \frac{\eta (kIdl)^2}{12\pi}. \tag{3.3.14}$$

The radiation intensity is defined as $U(\theta, \phi) = r^2 \vec{S}_{\text{avg}} \cdot \hat{r}$, which for the Hertzian dipole becomes

$$U(\theta) = \frac{\eta (kIdl)^2}{32\pi^2} \sin^2\theta. \tag{3.3.15}$$

The *directivity pattern* is then defined as the ratio of the radiation intensity in a given direction to this average value:

$$D(\theta) = \frac{U(\theta)}{P_{\text{rad}}/(4\pi)} = \frac{3}{2} \sin^2\theta. \tag{3.3.16}$$

Here, the term $P_{\text{rad}}/4\pi$ represents the *average radiated power per unit solid angle*. A solid angle $d\Omega = \sin\theta \, d\theta \, d\phi$ measures the angular extent of a surface as seen from the origin, and a full sphere subtends a total solid angle of 4π steradians. This shows that the Hertzian dipole radiates with maximum intensity in the plane perpendicular to its

Radiation from a Hertzian Dipole (cont.)

axis ($\theta = \pi/2$) and no radiation along the axis itself ($\theta = 0$ or π). The three-dimensional directivity pattern of a Hertzian dipole is illustrated in Fig. 1.

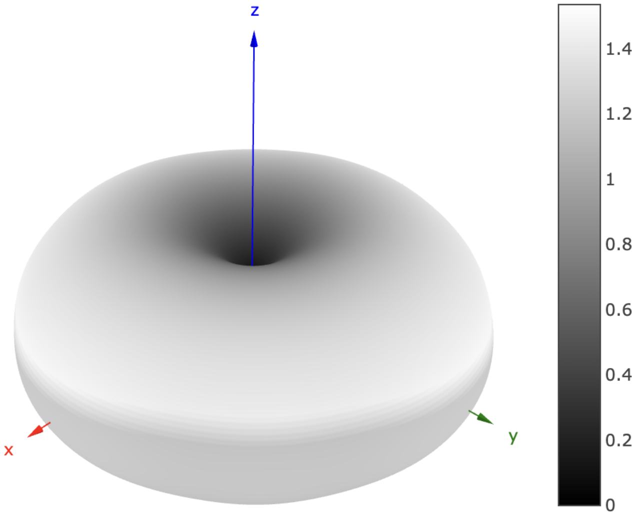
From the duality principle, the far field of a magnetic dipole source can be expressed as

$$\vec{E} \approx -ikI_m dl \sin\theta G\hat{\phi}, \tag{3.3.17}$$

$$\vec{H} \approx ik\frac{I_m}{\eta} dl \sin\theta G\hat{\theta}, \tag{3.3.18}$$

where I_m denotes the magnitude of the magnetic current. Although magnetic currents do not exist in the physical world, it can be shown that the radiation produced by an electric current loop is mathematically equivalent to that of a magnetic current dipole.

Radiation from a Hertzian Dipole (cont.)



Radiation from a Hertzian Dipole (cont.)

Figure: Directivity pattern of an Hertzian dipole.

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Image Theory

The image theory, discussed in this section, is a direct consequence of the uniqueness theorem. Once the field equations and boundary conditions at material interfaces are satisfied, the resulting field distribution is guaranteed to be unique. As a simple example, consider the electrostatic potential of a point charge located above a PEC plane. Let the conducting surface coincide with the plane $z = 0$, and a point charge q be placed at $(0, 0, h)$, where $h > 0$. The potential in the region $z > 0$ can be represented as the superposition of the field produced by the charge q and that produced by an image charge q' placed at $(0, 0, -h)$. In electrostatic cases, the magnitude of the image charge is determined by enforcing the PEC boundary condition at $z = 0$,

$$\varphi(z = 0) = 0,$$

which yields

$$q' = -q.$$

The resulting potential satisfies Laplace's equation in the upper half-space and the required boundary condition on the conducting surface, ensuring that it is the unique solution to the problem according to the uniqueness theorem.

Image Theory (cont.)

Extending the electrodynamic cases, the method allows us to replace the complex problem of a source and a conducting plane with a simpler equivalent problem of the original source and its image in free space. The key is to determine the correct orientation and phase of the image source such that the boundary conditions on the original conducting plane are satisfied. For a PEC, the tangential component of the electric field must be zero on its surface. For a PMC, the tangential component of the magnetic field must be zero on its surface.

We first consider a vertical electric dipole, oriented along the z-axis, located at a height h above a PEC plane at $z = 0$. From (3.3.11), it can be shown that for a vertical electric dipole, the image dipole is positioned at $z = -h$ and oriented in the same direction ($\hat{z}$) as the original dipole, ensuring that the total tangential electric field on the $z = 0$ plane is zero.

Next, consider a horizontal electric dipole, oriented along the x-axis, at a height h above the PEC plane. In this case, cancellation of the tangential electric field on the $z = 0$ plane requires an image dipole located at $z = -h$ and oriented in the opposite direction ($-\hat{x}$) to the original dipole.

Image Theory (cont.)

To verify these prescriptions, write the far field of a dipole with unit moment direction $\hat{p}$ in transverse form. Using $\sin\theta\,\hat{\theta} = -[\hat{p} - (\hat{p}\cdot\hat{R})\hat{R}]$, where $\hat{R}$ points from the dipole to the observation point,

$$\vec{E} \propto -G\,[\hat{p} - (\hat{p}\cdot\hat{R})\hat{R}].$$

For a source at $(0,0,h)$ and image at $(0,0,-h)$, a point $(x,y,0)$ on the plane has

$$\hat{R}_\pm = \frac{1}{R}(x,y,\mp h), \qquad R = \sqrt{x^2+y^2+h^2},$$

so the two Green functions are equal. For a vertical dipole, $\hat{p} = \hat{p}' = \hat{z}$, and the tangential x component is proportional to

$$-G\left[\left(0 - \frac{-h}{R}\frac{x}{R}\right) + \left(0 - \frac{h}{R}\frac{x}{R}\right)\right] = 0,$$

with the same cancellation for E_y. For a horizontal dipole, $\hat{p} = \hat{x}$ and $\hat{p}' = -\hat{x}$, giving

$$-G\left[\left(1 - \frac{x^2}{R^2}\right) + \left(-1 + \frac{x^2}{R^2}\right)\right] = 0.$$

Image Theory (cont.)

Thus the tangential electric field vanishes on the PEC plane, and uniqueness gives the image solution in the upper half-space.

Using analogous reasoning, the image configurations for a vertical magnetic dipole and a horizontal magnetic dipole above both PEC and PMC planes can also be determined.

The complete set of image configurations is summarized in Fig. 2.

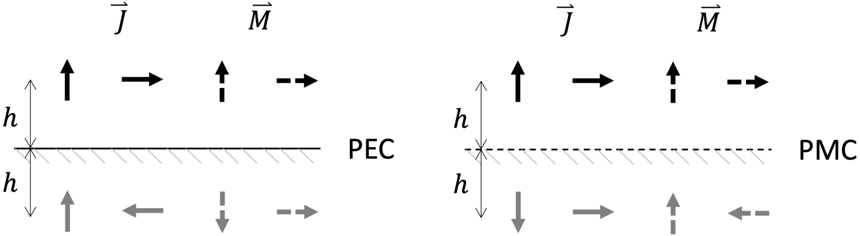

Figure: Illustration of image theory.

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Radiation from an Aperture

When an electromagnetic wave passes through an opening or aperture in a conducting screen, it radiates into the space beyond. According to the equivalence principle, the field in the region beyond the screen can be found by placing equivalent surface currents on the aperture.

Let the aperture be in the $z = 0$ plane. The equivalent surface currents are given by:

$$\vec{J}_s = \hat{z} \times \vec{H}_a, \quad \vec{M}_s = -\hat{z} \times \vec{E}_a, \tag{3.5.1}$$

where $\vec{E}_a$ and $\vec{H}_a$ are the electric and magnetic fields on the aperture. To simplify the analysis, consider the transformation of the original problem illustrated in Fig. 3. From the surface equivalence principle (1.5.29a), the PEC in the original problem is first replaced by surface currents on the z-plane. Since the tangential electric field on the PEC is zero, the corresponding equivalent magnetic surface current is also zero. In the second step, the surface equivalence principle (1.5.29b) is applied to replace V_1 with a PEC, and the equivalent electric surface current is set to zero in the original PEC region. At this stage, only the magnetic surface current remains on the original aperture. Finally, applying the image theory removes the PEC and doubles the equivalent magnetic surface current:

$$\vec{M}_s = -2\hat{z} \times \vec{E}_a, \tag{3.5.2}$$

yielding the final representation of the problem.

Radiation from an Aperture (cont.)

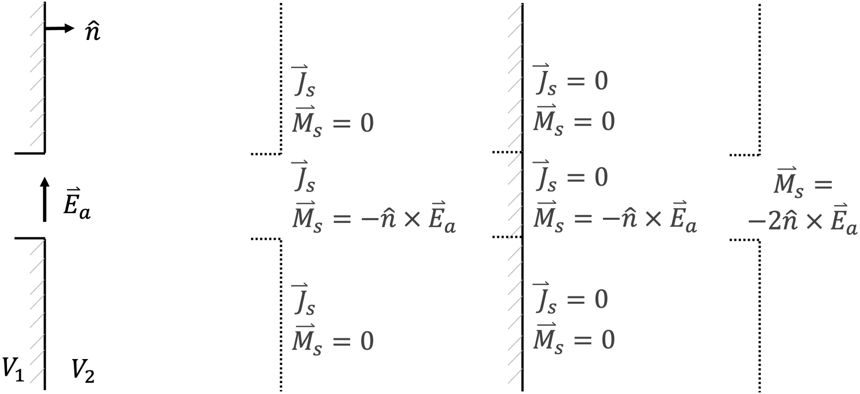

Figure: Illustration of the aperture radiation problem.

Now, the far-field electric field can then be found from the radiation integral (3.2.14) applied to the surface magnetic current:

$$\vec{E} \approx -ik\frac{e^{-ikr}}{2\pi r}\int_{S_A}\hat{r}\times\left(\hat{z}\times\vec{E}_a\right)e^{ik\hat{r}\cdot\vec{r}\,'}ds', \tag{3.5.3}$$

where S_A denotes the aperture area.
Consider a rectangular aperture mounted on an infinite conducting plane with dimensions a and b along the x and y axes, respectively. Assume the aperture is illuminated by a uniform plane wave traveling in the z-direction, such that the aperture electric field is given by $\vec{E}_a = E_0\hat{y}$. Then, (3.5.3) becomes:

$$\vec{E} \approx ikE_0\frac{e^{-ikr}}{2\pi r}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\begin{pmatrix}\sin\phi\,\hat{\theta}\\ +\cos\theta\cos\phi\,\hat{\phi}\end{pmatrix}e^{ik(x'u+y'v)}dx'dy', \tag{3.5.4}$$

where $u = \sin\theta\cos\phi$ and $v = \sin\theta\sin\phi$.

Radiation from an Aperture (cont.)

The polarization vector is independent of x' and y', so it factors out and the integral separates. Each factor is elementary:

$$\int_{-a/2}^{a/2} e^{ikux'}\,dx' = \frac{e^{ikua/2} - e^{-ikua/2}}{iku} = \frac{2\sin\left(\frac{kua}{2}\right)}{ku} = a\,\mathrm{sinc}\left(\frac{ka}{2}u\right),$$

and likewise $\int_{-b/2}^{b/2} e^{ikvy'}\,dy' = b\,\mathrm{sinc}\left(\frac{kb}{2}v\right)$, with $\mathrm{sinc}(x) = \sin x/x$. Multiplying the two,

$$\vec{E} \approx ikE_0 ab\frac{e^{-ikr}}{2\pi r}\begin{pmatrix} \sin\phi\,\hat{\theta} \\ +\cos\theta\cos\phi\,\hat{\phi}\end{pmatrix}\mathrm{sinc}\left(\frac{ka}{2}u\right)\mathrm{sinc}\left(\frac{kb}{2}v\right), \tag{3.5.5}$$

where $\mathrm{sinc}(x) = \sin(x)/x$. The radiation pattern is the product of two sinc functions, which is characteristic of rectangular apertures.

Further Reading

The exposition of radiation solutions within this chapter primarily draws upon Sheng's works Xin-Qing Sheng. *Electromagnetic Theory, Techniques, and Applications*. 2nd. In Chinese. Beijing: Tsinghua University Press, 2023; Xin-Qing Sheng and Wei Song. *Essentials of Computational Electromagnetics*. John Wiley & Sons, 2011. For comprehensive details regarding notations employing dyadic Green functions, the reader is directed to pertinent chapters in Pathak's text P. H. Pathak and R. J. Burkholder. *Electromagnetic Radiation, Scattering, and Diffraction*. Wiley, 2021 and Tai's authoritative book on this topicChen-To Tai. *General Vector and Dyadic Analysis: Applied Mathematics in Field Theory*. 2nd. IEEE Press, 1997. The Stratton–Chu formulation is extensively presented in Ishimaru's treatise Akira Ishimaru. *Electromagnetic Wave Propagation, Radiation and Scattering*. 2nd. Wiley, 2017. The application of the equivalence principle to aperture radiation is also demonstrated in S. K. Jeng's paper Shyh-Kang Jeng. "Scattering from a cavity-backed slit in a ground plant-TE case". In: *IEEE Transactions on Antennas and Propagation* 38.10 (1990), pp. 1523–1529, which shows how the aperture field due to a plane wave incident on a cavity-backed slit can be obtained, and how the corresponding radiation from the aperture, modeled as a magnetic current, may be computed using the same method described in this section.

Further Reading (cont.)

Certain advanced topics, including Liénard–Wiechert potentials and radiation from moving charges, fall outside the scope of this discussion. For these subjects, readers are encouraged to consult canonical graduate-level electrodynamics textbooks such as that by Zangwill Andrew Zangwill. *Modern Electrodynamics*. Cambridge University Press, 2013, or specialized volumes on radiation, such as Smith Glenn S Smith. *An Introduction to Classical Electromagnetic Radiation*. Cambridge University Press, 1997.

Problems

1. Verify (3.1.28) and (3.1.29) by applying the duality transform to (3.1.26) and (3.1.27).

2. Following the subsection 5, when magnetic sources are present but electric sources are absent, the electric flux density may be written using the electric vector potential

$$\vec{D} = -\vec{\nabla} \times \vec{F}.$$

Assuming that the Lorenz condition holds for electric vector potentials,

$$\vec{\nabla} \cdot \vec{F} = -i\omega\mu\epsilon\,\vartheta,$$

where $\vec{H} = -\vec{\nabla}\vartheta - i\omega\vec{F}$. Show that the electric and magnetic fields may be expressed in the general forms

$$\vec{E} = -i\omega\vec{A} + \frac{1}{i\omega\mu\epsilon}\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \frac{1}{\epsilon}\vec{\nabla} \times \vec{F},$$

$$\vec{H} = -i\omega\vec{F} + \frac{1}{i\omega\mu\epsilon}\vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) + \frac{1}{\mu}\vec{\nabla} \times \vec{A},$$

Problems (cont.)

3. Show that (3.1.32) is equal to

$$-ik\eta \int_{V'} \left[\left(\bar{\bar{I}} + \frac{1}{k^2} \vec{\nabla}\vec{\nabla} \right) G(\vec{r};\vec{r}\,') \right] \cdot \vec{J}(\vec{r}\,') dv'$$

$$= ik\eta \int_{V'} \bar{\bar{G}}(\vec{r};\vec{r}\,') \cdot \vec{J}(\vec{r}\,') dv',$$

where

$$\bar{\bar{G}}(\vec{r};\vec{r}\,') = - \left[\left(\bar{\bar{I}} + \frac{1}{k^2} \vec{\nabla}\vec{\nabla} \right) G(\vec{r};\vec{r}\,') \right]$$

is called the *dyadic Green function*.

4. Complete the intermediate steps in (3.3.3)–(3.3.8).