Wave Propagation and Transmission

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Chapter Overview

This chapter examines the fundamental principles of electromagnetic wave propagation and transmission. We begin with plane waves—the basic solutions to Maxwell equations—covering their angular-spectrum representation and polarization states. Wave behavior in unbounded media is then analyzed for both lossless and lossy cases. This leads to the study of oblique incidence on dielectric interfaces, including reflection and transmission, the critical angle, and Brewster's angle. The chapter concludes with guided-wave propagation in waveguides, outlining solution methods, mode characteristics, and the use of Green's identities and modal orthogonality.

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The Plane Wave Solution

Plane waves are the most fundamental solutions to the homogeneous Helmholtz equation (1.4.16) in Cartesian coordinates. The analysis begins with the Laplacian operator:

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{2.1.1}$$

Using the method of separation of variables, we first seek an elementary solution of the form

$$\psi(x, y, z) = X(x)Y(y)Z(z). \tag{2.1.2}$$

In general, the complete solution can be expressed as a weighted summation or integral of these elementary solutions, depending on whether the spectrum of allowable separation constants is discrete or continuous. Substituting this into the Helmholtz equation yields:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0. \tag{2.1.3}$$

For this equation to be valid for all x, y, z, each term must be a constant. This requirement separates the partial differential equation into three ordinary differential equations:

$$\frac{d^2\Psi}{d\xi^2} + k_\xi^2\Psi = 0, \tag{2.1.4}$$

The Plane Wave Solution (cont.)

where $\Psi \in \{X, Y, Z\}$ and $\xi \in \{x, y, z\}$. The separation constants must satisfy the relation:

$$k_x^2 + k_y^2 + k_z^2 = k^2. \tag{2.1.5}$$

These constants form the components of the wavevector, defined as:

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = k\hat{k}. \tag{2.1.6}$$

The elementary solution is a complex exponential, representing a plane wave:

$$\psi = e^{\pm i(k_x x + k_y y + k_z z)} = e^{\pm i\vec{k}\cdot\vec{r}}. \tag{2.1.7}$$

When converted to the time domain using the $e^{i\omega t}$ convention, the term $e^{-i\vec{k}\cdot\vec{r}}$ represents a wave propagating in the $+\vec{k}$ direction, while $e^{i\vec{k}\cdot\vec{r}}$ corresponds to a wave propagating in the $-\vec{k}$ direction. For the remainder of this text, we will adopt the former for illustration. The electric field of such a plane wave is thus expressed as:

$$\vec{E} = \vec{E}_0\psi = \vec{E}_0 e^{-i\vec{k}\cdot\vec{r}}, \tag{2.1.8}$$

where $\vec{E}_0$ is a constant vector representing the wave's amplitude and polarization.

The Plane Wave Solution (cont.)

For plane waves, the del operator $\vec{\nabla}$, when applied to (2.1.8), and notice that

$$\begin{aligned}\vec{\nabla}\, e^{-i\vec{k}\cdot\vec{r}} &= \left(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z\right) e^{-i(k_x x + k_y y + k_z z)} \\ &= \left(-ik_x\hat{x} + -ik_y\hat{y} + -ik_z\hat{z}\right) e^{-i\vec{k}\cdot\vec{r}},\end{aligned}$$

simplifies to multiplication by $-i\vec{k}$:

$$\vec{\nabla} \Rightarrow -i\vec{k}. \tag{2.1.9}$$

In a source-free region, Maxwell equations require that $\vec{\nabla}\cdot\vec{E} = 0$. Applying this condition to the plane-wave solution gives

$$\vec{\nabla}\cdot\left(\vec{E}_0 e^{-i\vec{k}\cdot\vec{r}}\right) = -i\left(\vec{k}\cdot\vec{E}_0\right) e^{-i\vec{k}\cdot\vec{r}} = 0. \tag{2.1.10}$$

For a homogeneous plane wave, where the wave vector $\vec{k}$ is real, this result implies that $\vec{k}\cdot\vec{E}_0 = 0$, meaning the electric field is perpendicular to the direction of propagation. When $\vec{k}$ is complex, corresponding to an inhomogeneous plane wave, the field vector is generally not orthogonal to $\vec{k}$. Such cases will be discussed in detail in Section 23. Special case where k is purely imaginary will be discussed in Section 9.

The Plane Wave Solution (cont.)

The magnetic field is then given by:

$$\begin{aligned}\vec{H} &= \frac{i}{\omega\mu}\vec{\nabla}\times\vec{E} = \frac{1}{\omega\mu}\vec{k}\times\vec{E} \\ &= \frac{1}{\eta}\hat{k}\times\vec{E} = \vec{H}_0 e^{-i\vec{k}\cdot\vec{r}},\end{aligned} \tag{2.1.11}$$

where $\vec{H}_0 = \frac{1}{\eta}\hat{k}\times\vec{E}_0$ and $\eta = \sqrt{\mu/\epsilon}$. This relationship implies that the wavevector $\vec{k}$, the electric field $\vec{E}$, and the magnetic field $\vec{H}$ are all mutually orthogonal, forming a right-handed system. By substituting these plane wave solutions back into the source-free Maxwell equations, we obtain the following general relationships for plane waves:

$$\vec{k}\times\vec{E} = \omega\mu\vec{H}, \tag{2.1.12}$$

$$\vec{k}\times\vec{H} = -\omega\epsilon\vec{E}, \tag{2.1.13}$$

$$\vec{k}\cdot\vec{E} = 0, \tag{2.1.14}$$

$$\vec{k}\cdot\vec{H} = 0. \tag{2.1.15}$$

Notice that the above equations also apply to $(\vec{E}_0, \vec{H}_0)$.

The Angular Spectrum Representation

The angular spectrum representation is a powerful technique that expresses any electromagnetic field as a superposition of plane waves propagating in various directions. In this subsection, we consider the problem of determining the electric field in the half spaces $z > 0$ and $z < 0$ produced by a prescribed tangential electric field distribution on the x–y plane. According to (1.2.3), this tangential field can be regarded as a surface magnetic current that radiates into the two half spaces. The goal is to represent the resulting field in terms of its angular spectrum components, which correspond to plane-wave contributions with different propagation directions. A similar approach will be applied later in Section 40 to analyze the radiation from an aperture.

We expand an electromagnetic field by evaluating its two-dimensional inverse Fourier transform on a plane perpendicular to the z-axis:

$$\vec{\mathcal{E}}(k_x, k_y, z) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \vec{E}(x, y, z) e^{i(k_x x + k_y y)} dxdy, \tag{2.1.16}$$

where x, y are the transverse position components in Cartesian coordinates and k_x, k_y are the corresponding spatial frequencies in k domain. The electromagnetic field itself

The Angular Spectrum Representation (cont.)

can then be reconstructed by taking the two-dimensional Fourier transform of this spectral field:

$$\vec{E}(x,y,z) = \iint_{-\infty}^{\infty} \vec{\mathcal{E}}(k_x,k_y,z)e^{-i(k_x x+k_y y)}dk_x dk_y. \tag{2.1.17}$$

By substituting this representation into the homogeneous vector Helmholtz equation, and defining the z-component of the wavevector as:

$$k_z = \begin{cases} \sqrt{k^2-k_x^2-k_y^2}, & k_x^2+k_y^2 \le k^2, \\ -i\sqrt{k_x^2+k_y^2-k^2}, & k_x^2+k_y^2 > k^2, \end{cases} \tag{2.1.18}$$

we can obtain the scalar Helmholtz equation for each Cartesian component of $\vec{\mathcal{E}}$, that is

$$\left(\frac{d^2}{dz^2}+k_z^2\right)\mathcal{E}_\xi = 0, \tag{2.1.19}$$

where $\xi \in \{x,y,z\}$. Thus, we find that the spectral field propagates along the z-axis according to:

$$\vec{\mathcal{E}}(k_x,k_y,z) = \vec{\mathcal{E}}(k_x,k_y,0)e^{\mp ik_z z}. \tag{2.1.20}$$

The Angular Spectrum Representation (cont.)

When $k_x^2 + k_y^2 > k^2$, write $\gamma = \sqrt{k_x^2 + k_y^2 - k^2} > 0$, so $k_z = -i\gamma$. The outgoing bounded choice depends on the half-space:

$$e^{-ik_z z} = e^{-\gamma z} \quad (z > 0), \qquad e^{+ik_z z} = e^{\gamma z} \quad (z < 0).$$

Thus a two-sided field radiated by data on the plane is written with the decaying factor $e^{-\gamma|z|}$; no single nonzero exponential is finite as $z \to +\infty$ and $z \to -\infty$.
Finally, the complete angular spectrum representation for the electric field is given by:

$$\vec{E}(x, y, z) = \iint_{-\infty}^{\infty} \vec{\mathcal{E}}(k_x, k_y, 0) e^{-i(k_x x + k_y y \pm k_z z)} dk_x dk_y. \tag{2.1.21}$$

While this representation inherently satisfies the Helmholtz equation, ensuring consistency with the full set of Maxwell equations requires that the wavevector $\vec{k}$ be perpendicular to the spectral amplitudes.
A further analysis of the plane wave factor $e^{-i(k_x x + k_y y \pm k_z z)}$ gives two characteristic solutions:

$$\begin{cases} e^{-i(k_x x + k_y y \pm k_z z)}, & k_x^2 + k_y^2 \leq k^2, \\ e^{-i(k_x x + k_y y)} \begin{cases} e^{-\gamma z}, & z > 0, \\ e^{\gamma z}, & z < 0, \end{cases} \quad \gamma = \sqrt{k_x^2 + k_y^2 - k^2}, & k_x^2 + k_y^2 > k^2. \end{cases} \tag{2.1.22}$$

The Angular Spectrum Representation (cont.)

The first represents propagating waves, which carry energy, the second represents evanescent waves, which are spatially decaying and do not propagate energy into the far field.

Polarization of Plane Waves

The polarization of an electromagnetic wave describes the orientation and time-variation of the electric field vector in the plane perpendicular to the direction of propagation. Consider a plane wave propagating in the $+z$-direction:

$$\vec{E} = (E_x \hat{x} + E_y \hat{y}) e^{-ikz}, \tag{2.1.23}$$

where

$$\begin{cases} E_x = |E_x| e^{i\delta_x} \\ E_y = |E_y| e^{i\delta_y} \end{cases} \tag{2.1.24}$$

are the complex amplitudes of the x and y components of the electric field. We can classify the polarization of the wave into three main types:

Linear Polarization.

A wave is linearly polarized if the phase difference between its x and y components is an integer multiple of π:

$$\delta_x - \delta_y = n\pi, \quad n \in \mathbb{Z}. \tag{2.1.25}$$

In this case, the electric field vector oscillates along a fixed line in the $x - y$ plane.

Circular Polarization.

Polarization of Plane Waves (cont.)

A wave is circularly polarized if the amplitudes of its x and y components are equal and the phase difference is an odd multiple of $\pi/2$:

$$|E_x| = |E_y|, \quad \delta_x - \delta_y = \frac{(2m+1)\pi}{2}, \quad m \in \mathbb{Z}. \tag{2.1.26}$$

Right-hand circular polarization (RHCP) and left-hand circular polarization (LHCP) are described by the following expressions:

$$\vec{E}_{RC} = E_0 \left(\hat{x} - i\hat{y}\right) e^{-ikz}, \tag{2.1.27a}$$

$$\vec{E}_{LC} = E_0 \left(\hat{x} + i\hat{y}\right) e^{-ikz}, \tag{2.1.27b}$$

where E_0 is a real constant.
In order to characterize the rotation direction, by applying (1.4.4), we can transform (2.1.27) into time domain:

$$\vec{E}_{RC}(z,t) = E_0 \left[\cos(\omega t - kz)\hat{x} + \sin(\omega t - kz)\hat{y}\right], \tag{2.1.28a}$$

$$\vec{E}_{LC}(z,t) = E_0 \left[\cos(\omega t - kz)\hat{x} - \sin(\omega t - kz)\hat{y}\right]. \tag{2.1.28b}$$

Polarization of Plane Waves (cont.)

Leaving $z = 0$, as t progresses, we can show that $\vec{E}_{RC}$ rotates counterclockwise and $\vec{E}_{LC}$ rotates clockwise on the $x - y$ plane when seen from the $+z$-axis. Notice that the optics community follows the opposite convention.

Elliptical Polarization.

Elliptical polarization is the most general case, occurring when the conditions for linear or circular polarization are not met. In this case, the tip of the electric field vector traces out an ellipse in the plane perpendicular to the direction of propagation.

Notice that most natural electromagnetic waves, such as daylight, are unpolarized, meaning they do not exhibit a fixed polarization state over time. Unlike idealized cases that are classified as linearly, circularly, or elliptically polarized, real-world electromagnetic waves often consist of rapidly fluctuating electric field orientations. As a result, their polarization cannot be described by a single deterministic pattern, but rather as a statistical mixture of all possible states.

Phase and Group Velocities

For a plane wave propagating in the z-direction, its time-domain representation can be written as

$$\vec{E}(z,t) = \Re\left\{\vec{E}_0 e^{i(\omega t - kz)}\right\}. \tag{2.1.29}$$

In this discussion, the wave number k is assumed to be real, corresponding to propagation in a lossless medium. This assumption allows us to define the phase and group velocities in a straightforward manner. The planes of constant phase are defined by the condition

$$\omega t - kz = C, \tag{2.1.30}$$

where C is a constant. By taking the time derivative of this expression, we can determine the *phase velocity*, which is the speed at which these planes of constant phase propagate:

$$v_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}. \tag{2.1.31}$$

In a *non-dispersive* medium, where the permittivity ϵ and permeability μ do not vary with frequency, the phase velocity remains constant for all frequencies.

A simple harmonic wave with a single frequency, characterized by a well-defined phase velocity, does not carry information on its own. Communication or signal transmission

Phase and Group Velocities (cont.)

requires modulating the carrier wave, which introduces a finite bandwidth or frequency spread. In such cases, the wave no longer consists of a single frequency but rather a superposition of closely spaced frequency components forming a wave packet. The velocity at which the envelope of this wave packet propagates, and hence the speed at which information or energy is transmitted, is known as the *group velocity*. To illustrate the concept, let us consider the following example.

For a narrow-band signal $s(t)$ modulated onto a high-frequency carrier $e^{i\omega_0 t}$, the overall envelope of the wave packet signal is $s(t)e^{i\omega_0 t}$. Let the Fourier transform of $s(t)$ be $S(\omega)$, then the Fourier transform of $s(t)e^{i\omega_0 t}$ will be $S(\omega - \omega_0)$. Suppose that the signal went through a distance z and was received by a receiver, then the field at the receiver can be expressed as

$$S_r(\omega, k) = S(\omega - \omega_0)e^{-ikz}. \tag{2.1.32}$$

Notice that the wavenumber k is also a function of ω. For narrow-band cases, we can expand k with the following expression

$$\begin{aligned} k &\approx k(\omega_0) + \left(dk/d\omega|_{\omega=\omega_0} \right) (\omega - \omega_0) \\ &= k_0 + \dot{k}\, (\omega - \omega_0)\,, \end{aligned} \tag{2.1.33}$$

Phase and Group Velocities (cont.)

where $k(\omega_0) = k_0$ and

$$\dot{k} = \left.\frac{dk}{d\omega}\right|_{\omega=\omega_0}$$

is the derivative of the wave number with respect to angular frequency, evaluated at the central frequency ω_0.

Plug the above equation into (2.1.32), we have

$$S_r(\omega, k) = S(\omega - \omega_0)e^{-ik_0 z}e^{-i(\omega-\omega_0)\dot{k}z}. \tag{2.1.34}$$

Inverse Fourier transform the above equation, we then get the time-domain signal at the receiver:

$$\begin{aligned} s_r(t) &= \mathfrak{F}^{-1}\left\{S_r(\omega)\right\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega-\omega_0)e^{-ik_0 z}e^{-i(\omega-\omega_0)\dot{k}z}e^{i\omega t}d\omega \\ &= \frac{e^{-ik_0 z}}{2\pi}\int_{-\infty}^{\infty} S(\Omega)e^{-i\dot{k}\Omega z}e^{i(\Omega+\omega_0)t}d\Omega \\ &= e^{i(\omega_0 t - k_0 z)}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\Omega)e^{i\Omega(t-\dot{k}z)}d\Omega \\ &= s(t-\dot{k}z)\,e^{i(\omega_0 t - k_0 z)}, \end{aligned} \tag{2.1.35}$$

Phase and Group Velocities (cont.)

which means that the wave packet travels with the group velocity of

$$v_g = \frac{1}{\dot{k}} = \frac{d\omega}{dk}. \tag{2.1.36}$$

The relationship between the phase and group velocities is given by:

$$v_g = \frac{1}{dk/d\omega} = \frac{1}{d(\frac{\omega}{v_p})/d\omega} = \frac{v_p}{1 - \frac{\omega}{v_p}\frac{dv_p}{d\omega}}. \tag{2.1.37}$$

The phenomenon of dispersion is characterized by the relationship between phase and group velocity:

1. *No Dispersion*: If $dv_p/d\omega = 0$, then $v_g = v_p$. All frequency components travel at the same speed.
2. *Normal Dispersion*: If $dv_p/d\omega < 0$, then $v_g < v_p$. Higher frequency components travel slower than lower frequency components.
3. *Anomalous Dispersion*: If $dv_p/d\omega > 0$, then $v_g > v_p$. Higher frequency components travel faster than lower frequency components.

Phase and Group Velocities (cont.)

An illustration of normal dispersion is shown in Fig. 1. In most transparent dielectrics, the phase velocity v_p decreases with increasing angular frequency, such that $dv_p/d\omega < 0$. According to (2.1.37), the group velocity is therefore smaller than the phase velocity ($v_g < v_p$). Typical examples include visible light propagation in glass or water, where higher-frequency (shorter-wavelength) components travel more slowly than lower-frequency components.

When $dv_p/d\omega > 0$, the group velocity becomes greater than the phase velocity ($v_g > v_p$), which defines anomalous dispersion. This condition commonly occurs in engineered metamaterials exhibiting negative dispersion.

Phase and Group Velocities (cont.)

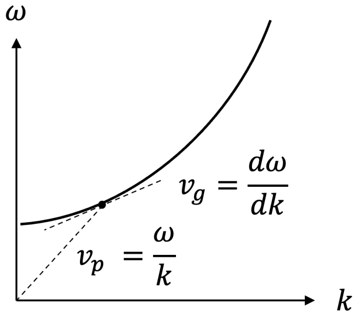

Figure: Illustration of phase velocity and group velocity.

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Propagation in an Unbounded Medium

In a lossless medium, the wavenumber $k = \omega\sqrt{\mu\epsilon}$ is a real number. However, the unit wavevector $\hat{k}$ can be a complex vector. If we express the complex unit wavevector as

$$\hat{k} = \vec{k}' - i\vec{k}'', \tag{2.2.1}$$

where $\vec{k}'$ and $\vec{k}''$ are real vectors, then the condition $\hat{k} \cdot \hat{k} = 1$ implies:

$$(k')^2 - (k'')^2 - 2i\vec{k}' \cdot \vec{k}'' = 1, \tag{2.2.2}$$

where $k' = |\vec{k}'|$ and $k'' = |\vec{k}''|$. This single complex equation is equivalent to two real equations:

$$(k')^2 - (k'')^2 = 1, \tag{2.2.3a}$$

$$\vec{k}' \cdot \vec{k}'' = 0. \tag{2.2.3b}$$

The simplest case is that of a uniform plane wave, where

$$k' = 1, \quad k'' = 0. \tag{2.2.4}$$

In the general case, where both $\vec{k}'$ and $\vec{k}''$ are non-zero, the solution $\vec{E} = \vec{E}_0 e^{-i\vec{k}\cdot\vec{r}}$ takes the form:

$$\vec{E} = \vec{E}_0 e^{-k\,\vec{k}''\cdot\vec{r}} e^{-ik\,\vec{k}'\cdot\vec{r}}. \tag{2.2.5}$$

Propagation in an Unbounded Medium (cont.)

This represents a non-uniform plane wave, in which the planes of constant phase are perpendicular to the real part of the wavevector, $\vec{k}'$, and the planes of constant amplitude are perpendicular to the imaginary part, $\vec{k}''$. According to (2.2.3b), $\vec{k}'$ and $\vec{k}''$ are orthogonal.

In a lossy medium, the wavenumber itself becomes a complex number, $k = \omega\sqrt{\mu\epsilon_c} = \beta - i\alpha$. If the wavevector $\hat{k}$ is real, the solution takes the form:

$$\vec{E} = \vec{E}_0 e^{-i\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-\alpha\hat{k}\cdot\vec{r}} e^{-i\beta\hat{k}\cdot\vec{r}}. \tag{2.2.6}$$

This represents an attenuated uniform plane wave, not an evanescent wave: the constant-phase and constant-amplitude planes are both normal to $\hat{k}$, and the time-averaged power still flows along $\hat{k}$ while decaying with distance. Using the expression in (1.4.10) and (1.4.11), the propagation constant β and the attenuation constant α are given by:

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+(\tan\delta)^2}+1\right]\right\}^{1/2}, \tag{2.2.7}$$

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+(\tan\delta)^2}-1\right]\right\}^{1/2}. \tag{2.2.8}$$

Propagation in an Unbounded Medium (cont.)

The intrinsic impedance and phase velocity are given by

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\,(1 - i\tan\delta)^{-1/2}\,, \tag{2.2.9}$$

$$v_p = \frac{\omega}{\beta} = \left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1 + (\tan\delta)^2} + 1\right]\right\}^{-1/2}. \tag{2.2.10}$$

Oblique Incidence at an Interface

We now examine the interaction of a plane electromagnetic wave with a planar boundary separating two simple media. Specifically, we derive the general expressions for reflection and transmission under oblique incidence, where the angle of incidence θ_i is arbitrary. This analysis forms the basis for understanding various wave-interface phenomena, including total internal reflection and polarization-dependent reflection. Notice that special cases such as normal incidence (by setting $\theta_i = 0$) and incidence upon a PEC interface (by taking the limit $\epsilon_2 \to -i\infty$) can be obtained directly from this more general analysis and provide useful limiting behavior that simplifies the resulting expressions. Let the amplitude of the incident electric field be E_i. In order to characterize the phenomena, we define the *reflection* and *transmission coefficients*, Γ and T, as the ratios of the amplitudes of the reflected and transmitted electric field to the incident electric field, respectively:

$$\Gamma = \frac{E_r}{E_i}, \tag{2.2.11}$$

$$T = \frac{E_t}{E_i}, \tag{2.2.12}$$

where E_r is the amplitude of the reflected electric field and E_t is the amplitude of the transmitted electric field. We will consider the two principal polarizations: *perpendicular*

Oblique Incidence at an Interface (cont.)

and *parallel*, illustrated in Fig. 2. Any arbitrarily polarized plane wave can be decomposed into these two cases, so this analysis provides a comprehensive understanding of wave interactions at boundaries.

Oblique Incidence at an Interface (cont.)

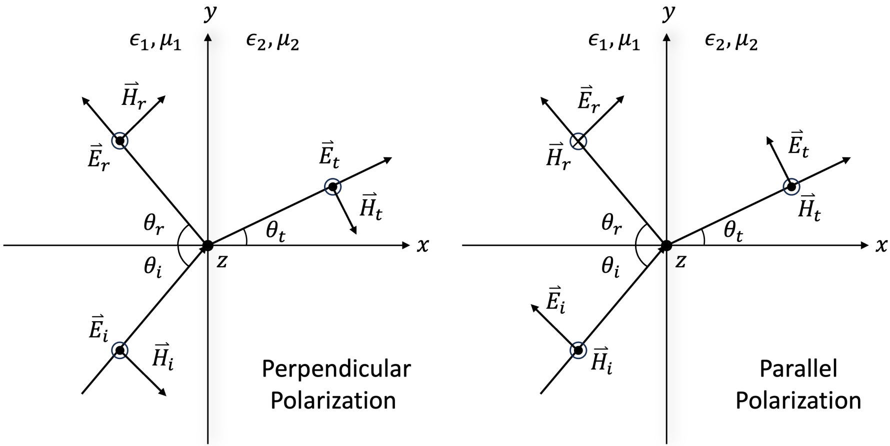


Figure: Illustration of the plane wave incident upon a plane interface with (i) perpendicular and (ii) parallel polarization.

Oblique Incidence at an Interface (cont.)

Perpendicular Polarization.
For perpendicular polarization, the incident electric field is oriented normal to the plane of incidence. From (2.1.8) and (2.1.12), the incident electric field can be expressed as:

$$\vec{E}_i = E_i e^{-i\vec{k}_i\cdot\vec{r}} = E_i e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}\hat{z}, \tag{2.2.13a}$$

$$\begin{aligned}\vec{H}_i &= \frac{1}{\eta_1}\hat{k}_1 \times \vec{E}_i = \frac{1}{\eta_1}(\cos\theta_i\hat{x} + \sin\theta_i\hat{y}) \times \vec{E}_i \\ &= \frac{E_i}{\eta_1} e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}(\sin\theta_i\hat{x} - \cos\theta_i\hat{y}),\end{aligned} \tag{2.2.13b}$$

where $\vec{k}_i = k_1\hat{k}_i$ and $k_n = \omega\sqrt{\epsilon_n\mu_n}$, $\quad \eta_n = \sqrt{\mu_n/\epsilon_n}$ $(n := 1, 2)$. The reflected and transmitted fields are expressed similarly:

$$\vec{E}_r = \Gamma_\perp E_i e^{-ik_1(-x\cos\theta_r + y\sin\theta_r)}\hat{z}, \tag{2.2.14a}$$

$$\vec{H}_r = \frac{\Gamma_\perp E_i}{\eta_1} e^{-ik_1(-x\cos\theta_r + y\sin\theta_r)}(\sin\theta_r\hat{x} + \cos\theta_r\hat{y}), \tag{2.2.14b}$$

Oblique Incidence at an Interface (cont.)

$$\vec{E}_t = T_\perp E_i e^{-ik_2(x\cos\theta_t + y\sin\theta_t)}\hat{z}, \tag{2.2.15a}$$

$$\vec{H}_t = \frac{T_\perp E_i}{\eta_2} e^{-ik_2(x\cos\theta_t + y\sin\theta_t)}(\sin\theta_t\hat{x} - \cos\theta_t\hat{y}). \tag{2.2.15b}$$

Applying the matching conditions for the tangential components of the electric and magnetic fields at the interface ($x = 0$):

$$(\vec{E}_i + \vec{E}_r)|^z_{x=0} = \vec{E}_t|^z_{x=0}, \tag{2.2.16a}$$

$$(\vec{H}_i + \vec{H}_r)|^y_{x=0} = \vec{H}_t|^y_{x=0}, \tag{2.2.16b}$$

we get

$$e^{-ik_1 y\sin\theta_i} + \Gamma_\perp e^{-ik_1 y\sin\theta_r} = T_\perp e^{-ik_2 y\sin\theta_t}, \tag{2.2.17a}$$

$$\frac{1}{\eta_1}\left(-\cos\theta_i e^{-ik_1 y\sin\theta_i} + \Gamma_\perp\cos\theta_r e^{-ik_1 y\sin\theta_r}\right) = -\frac{1}{\eta_2}T_\perp\cos\theta_t e^{-ik_2 y\sin\theta_t}. \tag{2.2.17b}$$

Oblique Incidence at an Interface (cont.)

For these boundary conditions to hold for all y on the interface, the phase terms must be equal. This requirement leads to the phase-matching condition, also known as *Snell's law*:

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t. \tag{2.2.18}$$

This result implies that the angle of reflection equals the angle of incidence, $\theta_r = \theta_i$. Using the phase-matching condition, the boundary condition equations simplify to:

$$1 + \Gamma_\perp = T_\perp, \tag{2.2.19a}$$

$$\frac{\cos\theta_i}{\eta_1}(-1 + \Gamma_\perp) = -\frac{\cos\theta_t}{\eta_2} T_\perp. \tag{2.2.19b}$$

Solving (2.2.19), we can derive the reflection and transmission coefficients for perpendicular polarization:

$$\Gamma_\perp = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}, \tag{2.2.20a}$$

$$T_\perp = \frac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}. \tag{2.2.20b}$$

The terms $\eta_1/\cos\theta_i$ and $\eta_2/\cos\theta_t$ can be interpreted as the wave impedances for the tangential field components.

Parallel Polarization.

For parallel polarization, the incident electric field lies within the plane of incidence. Following similar derivations, the corresponding fields are expressed as:

$$\vec{E}_i = E_i e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}(-\sin\theta_i\hat{x} + \cos\theta_i\hat{y}), \tag{2.2.21a}$$

$$\vec{H}_i = \frac{E_i}{\eta_1} e^{-ik_1(x\cos\theta_i + y\sin\theta_i)}\hat{z}, \tag{2.2.21b}$$

$$\vec{E}_r = \Gamma_\parallel E_i e^{-ik_1(-x\cos\theta_r + y\sin\theta_r)}(\sin\theta_r\hat{x} + \cos\theta_r\hat{y}), \tag{2.2.22a}$$

$$\vec{H}_r = -\frac{\Gamma_\parallel E_i}{\eta_1} e^{-ik_1(-x\cos\theta_r + y\sin\theta_r)}\hat{z}, \tag{2.2.22b}$$

$$\vec{E}_t = T_\parallel E_i e^{-ik_2(x\cos\theta_t + y\sin\theta_t)}(-\sin\theta_t\hat{x} + \cos\theta_t\hat{y}), \tag{2.2.23a}$$

$$\vec{H}_t = \frac{T_\parallel E_i}{\eta_2} e^{-ik_2(x\cos\theta_t + y\sin\theta_t)}\hat{z}. \tag{2.2.23b}$$

Oblique Incidence at an Interface (cont.)

Applying the boundary conditions at the interface $x = 0$ yields the same phase-matching relation as before, and the reflection and transmission coefficients are derived in a similar manner:

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}, \tag{2.2.24a}$$

$$T_\parallel = \frac{2\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \left(\frac{\cos\theta_i}{\cos\theta_t}\right). \tag{2.2.24b}$$

The seemingly additional factor of $\cos\theta_i / \cos\theta_t$ in the transmission coefficient arises from the definition of $T_\parallel$ as the ratio of the total electric field magnitudes, where the tangential components are projected differently: $T_\parallel = \frac{E_t}{E_i} = \frac{E_{yt} / \cos\theta_t}{E_{yi} / \cos\theta_i} = \frac{E_{yt}}{E_{yi}} \left(\frac{\cos\theta_i}{\cos\theta_t}\right)$.

The Critical Angle and Total Internal Reflection.

Total internal reflection can occur when a wave propagates from a denser medium to a less dense one (i.e., $\sqrt{\epsilon_1\mu_1} > \sqrt{\epsilon_2\mu_2}$). The onset of this phenomenon is the critical angle, θ_c, which is the angle of incidence that results in a transmission angle of $\theta_t = 90^\circ$. From (2.2.18), we have

$$\sqrt{\epsilon_1\mu_1} \sin\theta_c = \sqrt{\epsilon_2\mu_2}, \tag{2.2.25}$$

Oblique Incidence at an Interface (cont.)

or,

$$\theta_c = \sin^{-1}\left(\sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}}\right). \tag{2.2.26}$$

For incidence above the critical angle, $\theta_i > \theta_c$, the wave is totally reflected and the transmitted field is evanescent; at $\theta_i = \theta_c$ the transmitted wave is the grazing limiting case. To understand what happens to the transmitted field, let us examine the case of perpendicular polarization. The transmitted field is given by

$$\begin{aligned} E_t &= T_\perp E_i e^{-ik_2(x\cos\theta_t + y\sin\theta_t)} \\ &= T_\perp E_i e^{-ik_1 y\sin\theta_i} e^{-ik_2 x\cos\theta_t}, \end{aligned} \tag{2.2.27}$$

where

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}\sin^2\theta_i}. \tag{2.2.28}$$

When $\theta_i > \theta_c$, the term under the square root in (2.2.28) becomes negative, making $\cos\theta_t$ purely imaginary, as illustrated in Fig. 3. This figure shows that when the incident angle exceeds the critical angle, the refracted ray no longer propagates into the second

Oblique Incidence at an Interface (cont.)

medium but instead gives rise to an evanescent field that decays exponentially away from the interface.

The complex transmission angle can be expressed as

$$\begin{aligned}\cos\theta_t &= \cos\left(\frac{\pi}{2} + i\delta_t\right) = -i\sinh\delta_t \\ &= -i\sqrt{\frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}\sin^2\theta_i - 1},\end{aligned} \tag{2.2.29}$$

where $\sinh\delta_t > 0$. The quantity δ_t characterizes the rate of exponential decay of the evanescent wave in the second medium. Physically, the complex angle $\theta_t = \frac{\pi}{2} + i\delta_t$ indicates that the wavefronts remain parallel to the interface, while the field amplitude decreases exponentially in the direction normal to it.

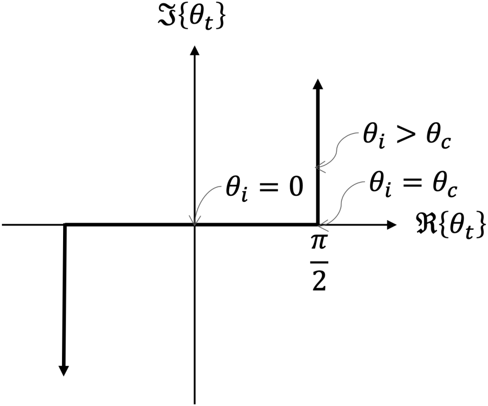
Substituting this expression into (2.2.27), the transmitted field becomes

$$E_t = T_\perp E_i e^{-\sinh\delta_t k_2 x} e^{-ik_1 y\sin\theta_i}. \tag{2.2.30}$$

This represents an evanescent field confined near the interface, with its amplitude decaying into the optically rarer medium.

Oblique Incidence at an Interface (cont.)

This is the mathematical form of a non-uniform plane wave, as seen in (2.2.5). For incident angles greater than the critical angle, the transmitted field becomes an evanescent wave that decays exponentially from the interface and does not propagate energy into the second medium.



Oblique Incidence at an Interface (cont.)

Figure: Illustration of the complex θ_t plane.

Brewster's Angle.

Brewster's angle, θ_b, is the specific angle of incidence at which the reflection coefficient is zero, meaning no wave is reflected. This condition, $\Gamma = 0$, is met when the numerators of the reflection coefficient equations, (2.2.20a) or (2.2.24a), are zero.
For perpendicular polarization, we need $\eta_2/\cos\theta_t = \eta_1/\cos\theta_i$, or

$$(1-\sin^2\theta_i) = \frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}(1-\sin^2\theta_t). \tag{2.2.31}$$

From (2.2.18), we have $\sin^2\theta_t = \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i$. Thus,

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\mu_1/\mu_2 - \mu_2/\mu_1}}. \tag{2.2.32}$$

A real solution for θ_i exists only if $\sqrt{\frac{\epsilon_2/\epsilon_1-\mu_2/\mu_1}{\mu_1/\mu_2-\mu_2/\mu_1}} \leq 1$, or

$$|\epsilon_2/\epsilon_1 - \mu_2/\mu_1| \leq |\mu_1/\mu_2 - \mu_2/\mu_1|. \tag{2.2.33}$$

Oblique Incidence at an Interface (cont.)

However, for most non-magnetic materials where $\mu_1 = \mu_2 = \mu_0$, the denominator of (2.2.32) becomes zero. This implies that a Brewster's angle does not exist for perpendicular polarization in non-magnetic media.
For the parallel polarization we need $\eta_2 \cos\theta_t = \eta_1 \cos\theta_i$, or

$$(1 - \sin^2\theta_i) = \frac{\mu_2\epsilon_1}{\mu_1\epsilon_2}(1 - \sin^2\theta_t). \tag{2.2.34}$$

Following a similar analysis and using Snell's law, we find:

$$\sin\theta_i = \sqrt{\frac{\epsilon_2/\epsilon_1 - \mu_2/\mu_1}{\epsilon_2/\epsilon_1 - \epsilon_1/\epsilon_2}}. \tag{2.2.35}$$

For θ_i to have real solution, it is required that

$$|\epsilon_2/\epsilon_1 - \mu_2/\mu_1| \leq |\epsilon_2/\epsilon_1 - \epsilon_1/\epsilon_2|. \tag{2.2.36}$$

For the common case of non-magnetic materials, this simplifies to:

$$\begin{aligned}\theta_b &= \sin^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1+\epsilon_2}}\right) = \cos^{-1}\left(\sqrt{\frac{\epsilon_1}{\epsilon_1+\epsilon_2}}\right) \\ &= \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right),\end{aligned} \tag{2.2.37}$$

Oblique Incidence at an Interface (cont.)

where θ_b is called the Brewster's angle.
It is worth noting that since $\sqrt{\epsilon_2}\sin\theta_t = \sqrt{\epsilon_1}\sin\theta_b = \sqrt{\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}}$, we have

$$\begin{aligned}\theta_t &= \sin^{-1}\left(\sqrt{\frac{\epsilon_1}{\epsilon_1+\epsilon_2}}\right) = \cos^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1+\epsilon_2}}\right) \\ &= \tan^{-1}\left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right).\end{aligned} \tag{2.2.38}$$

Thus, we have the following relationship:

$$\theta_b + \theta_t = \frac{\pi}{2}. \tag{2.2.39}$$

The relationship between θ_b and θ_t can be visualized, as shown in Fig. 4.

Oblique Incidence at an Interface (cont.)

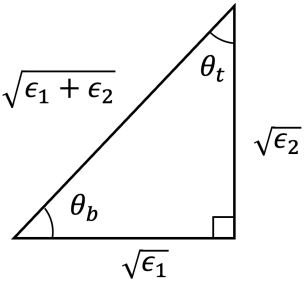


Figure: Illustration of the relationship between θ_b and θ_t.

Oblique Incidence at an Interface (cont.)

An intuitive physical explanation for Brewster's angle is as follows: the transmitted electric field induces oscillating charges in the second medium. These oscillating charges behave as small *dipole antennas* that re-radiate electromagnetic waves, producing the reflected field. However, a dipole antenna does not radiate along its axis of oscillation, which will be discussed in greater detail in Section 27. At Brewster's angle, the direction of the would-be reflected wave coincides with the dipole's oscillation axis, making this transverse component zero. As a result, no energy is radiated in the reflection direction, and the reflected wave vanishes.

Table of contents

Solving Waveguide Problems

Waveguides are structures designed to guide the propagation of electromagnetic waves. Beyond simple Transverse Electro-Magnetic (TEM) waves, they can support a variety of complex wave patterns, or modes. For a waveguide that is uniform along the z-direction, a common analytical approach is to first determine the longitudinal field components (E_z and H_z) and subsequently derive the transverse components from them. We assume that the fields have the following form:

$$\vec{E}(x, y, z) = \vec{E}_0(x, y)e^{-ik_z z}, \tag{2.3.1a}$$

$$\vec{H}(x, y, z) = \vec{H}_0(x, y)e^{-ik_z z}. \tag{2.3.1b}$$

By separating the fields into their transverse and longitudinal components (e.g., $\vec{E}_0 = \vec{E}^0_\perp + E^0_z \hat{z}$) and substituting these into the source-free Maxwell equations with $\partial_z \to -ik_z$, we can express the transverse field components in terms of the longitudinal components E^0_z and H^0_z:

$$\vec{E}^0_\perp = -\frac{i}{k_c^2}\left(k_z \vec{\nabla}_\perp E^0_z - \omega\mu\, \hat{z} \times \vec{\nabla}_\perp H^0_z\right), \tag{2.3.2a}$$

$$\vec{H}^0_\perp = -\frac{i}{k_c^2}\left(k_z \vec{\nabla}_\perp H^0_z + \omega\epsilon\, \hat{z} \times \vec{\nabla}_\perp E^0_z\right). \tag{2.3.2b}$$

where $\vec{\nabla}_\perp = (\hat{x}\partial_x + \hat{y}\partial_y)$ and

$$k_c^2 = k^2 - k_z^2. \tag{2.3.3}$$

These general expressions can be simplified for specific mode types: **TM Modes.**
For Transverse Magnetic (TM) modes, the magnetic field is purely transverse to the direction of propagation, so $H_z^0 = 0$ and $E_z^0 \neq 0$. The transverse fields are then given by:

$$H_z^0 = 0, \tag{2.3.4a}$$

$$\vec{E}_\perp^0 = -\frac{i}{k_c^2} k_z \vec{\nabla}_\perp E_z^0, \tag{2.3.4b}$$

$$\vec{H}_\perp^0 = -\frac{i}{k_c^2}\omega\epsilon\, \hat{z} \times \vec{\nabla}_\perp E_z^0. \tag{2.3.4c}$$

TE Modes.
For Transverse Electric (TE) modes, the electric field is purely transverse, so $E_z^0 = 0$ and $H_z^0 \neq 0$. The transverse fields are then:

$$E_z^0 = 0, \tag{2.3.5a}$$

$$\vec{E}_\perp^0 = \frac{i}{k_c^2}\omega\mu\, \hat{z} \times \vec{\nabla}_\perp H_z^0, \tag{2.3.5b}$$

Solving Waveguide Problems (cont.)

$$\vec{H}_\perp^0 = -\frac{i}{k_c^2} k_z \vec{\nabla}_\perp H_z^0. \tag{2.3.5c}$$

By substituting (2.3.1) into the vector Helmholtz equation, we can show that the longitudinal field components, E_z^0 and H_z^0, must satisfy the two-dimensional Helmholtz equation:

$$\left(\nabla_\perp^2 + k_c^2\right)\psi = 0, \tag{2.3.6}$$

where ψ represents either E_z^0 or H_z^0. The longitudinal fields can be determined from this equation based on the specific matching conditions, after which the transverse fields can be obtained using (2.3.2).

For a waveguide constructed from a PEC, referring to the geometry shown in Fig. 5, the tangential component of the electric field must be zero at the waveguide walls:

$$\left(\hat{n} \times \vec{E}\right)\Big|_W = 0, \tag{2.3.7}$$

where the subscript W denotes the surface of the waveguide and $\hat{n}$ is the normal unit vector pointing outwards. This implies that for TM waves, we must have

$$E_z^0\Big|_W = 0, \tag{2.3.8}$$

Solving Waveguide Problems (cont.)

which is a Dirichlet boundary condition. For TE waves, considering the tangential component of the electric field $E_\tau^0 = \frac{i\omega\mu}{k_c^2}\partial_n H_z^0$, where $\hat{\tau} = \hat{z} \times \hat{n}$, we must have

$$\left(\partial_n H_z^0\right)\Big|_W = 0, \tag{2.3.9}$$

which is a Neumann boundary condition.

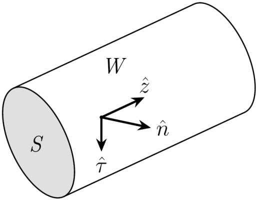


Figure: Illustration of a waveguide cross section.

Solving Waveguide Problems (cont.)

Imposing these boundary conditions restricts solutions to a set of discrete values of k_c, known as eigenvalues. Each eigenvalue characterizes a distinct waveguide mode. The quantity k_c is related to the cutoff frequency f_c of the waveguide mode by

$$k_c = \omega_c \sqrt{\mu\epsilon}. \tag{2.3.10}$$

From (2.3.3), we can get

$$k_z = k_c \sqrt{(f/f_c)^2 - 1} = k\sqrt{1 - (f_c/f)^2}, \tag{2.3.11}$$

where

$$f_c = \frac{\omega_c}{2\pi} = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} \tag{2.3.12}$$

is the cut-off frequency. A propagating solution exists only if k_z is real, which requires the operating frequency f to be greater than the cutoff frequency f_c. This behavior demonstrates that waveguides function as high-pass filters.

Generally, $k_z = \beta - i\alpha$. If the waveguide is lossless, then $\beta = k_z$. For a lossless waveguide operating above the cutoff frequency ($f > f_c$), the guided wavelength in the z-direction is given by

$$\lambda_z = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} > \lambda, \tag{2.3.13}$$

Solving Waveguide Problems (cont.)

where $\lambda = 2\pi/k$ is the wavelength of a plane wave in an unbounded medium of the same material. The phase velocity is then calculated by

$$v_{pz} = \frac{\omega}{\beta} = \frac{v}{\sqrt{1 - (f_c/f)^2}} > v \tag{2.3.14}$$

and the group velocity by

$$\begin{aligned} v_{gz} &= \frac{d\omega}{d\beta} = \frac{1}{d\beta/d\omega} = \frac{1}{\frac{d}{d\omega}\left[\frac{\omega}{v}\sqrt{1 - (\omega_c/\omega)^2}\right]} \\ &= \frac{v}{d(\sqrt{\omega^2 - \omega_c^2})/d\omega} = \frac{v}{\omega/\sqrt{\omega^2 - \omega_c^2}} \\ &= v\sqrt{1 - (f_c/f)^2} < v, \end{aligned} \tag{2.3.15}$$

where $v = \omega/k$ is phase velocity of a plane wave in an unbounded medium of the same material. From (2.3.14), the phase velocity in a waveguide is always higher than in an unbounded medium and varies with frequency, indicating dispersion. Its relation to group velocity is given by:

$$v_{pz} v_{gz} = v^2. \tag{2.3.16}$$

Characteristics of Waveguide Modes

This section introduces the concept of modal orthogonality in waveguides. We will establish that the various modes within a waveguide form an orthogonal set of solutions. This orthogonality, which is a direct consequence of Green's identities, is demonstrated with respect to a defined inner product.

A. Green's Identities.

Green's identities are fundamental results in vector calculus derived from the divergence theorem. For two scalar fields, f and g, the divergence theorem (1.1.9) leads to:

$$\int_V \vec{\nabla} \cdot \left(g \vec{\nabla} f \right) dv = \oint_S g \vec{\nabla} f \cdot d\vec{s}. \tag{2.3.17}$$

Expanding the integrand on the right-hand side, we obtain Green's first identity:

$$\int_V \left(g \nabla^2 f + \vec{\nabla} g \cdot \vec{\nabla} f \right) dv = \oint_S g \vec{\nabla} f \cdot d\vec{s} = \oint_S g \partial_n f ds, \tag{2.3.18}$$

where n denotes the outward normal direction. In two dimensions, this becomes:

$$\int_S \left(g \nabla_\perp^2 f + \vec{\nabla}_\perp g \cdot \vec{\nabla}_\perp f \right) ds = \oint_C g \vec{\nabla}_\perp f \cdot d\vec{l} = \oint_C g \partial_n f dl. \tag{2.3.19}$$

Characteristics of Waveguide Modes (cont.)

Green's second identity is obtained by interchanging the roles of f and g in the first identity and subtracting the two resulting equations:

$$\begin{aligned}\int_V \left(g\nabla^2 f - f\nabla^2 g\right) dv &= \oint_S \left(g\vec{\nabla} f - f\vec{\nabla} g\right)\cdot d\vec{s} \\ &= \oint_S \left(g\partial_n f - f\partial_n g\right) ds.\end{aligned} \tag{2.3.20}$$

In two dimensions, this takes the form:

$$\begin{aligned}\int_S \left(g\nabla_\perp^2 f - f\nabla_\perp^2 g\right) ds &= \oint_C \left(g\vec{\nabla}_\perp f - f\vec{\nabla}_\perp g\right)\cdot d\vec{l} \\ &= \oint_C \left(g\partial_n f - f\partial_n g\right) dl.\end{aligned} \tag{2.3.21}$$

Characteristics of Waveguide Modes (cont.)

Another useful relation can be derived from Green's first identity. By interchanging f and g in (2.3.18) and adding the resulting equations, we obtain:

$$\int_V \vec{\nabla} g \cdot \vec{\nabla} f dv = \frac{1}{2}\left[\oint_S \left(g\partial_n f + f\partial_n g\right) ds - \int_V \left(f\nabla^2 g + g\nabla^2 f\right) dv\right]. \tag{2.3.22}$$

In two dimensions, we have

$$\int_S \vec{\nabla}_\perp g \cdot \vec{\nabla}_\perp f ds = \frac{1}{2}\left[\oint_C \left(g\partial_n f + f\partial_n g\right) dl - \int_S \left(f\nabla_\perp^2 g + g\nabla_\perp^2 f\right) ds\right]. \tag{2.3.23}$$

B. Modal Orthogonality.

Characteristics of Waveguide Modes (cont.)

Waveguide problems can be reduced to two-dimensional eigenfunction problems defined over the cross-section of the waveguide. Let

$$\psi_{pq}, \quad p, q \in \mathbb{N} \tag{2.3.24}$$

represent a doubly infinite set of eigenfunction solutions for either E_z^0 or H_z^0, which satisfies the Helmholtz equation

$$\left(\nabla_\perp^2 + k_{pq}^2\right) \psi_{pq} = 0 \tag{2.3.25}$$

with either Dirichlet boundary conditions:

$$\psi_{pq}\big|_W = 0 \tag{2.3.26a}$$

or Neumann boundary conditions:

$$\partial_n \psi_{pq}\big|_W = 0. \tag{2.3.26b}$$

Characteristics of Waveguide Modes (cont.)

First, consider two distinct eigenfunctions, ψ_{pq} and $\psi_{p'q'}$. Substituting them into the Helmholtz equation (2.3.25) and subtracting the resulting expressions—each multiplied by the other function—yields:

$$\psi_{p'q'} \nabla_\perp^2 \psi_{pq} - \psi_{pq} \nabla_\perp^2 \psi_{p'q'} = \left(k_{p'q'}^2 - k_{pq}^2\right) \psi_{pq} \psi_{p'q'}. \tag{2.3.27}$$

Applying (2.3.21) we have

$$\oint_C \left(\psi_{p'q'} \partial_n \psi_{pq} - \psi_{pq} \partial_n \psi_{p'q'}\right) dl = \int_S \left(k_{p'q'}^2 - k_{pq}^2\right) \psi_{pq} \psi_{p'q'} \, ds. \tag{2.3.28}$$

When ψ_{pq} and $\psi_{p'q'}$ belong to the same scalar family, they satisfy the same boundary condition, either (2.3.26a) or (2.3.26b). In that case the left-hand side of (2.3.28) vanishes.

For non-degenerate modes within the same scalar family, where $k_{pq}^2 \neq k_{p'q'}^2$, this equation implies the orthogonality relation:

$$\int_S \psi_{pq} \psi_{p'q'} \, ds = 0. \tag{2.3.29}$$

Characteristics of Waveguide Modes (cont.)

This implies the orthogonality of the longitudinal electric fields within the TM family, and of the longitudinal magnetic fields within the TE family, between two distinct non-degenerate modes:

$$\int_S \vec{E}^0_{z(pq)} \cdot \vec{E}^0_{z(p'q')} ds = 0, \tag{2.3.30a}$$

$$\int_S \vec{H}^0_{z(pq)} \cdot \vec{H}^0_{z(p'q')} ds = 0. \tag{2.3.30b}$$

Let us calculate another set of surface integration. By applying (2.3.19) and (2.3.25):

$$\begin{aligned}
&\int_S \vec{\nabla}_\perp \psi_{pq} \cdot \vec{\nabla}_\perp \psi_{p'q'} ds \\
&= \oint_C \psi_{pq} \partial_n \psi_{p'q'} dl - \int_S \psi_{pq} \nabla^2_\perp \psi_{p'q'} ds \\
&= \oint_C \psi_{pq} \partial_n \psi_{p'q'} dl + k^2_{p'q'} \int_S \psi_{pq} \psi_{p'q'} ds
\end{aligned}$$

Characteristics of Waveguide Modes (cont.)

$$= \oint_C \psi_{pq} \partial_n \psi_{p'q'} dl = \oint_C \psi_{p'q'} \partial_n \psi_{pq} dl.$$

The last equality holds since surface integral is symmetric when interchanging pq and $p'q'$. From the boundary conditions (2.3.26), we get:

$$\int_S \vec{\nabla}_\perp \psi_{pq} \cdot \vec{\nabla}_\perp \psi_{p'q'} ds = 0. \tag{2.3.31}$$

This can be utilized to prove the orthogonality of the transverse fields between two distinct modes:

$$\int_S \vec{E}^0_{\perp(pq)} \cdot \vec{E}^0_{\perp(p'q')} ds = 0, \tag{2.3.32a}$$

$$\int_S \vec{H}^0_{\perp(pq)} \cdot \vec{H}^0_{\perp(p'q')} ds = 0, \tag{2.3.32b}$$

We first divide (2.3.32) into the following three cases:

1. Both ψ_{pq} and $\psi_{p'q'}$ are TM modes,
2. Both ψ_{pq} and $\psi_{p'q'}$ are TE modes,

Characteristics of Waveguide Modes (cont.)

3. ψ_{pq} is TM/TE mode and $\psi_{p'q'}$ is TE/TM mode.

For Cases 1 and 2, and also notice that

$$\left(\hat{z} \times \vec{\nabla}_\perp \psi_{pq}\right) \cdot \left(\hat{z} \times \vec{\nabla}_\perp \psi_{p'q'}\right) = \hat{z} \cdot \left[\vec{\nabla}_\perp \psi_{p'q'} \times \left(\hat{z} \times \vec{\nabla}_\perp \psi_{pq}\right)\right]$$
$$= \hat{z} \cdot \left[\hat{z}\left(\vec{\nabla}_\perp \psi_{pq} \cdot \vec{\nabla}_\perp \psi_{p'q'}\right)\right] = \vec{\nabla}_\perp \psi_{pq} \cdot \vec{\nabla}_\perp \psi_{p'q'},$$

it readily demonstrated that the equations in (2.3.32) are satisfied by (2.3.31).

Characteristics of Waveguide Modes (cont.)

For Case 3, we encounter the following form

$$\vec{\nabla}_\perp \psi_{pq} \cdot \left(\hat{z} \times \vec{\nabla}_\perp \psi_{p'q'}\right).$$

Apply the vector identity $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$, we get the following identity:

$$\vec{\nabla}_\perp \cdot \left(\psi_{p'q'} \hat{z} \times \vec{\nabla}_\perp \psi_{pq}\right) = \vec{\nabla}_\perp \psi_{pq} \cdot \left(\vec{\nabla}_\perp \psi_{p'q'} \times \hat{z}\right).$$

Thus,

$$\begin{aligned} &\int_S \vec{\nabla}_\perp \psi_{pq} \cdot \left(\hat{z} \times \vec{\nabla}_\perp \psi_{p'q'}\right) ds \\ &= -\int_S \vec{\nabla}_\perp \cdot \left(\psi_{p'q'} \hat{z} \times \vec{\nabla}_\perp \psi_{pq}\right) ds \\ &= -\oint_C \psi_{p'q'} \left(\hat{z} \times \vec{\nabla}_\perp \psi_{pq}\right) \cdot \hat{n} dl. \end{aligned} \tag{2.3.33}$$

If ψ_{pq} corresponds to a TM mode and $\psi_{p'q'}$ to a TE mode, we consider the following identity:

$$(\hat{z} \times \vec{\nabla}_\perp \psi_{pq}) \cdot \hat{n} = \vec{\nabla}_\perp \psi_{pq} \cdot (\hat{n} \times \hat{z}) = -\vec{\nabla}_\perp \psi_{pq} \cdot \hat{\tau}.$$

Characteristics of Waveguide Modes (cont.)

Since $\vec{\nabla}_\perp \psi_{pq}$ represents the transverse electric field $\vec{E}^0_{\perp(pq)}$, (2.3.33) vanishes because the tangential electric field must be zero on the PEC boundary of the waveguide. On the other hand, if ψ_{pq} corresponds to a TE mode and $\psi_{p'q'}$ to a TM mode, then from (2.3.26a), $\psi_{p'q'}$ vanishes along the conducting boundary. Consequently, (2.3.33) also evaluates to zero. The same argument holds when the roles of ψ_{pq} and $\psi_{p'q'}$ are interchanged.

Characteristics of Waveguide Modes (cont.)

Finally, we consider the evaluation of $\vec{E}^0_{\perp(pq)} \times \vec{H}^0_{\perp(p'q')}$, which requires computing integrands of the following forms for Cases 1–3:

$$\vec{\nabla}_\perp \psi_{pq} \times \left(\hat{z} \times \vec{\nabla}_\perp \psi_{p'q'} \right),$$
$$\left(\hat{z} \times \vec{\nabla}_\perp \psi_{pq} \right) \times \vec{\nabla}_\perp \psi_{p'q'},$$
$$\vec{\nabla}_\perp \psi_{pq} \times \vec{\nabla}_\perp \psi_{p'q'} \cdot \hat{z}.$$

By applying the orthogonality relation (2.3.31) and the analysis in (2.3.33), it can be shown that the integrals of these terms are identically zero. Thus, in general,

$$\int_S \left(\vec{E}^0_{\perp(pq)} \times \vec{H}^0_{\perp(p'q')} \right) \cdot d\vec{s} = 0. \tag{2.3.34}$$

The above analysis is valid when the magnetic field is conjugated. This has the important physical consequence that distinct modes propagating within a waveguide do not couple with one another. However, for lossy walls, the modes may become coupled; additional complications also arise in the case of degenerate modes.

Further Reading

Advanced topics such as wave propagation in inhomogeneous, layered, or anisotropic media are presented in Ishimaru's text Akira Ishimaru. *Electromagnetic Wave Propagation, Radiation and Scattering*. 2nd. Wiley, 2017 and Yeh's monograph *Optical Waves in Layered Media* Pochi Yeh. *Optical Waves in Layered Media*. John Wiley & Sons, 2005. For a more detailed discussion of inhomogeneous media, Chew's book Weng Cho Chew. *Waves and Fields in Inhomogenous Media*. John Wiley & Sons, 1999 provides an excellent treatment and also serves as a solid general introduction to electromagnetic theory.

The angular spectrum representation (plane-wave expansion) is discussed in independent chapters by Novotny and Hecht Lukas Novotny and Bert Hecht. *Principles of Nano-Optics*. Cambridge University Press, 2012, Pathak P. H. Pathak and R. J. Burkholder. *Electromagnetic Radiation, Scattering, and Diffraction*. Wiley, 2021, and in depth in Clemmow's monograph *The Plane Wave Spectrum Representation of Electromagnetic Fields* Phillip C Clemmow. *The Plane Wave Spectrum Representation of Electromagnetic Fields: International Series of Monographs in Electromagnetic Waves*. Elsevier, 2013. Applications of this formulation to planar near-field measurement techniques are presented in Gregson et al. Stuart Gregson, John McCormick, and Clive Parini. *Principles of Planar Near-field Antenna Measurements*. Vol. 53. IET, 2007.

Further Reading (cont.)

For guided-wave propagation, Both Ishimaru Ishimaru, *Electromagnetic Wave Propagation, Radiation and Scattering* and Zangwill Andrew Zangwill. *Modern Electrodynamics*. Cambridge University Press, 2013 offer a concise overview in relevant chapters, whereas Collin's authorative text *Field Theory of Guided Waves* Robert E Collin. *Field Theory of Guided Waves*. John Wiley & Sons, 1990 provides a comprehensive and rigorous treatment of waveguide transmission.

Problems

1. Verify (2.1.9) with gradient, divergence and curl in Cartesian coordinate system.
2. Derive (2.2.7) and (2.2.8).
3. Show that (2.2.30) satisfies (2.2.3a) and (2.2.3b).
4. Derive (2.3.2) and express them in Cartesian coordinates.
5. Show that for TM mode, we have

$$\vec{E}_\perp^0 = -\frac{k_z}{\omega\epsilon}\hat{z} \times \vec{H}_\perp^0,$$
$$\vec{H}_\perp^0 = \frac{\omega\epsilon}{k_z}\hat{z} \times \vec{E}_\perp^0,$$

   and for TE mode, we have

$$\vec{E}_\perp^0 = -\frac{\omega\mu}{k_z}\hat{z} \times \vec{H}_\perp^0,$$
$$\vec{H}_\perp^0 = \frac{k_z}{\omega\mu}\hat{z} \times \vec{E}_\perp^0.$$

6. Show that TEM mode cannot exist in a hollow waveguide with arbitrary cross section.