Maxwell Equations

Jake W. Liu

June 13, 2026

Outline

Chapter Overview

This chapter lays the groundwork for understanding electromagnetic phenomena by introducing Maxwell equations in both integral and differential formulations, highlighting their inherent symmetry and duality. We then explore constitutive relations and matching conditions, crucial for describing field behavior at interfaces. From Maxwell equations, we derive the wave equations and introduce time-harmonic fields for simplified analysis. The chapter concludes with fundamental theorems: Poynting theorem for energy flow, the uniqueness theorem for solution conditions, the reciprocity theorem for source-field interaction symmetry, and the equivalence principle for simplifying radiation and scattering problems.

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Integral and Differential Formulations

At the core of electromagnetic theory lie Maxwell equations, a concise set of fundamental relations that, in conjunction with the Lorentz force law, provide a complete description of all classical electromagnetic phenomena. It is more intuitive to begin with the *integral formulation* of the Faraday's law and the Ampère's law, which connects field quantities with physically measurable fluxes and circulations over finite surfaces and loops:

$$\oint_C \vec{E} \cdot d\vec{l} = -\int_S \left(\partial_t \vec{B} + \vec{M}\right) \cdot d\vec{s}, \tag{1.1.1}$$

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \left(\partial_t \vec{D} + \vec{J}\right) \cdot d\vec{s}, \tag{1.1.2}$$

These integral forms represent global field behavior: Faraday's law relates the electromotive force around a closed path to the rate of change of magnetic flux and magnetic current through the enclosed surface, while Ampère's law relates the magnetomotive force to the time-varying electric flux and conduction current.
These primary equations are supplemented by the continuity equations, which express the conservation of electric and magnetic charge:

$$\oint_S \vec{J} \cdot d\vec{s} = -\partial_t \int_V \rho \, dv, \tag{1.1.3}$$

Integral and Differential Formulations (cont.)

$$\oint_S \vec{M} \cdot d\vec{s} = -\partial_t \int_V \varrho \, dv, \tag{1.1.4}$$

and the Lorentz force equation, which expresses the action of the electromagnetic field on a moving charge and connects the field quantities with measurable mechanical effects :

$$\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right). \tag{1.1.5}$$

The physical quantities in these equations are defined as follows:

- $\vec{E}$: Electric Field Intensity (V/m)
- $\vec{H}$: Magnetic Field Intensity (A/m)
- $\vec{D}$: Electric Flux Density (C/m^2)
- $\vec{B}$: Magnetic Flux Density (T)
- $\vec{J}$: Volumetric Electric Current Density (A/m^2)
- ρ: Electric Charge Density (C/m^3)
- $\vec{M}$: Volumetric Magnetic Current Density (V/m^2)
- ϱ: Magnetic Charge Density (Wb/m^3)

Integral and Differential Formulations (cont.)

Notice that the inclusion of magnetic sources $\vec{M}$ and ϱ provides mathematical symmetry, though such sources have not been observed in nature.

To obtain the local, pointwise relations, the integral surfaces and volumes are conceptually shrunk to infinitesimal elements. This process yields the *differential formulation* of Maxwell equations, which describe the field behavior at every point in space.
From (1.1.1)–(1.1.2), and with reference to the illustration in Fig. 1, applying the Stokes' theorem:

$$\int_S \left(\vec{\nabla} \times \vec{\Psi}\right) \cdot d\vec{s} = \oint_C \vec{\Psi} \cdot d\vec{l}, \tag{1.1.6}$$

we obtain the curl-form differential equations:

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} - \vec{M}, \tag{1.1.7}$$

$$\vec{\nabla} \times \vec{H} = \partial_t \vec{D} + \vec{J}. \tag{1.1.8}$$

These represent the local versions of Faraday's and Ampère's laws, expressing how the fields curl in response to time-varying fluxes and sources.

Integral and Differential Formulations (cont.)

From (1.1.3)–(1.1.4), and with reference to the illustration in Fig. 1, applying the divergence theorem:

$$\int_V \vec{\nabla} \cdot \vec{\Psi} dv = \oint_S \vec{\Psi} \cdot d\vec{s}, \tag{1.1.9}$$

we get the continuity equations in differential formulation

$$\vec{\nabla} \cdot \vec{J} = -\partial_t \rho, \tag{1.1.10}$$

$$\vec{\nabla} \cdot \vec{M} = -\partial_t \varrho. \tag{1.1.11}$$

Integral and Differential Formulations (cont.)

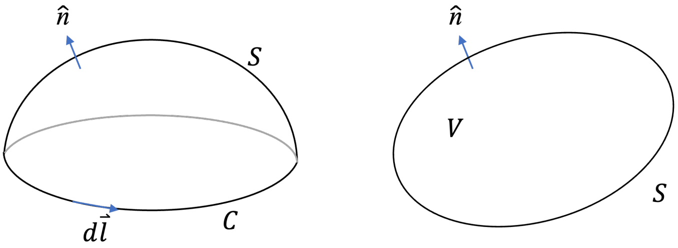

Figure: An open and a closed surface applied to the Stokes' theorem and divergence theorem.

Integral and Differential Formulations (cont.)

To obtain the divergence forms of Maxwell equations (Gauss' laws), we begin with Ampère's law (1.1.8). Taking the divergence of both sides and using the vector identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) \equiv 0$ yields

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{H}\right) = \partial_t \left(\vec{\nabla} \cdot \vec{D}\right) + \vec{\nabla} \cdot \vec{J} \equiv 0. \tag{1.1.12}$$

By substituting the continuity relation (1.1.10), we obtain

$$\partial_t \left(\vec{\nabla} \cdot \vec{D} - \rho\right) \equiv 0, \tag{1.1.13}$$

which implies that the quantity $(\vec{\nabla} \cdot \vec{D} - \rho)$ remains constant in time:

$$\vec{\nabla} \cdot \vec{D} - \rho = C(x, y, z). \tag{1.1.14}$$

A similar argument applied to Faraday's law leads to

$$\vec{\nabla} \cdot \vec{B} - \varrho = C'(x, y, z). \tag{1.1.15}$$

Integral and Differential Formulations (cont.)

Any static charge distributions can be absorbed into ρ and ϱ, allowing the constants C and C' to be set to zero. We therefore obtain Gauss' laws in differential formulation:

$$\vec{\nabla} \cdot \vec{D} = \rho, \tag{1.1.16}$$

$$\vec{\nabla} \cdot \vec{B} = \varrho, \tag{1.1.17}$$

and their corresponding integral formulation:

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dv, \tag{1.1.18}$$

$$\oint_S \vec{B} \cdot d\vec{s} = \int_V \varrho \, dv. \tag{1.1.19}$$

Thus, it has been shown that when Faraday's law, Ampère's law, and the continuity equation are introduced, Gauss' laws are not independent but can be derived from them.

Another useful theorem for the curl operator, also derivable from the divergence theorem, is

$$\int_V \vec{\nabla} \times \vec{\Psi} dv = \oint_S \hat{n} \times \vec{\Psi} ds. \tag{1.1.20}$$

Integral and Differential Formulations (cont.)

Applying this theorem to the curl equations (1.1.7)–(1.1.8) produces

$$\oint_S \hat{n} \times \vec{E} ds = -\int_V \left(\partial_t \vec{B} + \vec{M}\right) dv, \tag{1.1.21}$$

$$\oint_S \hat{n} \times \vec{H} ds = \int_V \left(\partial_t \vec{D} + \vec{J}\right) dv. \tag{1.1.22}$$

(1.1.21) and (1.1.22) are particularly useful for deriving the matching conditions, as shown in Section 1.2.

Constitutive Relations

For Maxwell equations to form a complete and solvable system, the field vectors must be related through constitutive relations, which describe how the medium responds to electric and magnetic fields. These relations are essential for characterizing the macroscopic electromagnetic properties of materials.

In the absence of non-physical quantities $\vec{M}$ and ϱ, the system consists of 5 vector quantities—$\vec{E}, \vec{H}, \vec{D}, \vec{B}, \vec{J}$—and 1 scalar quantity—$\rho$—leading to a total of 16 scalar unknowns. Maxwell equations and the continuity equation, namely (1.1.7)–(1.1.10), provide only 7 independent scalar equations: 3 from Faraday's law, 3 from Ampère's law, and 1 from the continuity equation. Therefore, 9 additional equations are required to close the system. These are supplied by the constitutive relations, which link $\vec{D}$, $\vec{B}$ and $\vec{J}$ to $\vec{E}$ and $\vec{H}$ as follows:

$$\vec{D} = \bar{\bar{C}}_1(\vec{E}, \partial_t \vec{E}, \partial_t^2 \vec{E}, \ldots;\ \vec{H}, \partial_t \vec{H}, \partial_t^2 \vec{H}, \ldots), \tag{1.1.23a}$$

$$\vec{B} = \bar{\bar{C}}_2(\vec{E}, \partial_t \vec{E}, \partial_t^2 \vec{E}, \ldots;\ \vec{H}, \partial_t \vec{H}, \partial_t^2 \vec{H}, \ldots), \tag{1.1.23b}$$

$$\vec{J} = \bar{\bar{C}}_3(\vec{E}, \partial_t \vec{E}, \partial_t^2 \vec{E}, \ldots;\ \vec{H}, \partial_t \vec{H}, \partial_t^2 \vec{H}, \ldots). \tag{1.1.23c}$$

In general, $\bar{\bar{C}}_j$ $(j = 1, 2, 3)$ are second-rank tensor functions of time. To simplify the discussion of material response, we impose the following restrictions:

Constitutive Relations (cont.)

1. *Stationary*: $\bar{\bar{C}}_j$ are independent of time.
2. *Non-chiral*: $\bar{\bar{C}}_1$ and $\bar{\bar{C}}_3$ depend only on the electric field $\vec{E}$, and $\bar{\bar{C}}_2$ depends only on the magnetic field $\vec{H}$.
3. *Linear*: The response tensors $\bar{\bar{C}}_j$ are linear functions of $\vec{E}$ and $\vec{H}$ and exclude higher-order derivatives or nonlinear dependencies.
4. *Isotropic*: The tensors $\bar{\bar{C}}_j$ reduce to scalar multiples of the identity tensor, denoted C_j (otherwise, the medium is said to be *anisotropic*).
5. *Homogeneous*: The coefficients $\bar{\bar{C}}_j$ are spatially uniform.

Constitutive Relations (cont.)

In this lecture note, restrictions 1 and 2 are implicitly assumed. A *simple medium* is defined as one that is linear, isotropic, and homogeneous. Without stating it explicitly each time, we assume a simple medium throughout this text.

When a linear dielectric medium is perturbed by an electric field, the constitutive relation between $\vec{D}$ and $\vec{E}$ is given by

$$\vec{D} = \epsilon_0 \vec{E} + \vec{\mathcal{P}}, \tag{1.1.24}$$

where ϵ_0 is the vacuum permittivity (8.854×10^{-12} F/m), and

$$\vec{\mathcal{P}} = \epsilon_0 \bar{\bar{\chi}}_e \cdot \vec{E} \tag{1.1.25}$$

is the electric polarization, with $\bar{\bar{\chi}}_e$ denoting the electric susceptibility tensor.

When an external electric field is applied to a dielectric, the bound charges within each molecule are slightly displaced relative to their nuclei. This displacement induces microscopic electric dipoles whose collective average gives rise to the polarization vector $\vec{\mathcal{P}}$. In isotropic media, the induced dipoles align with the applied field, keeping $\vec{D}$ parallel to $\vec{E}$; in anisotropic media, the response depends on the direction through $\bar{\bar{\chi}}_e$.

Constitutive Relations (cont.)

For isotropic materials, $\bar{\bar{\chi}}_e$ reduces to a scalar, making $\vec{D}$ parallel to $\vec{E}$. Combining (1.1.24) and (1.1.25), we define

$$\vec{D} = \bar{\bar{\epsilon}} \cdot \vec{E}, \tag{1.1.26}$$

with

$$\bar{\bar{\epsilon}} = \epsilon_0 \left(\bar{\bar{I}} + \bar{\bar{\chi}}_e \right), \tag{1.1.27}$$

where $\bar{\bar{\epsilon}}$ is the permittivity tensor and $\bar{\bar{I}}$ is the identity tensor, which returns any vector unchanged when applied to it.

Similarly, the magnetic flux density relates to the magnetic field as

$$\vec{B} = \mu_0 \vec{H} + \vec{\mathcal{M}}, \tag{1.1.28}$$

where μ_0 is the vacuum permeability ($4\pi \times 10^{-7}$ H/m), and $\vec{\mathcal{M}}$ is the magnetic polarization given by

$$\vec{\mathcal{M}} = \mu_0 \bar{\bar{\chi}}_m \cdot \vec{H}, \tag{1.1.29}$$

with $\bar{\bar{\chi}}_m$ being the magnetic susceptibility tensor. This leads to the compact expression:

$$\vec{B} = \bar{\bar{\mu}} \cdot \vec{H}, \tag{1.1.30}$$

Constitutive Relations (cont.)

where

$$\bar{\bar{\mu}} = \mu_0 \left(\bar{\bar{I}} + \bar{\bar{\chi}}_m \right), \tag{1.1.31}$$

and $\bar{\bar{\mu}}$ is the permeability tensor.

Finally, the relation between current density and electric field is given by Ohm's law:

$$\vec{J} = \bar{\bar{\sigma}} \cdot \vec{E}, \tag{1.1.32}$$

where $\bar{\bar{\sigma}}$ is the conductivity tensor. In vacuum, the conductivity is zero.

For a simple medium, the tensors $\bar{\bar{\epsilon}}, \bar{\bar{\mu}}, \bar{\bar{\sigma}}$ reduce to scalar constants ϵ, μ, σ, respectively. Consequently, the primary curl equations can be simplified and expressed solely in terms of the electric and magnetic fields:

$$\vec{\nabla} \times \vec{E} = -\mu \partial_t \vec{H}, \tag{1.1.33}$$

$$\vec{\nabla} \times \vec{H} = (\sigma + \epsilon \partial_t) \vec{E}. \tag{1.1.34}$$

These constitutive relations serve as essential closures to Maxwell equations, linking material response to field behavior.

Symmetry and Duality

An important feature of Maxwell equations is their intrinsic symmetry, which becomes fully evident upon introducing fictitious magnetic sources, namely the magnetic charge density ϱ and the magnetic current density $\vec{M}$. By incorporating these hypothetical sources, the equations exhibit a dual structure that allows them to be separated into two symmetric and independent sets: one involving only electric sources, and the other involving only magnetic sources.

In simple medium, these two sets of Maxwell equations are given as follows:

$$\begin{cases} \vec{\nabla} \times \vec{E} = -\mu \partial_t \vec{H} \\ \vec{\nabla} \times \vec{H} = \epsilon \partial_t \vec{E} + \vec{J} \\ \vec{\nabla} \cdot \left(\epsilon \vec{E}\right) = \rho \\ \vec{\nabla} \cdot \left(\mu \vec{H}\right) = 0 \end{cases}, \qquad \text{(electric sources)}, \tag{1.1.35a}$$

$$\begin{cases} \vec{\nabla} \times \vec{E} = -\mu \partial_t \vec{H} - \vec{M} \\ \vec{\nabla} \times \vec{H} = \epsilon \partial_t \vec{E} \\ \vec{\nabla} \cdot \left(\epsilon \vec{E}\right) = 0 \\ \vec{\nabla} \cdot \left(\mu \vec{H}\right) = \varrho \end{cases}, \qquad \text{(magnetic sources)}. \tag{1.1.35b}$$

Symmetry and Duality (cont.)

When the following transformations are applied, equations (1.1.35a) and (1.1.35b) are interchanged:

$$\begin{aligned} \vec{E} &\to \vec{H}', \quad \vec{H} \to -\vec{E}', \\ \vec{J} &\to \vec{M}', \quad \vec{M} \to -\vec{J}', \\ \rho &\to \varrho', \quad \varrho \to -\rho', \\ \epsilon &\to \mu', \quad \mu \to \epsilon'. \end{aligned} \tag{1.1.36}$$

When applying the transformation within the same medium, characterized by its intrinsic impedance $\eta = \sqrt{\mu/\epsilon}$, the transformations become:

$$\begin{aligned} \vec{E} &\to \eta\vec{H}', \quad \vec{H} \to -\vec{E}'/\eta \\ \vec{J} &\to \vec{M}'/\eta, \quad \vec{M} \to -\eta\vec{J}' \\ \rho &\to \varrho'/\eta, \quad \varrho \to -\eta\rho'. \end{aligned} \tag{1.1.37}$$

This symmetry, known as the *duality principle*, highlights the structural equivalence between electric and magnetic fields in Maxwell equations when magnetic sources are introduced.

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Matching Conditions at Material Interfaces

To derive the matching conditions that govern the behavior of electromagnetic fields at an interface between two different media, we employ the integral forms of Maxwell equations. By considering an infinitesimally small, pillbox-shaped Gaussian surface that straddles the interface shown in Fig. 2, we can deduce the constraints on the field components.

Applying (1.1.21) to the pillbox, we obtain

$$\begin{aligned}\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right) \Delta S + \int_{\text{side}} \left(\hat{t} \times \vec{E}\right)\, ds \\ = -\Delta S \int_{-h/2}^{h/2} \left(\partial_t \vec{B} + \vec{M}\right)\, d\zeta,\end{aligned} \tag{1.2.1}$$

where $\hat{n}$ denotes the unit normal vector pointing from medium 2 to medium 1, and $\hat{t}$ denotes the outward-pointing unit normal vector to the lateral surface, orthogonal to the sidewall contour.

Taking the limit $h \to 0$, we assume that the electric field and its time derivative remain finite within the pillbox, while the magnetic current density $\vec{M}$ may become large near the interface such that its integral across the infinitesimal interval $[-h/2, h/2]$ stays

Matching Conditions at Material Interfaces (cont.)

finite. In this sense, $\vec{M}$ behaves like a generalized function, analogous to the Dirac delta function defined as the limiting form of a narrow rectangle function whose area remains constant. The surface magnetic current density (units: V/m) is therefore defined as

$$\vec{M}_s = \lim_{h \to 0} \int_{-h/2}^{h/2} \vec{M} \, d\zeta, \tag{1.2.2}$$

which leads to the boundary condition for the tangential component of the electric field:

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2 \right) = -\vec{M}_s. \tag{1.2.3}$$

Matching Conditions at Material Interfaces (cont.)

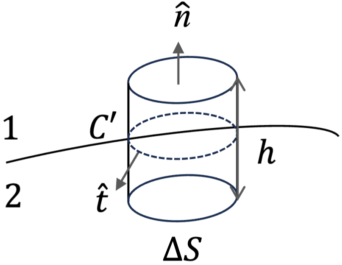

Figure: A pillbox-shaped Gaussian surface.

Matching Conditions at Material Interfaces (cont.)

Similarly, applying (1.1.22) and taking the limit $h \to 0$, under the same assumption that the field quantities and their time derivatives remain finite within the pillbox, we define the surface electric current density (units: A/m) as

$$\vec{J}_s = \lim_{h \to 0} \int_{-h/2}^{h/2} \vec{J}\, d\zeta, \tag{1.2.4}$$

yielding the boundary condition for the tangential component of the magnetic field:

$$\hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s. \tag{1.2.5}$$

The surface current $\vec{J}_s$ represents any current density whose integral through the vanishing pillbox height remains finite. A PEC surface is an important case, where the conductor can support an induced surface current, but an impressed current sheet may also be prescribed at an interface.
Next, applying the Gauss' law (1.1.18) to the same pillbox gives

$$\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right) \Delta S + \int_{\text{side}} \hat{t} \cdot \vec{D}\, ds = \Delta S \int_{-h/2}^{h/2} \rho\, d\zeta, \tag{1.2.6}$$

Matching Conditions at Material Interfaces (cont.)

Taking the limit $h \to 0$, we define the surface charge density (units: $\mathrm{C/m^2}$) as

$$\rho_s = \lim_{h\to 0} \int_{-h/2}^{h/2} \rho \, d\zeta, \tag{1.2.7}$$

which gives the boundary condition for the normal component of the electric flux density:

$$\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s. \tag{1.2.8}$$

Applying (1.1.19) to the same surface yields the analogous condition for the magnetic flux density:

$$\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = \varrho_s. \tag{1.2.9}$$

Finally, applying the continuity equation (1.1.3) to the pillbox gives

$$\hat{n} \cdot \left(\vec{J}_1 - \vec{J}_2\right) \Delta S + \int_{\text{side}} \hat{t} \cdot \vec{J} \, ds = -\Delta S \int_{-h/2}^{h/2} \partial_t \rho \, d\zeta. \tag{1.2.10}$$

In the limit $h \to 0$, the side integral becomes

$$\oint_{C'} \left(\lim_{h\to 0} \int_{-h/2}^{h/2} \vec{J}\, d\zeta \right) \cdot \hat{t}\, dl = \oint_{C'} \vec{J}_s \cdot \hat{t}\, dl = \int_{S'} \vec{\nabla}_s \cdot \vec{J}_s\, ds, \tag{1.2.11}$$

where $\vec{\nabla}_s \cdot$ denotes the surface divergence operator, defined as the divergence taken in the tangent plane of the surface. Thus, the continuity condition for the electric current density becomes

$$\hat{n} \cdot \left(\vec{J}_1 - \vec{J}_2 \right) + \vec{\nabla}_s \cdot \vec{J}_s = -\partial_t \rho_s. \tag{1.2.12}$$

Applying the same procedure to (1.1.4), we obtain the corresponding condition for the magnetic current density:

$$\hat{n} \cdot \left(\vec{M}_1 - \vec{M}_2 \right) + \vec{\nabla}_s \cdot \vec{M}_s = -\partial_t \varrho_s. \tag{1.2.13}$$

These boundary conditions simplify in several important practical scenarios. For a perfect electric conductor (PEC), the electric field must vanish inside the conductor because any non-zero electric field would induce infinite current due to the infinite conductivity, violating physical constraints. In this case, since surface magnetic currents do not exist in practice, the tangential component of the electric field intensity must also

Matching Conditions at Material Interfaces (cont.)

vanish at the PEC surface, as required by (1.2.3). For a perfect magnetic conductor (PMC), a hypothetical material, the magnetic field is zero inside the conductor. Analogously, the absence of surface electric currents implies that the tangential component of the magnetic field must vanish at the PMC surface, as required by (1.2.5). At the interface between two lossless dielectric media, and in the absence of free surface charge or surface current (as the boundary cannot sustain such sources), all source terms in the electromagnetic boundary conditions reduce to zero. Consequently, the electromagnetic field components must satisfy specific continuity conditions across the interface. In particular, the tangential components of $\vec{E}$ and $\vec{H}$ must remain continuous, since any discontinuity would imply the presence of surface charge or current densities, which are not present by assumption. Similarly, the normal components of $\vec{D}$ and $\vec{B}$ must also be continuous across the interface. These continuity conditions are critical in determining the reflection and transmission behavior of electromagnetic waves at dielectric boundaries.

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The Wave Equation

A fundamental implication of Maxwell equations is the existence of electromagnetic waves. These waves arise naturally from the interplay between the electric and magnetic fields described by Maxwell curl equations. To derive the wave equations, we begin by considering a simple medium characterized by constant permittivity ϵ and permeability μ. Starting from Faraday's law (1.1.7), we take the curl of both sides:

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) &= \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} \\ &= -\mu \partial_t \left(\vec{\nabla} \times \vec{H}\right) - \vec{\nabla} \times \vec{M}. \end{aligned} \tag{1.3.1}$$

The right-hand side involves the time derivative of Ampère's law with Maxwell's correction. Substituting Ampère's law (1.1.8) into this expression yields:

$$\nabla^2 \vec{E} - \mu\epsilon \partial_t^2 \vec{E} = \mu \partial_t \vec{J} + \vec{\nabla}\left(\frac{\rho}{\epsilon}\right) + \vec{\nabla} \times \vec{M}. \tag{1.3.2}$$

This is the inhomogeneous wave equation for the electric field $\vec{E}$. The source terms on the right-hand side arise from the current density $\vec{J}$, the free charge density ρ, and the effective magnetization current $\vec{\nabla} \times \vec{M}$.

The Wave Equation (cont.)

To obtain the corresponding wave equation for the magnetic field, we apply the duality transformation to (1.3.2). This leads to:

$$\nabla^2 \vec{H} - \mu\epsilon\partial_t^2 \vec{H} = -\vec{\nabla} \times \vec{J} + \vec{\nabla}\left(\frac{\varrho}{\mu}\right) + \epsilon\partial_t \vec{M}. \quad (1.3.3)$$

The wave equations (1.3.2) and (1.3.3) can be expressed more compactly using the d'Alembert operator $\Box$, defined as

$$\Box \equiv \nabla^2 - \frac{1}{v^2}\partial_t^2,$$

where $v = 1/\sqrt{\mu\epsilon}$ is the phase velocity of electromagnetic waves in the medium. Using this notation, the wave equations can be written in a unified matrix form:

$$\Box \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} \mu\partial_t \vec{J} + \vec{\nabla}\left(\frac{\varrho}{\epsilon}\right) + \vec{\nabla} \times \vec{M} \\ -\vec{\nabla} \times \vec{J} + \vec{\nabla}\left(\frac{\varrho}{\mu}\right) + \epsilon\partial_t \vec{M} \end{pmatrix}. \quad (1.3.4)$$

The Wave Equation (cont.)

In regions where all source terms vanish, i.e., $\vec{J} = 0$, $\rho = 0$, $\vec{M} = 0$, and $\varrho = 0$, the wave equations reduce to their homogeneous forms:

$$\Box \vec{E} = 0, \quad \Box \vec{H} = 0. \tag{1.3.5}$$

These homogeneous wave equations describe the free propagation of electromagnetic waves in a source-free medium.

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Time-Harmonic Fields and Phasor Notation

In many practical electromagnetic problems, it is highly convenient to analyze fields that vary sinusoidally with time. Any time-domain signal $f(t)$ can be represented as a superposition of time-harmonic components through the use of the inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega \tag{1.4.1}$$

with the corresponding time-harmonic components given by:

$$g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{1.4.2}$$

The Fourier and inverse Fourier transform relationship is denoted as

$$\begin{aligned} g(\omega) &= \mathfrak{F}\left\{f(t)\right\}, \\ f(t) &= \mathfrak{F}^{-1}\left\{g(\omega)\right\}. \end{aligned} \tag{1.4.3}$$

For a time-harmonic electric field at a specific angular frequency $\omega = 2\pi f$, we can simplify the analysis by using its phasor form:

$$\vec{E}(\vec{r}, t) = \Re\left\{\vec{E}(\vec{r}, \omega) e^{i\omega t}\right\}, \tag{1.4.4}$$

Time-Harmonic Fields and Phasor Notation (cont.)

where $\vec{E}(\vec{r}, \omega)$ is a complex vector that contains information about the amplitude and phase of the field and $\Re\{\cdot\}$ denotes taking the real part ($\Im\{\cdot\}$ denotes taking the imaginary part). Notice that the same notation for the field is used for both the time-domain and frequency-domain fields. The functional dependence will be omitted when no ambiguity arises.

In this phasor representation, the time derivative operator ∂_t is simply replaced by multiplication by $i\omega$. This transforms Maxwell equations into the time-harmonic form:

$$\vec{\nabla} \times \vec{E} = -i\omega\vec{B} - \vec{M}, \tag{1.4.5}$$

$$\vec{\nabla} \times \vec{H} = i\omega\vec{D} + \vec{J}, \tag{1.4.6}$$

$$\vec{\nabla} \cdot \vec{D} = \rho, \tag{1.4.7}$$

$$\vec{\nabla} \cdot \vec{B} = \varrho. \tag{1.4.8}$$

Using the constitutive relations $\vec{D} = \epsilon\vec{E}$ and $\vec{J} = \sigma\vec{E} + \vec{J}_i$ (where $\vec{J}_i$ represents an impressed or external current source), Ampère's law (1.4.6) can be rewritten as:

$$\vec{\nabla} \times \vec{H} = i\omega\left(\epsilon - i\frac{\sigma}{\omega}\right)\vec{E} + \vec{J}_i. \tag{1.4.9}$$

Time-Harmonic Fields and Phasor Notation (cont.)

It is convenient to define a complex permittivity

$$\epsilon_c = \epsilon - i\frac{\sigma}{\omega} = \epsilon' - i\epsilon'', \quad \epsilon', \epsilon'' \in \mathbb{R}, \tag{1.4.10}$$

which incorporates both the permittivity and conductivity of the medium. In general the permittivity is also a complex valued: $\epsilon = \epsilon_R - i\epsilon_I$. The loss tangent of the medium is then defined as the ratio of the imaginary to the real part of the complex permittivity:

$$\tan\delta = \frac{\epsilon''}{\epsilon'} = \frac{\epsilon_I}{\epsilon_R} + \frac{\sigma}{\omega\epsilon_R}. \tag{1.4.11}$$

In this time-harmonic framework, Maxwell equations can be expressed in a highly compact and symmetric form:

$$\vec{\nabla} \times \vec{E} = -i\omega\mu\vec{H} - \vec{M}_i, \tag{1.4.12}$$

$$\vec{\nabla} \times \vec{H} = i\omega\epsilon_c\vec{E} + \vec{J}_i. \tag{1.4.13}$$

The d'Alembert operator in the frequency domain becomes the Helmholtz operator:

$$\nabla^2 - \frac{(i\omega)^2}{1/\mu\epsilon_c} = \nabla^2 + k^2, \tag{1.4.14}$$

where

$$k = \omega\sqrt{\mu\epsilon_c} = k_R + ik_I \tag{1.4.15}$$

is the complex wavenumber. The square root of ϵ_c is chosen so that k_I relates to the physical attenuation of the wave propagation. Several literatures use the notation γ which is related to the complex wavenumber with $\gamma^2 = (\alpha + i\beta)^2 = -\omega^2\mu\epsilon_c$. The relationship between the real and imaginary parts are $\alpha = -k_I$ and $\beta = k_R$. Thus $\gamma = ik$ or $k = \beta - i\alpha$.

In a source-free region, Maxwell equations reduce to wave equations for the electric and magnetic field components. Each Cartesian component of the fields satisfies the scalar homogeneous Helmholtz equation:

$$\nabla^2\psi + k^2\psi = 0. \tag{1.4.16}$$

This decoupling into scalar equations occurs because, in Cartesian coordinates, the vector Laplacian acting on $\vec{E}$ or $\vec{H}$ reduces to scalar Laplacians on individual components since $\hat{x}, \hat{y}, \hat{z}$ are constant vectors throughout space.

However, in cylindrical or spherical coordinates the situation is more intricate. The unit vectors $\hat{\rho}, \hat{\phi}$ in cylindrical coordinates and $\hat{r}, \hat{\theta}, \hat{\phi}$ in spherical coordinates vary with position, so their directions are not fixed throughout space. As a result, the components of $\vec{E}$ and $\vec{H}$ generally satisfy coupled vector wave equations. Only in certain symmetric and separable configurations can individual field components be shown to satisfy decoupled scalar Helmholtz equations. In such cases, scalar field solutions can still be used to construct the full vector field solutions through appropriate coordinate-dependent relations. This will be demonstrated in more detail in Chapter 3.

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Poynting Theorem

The Poynting theorem expresses the conservation of electromagnetic energy. It is derived from Maxwell equations using the vector identity:

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{H}\right). \tag{1.5.1}$$

Substituting Maxwell equations into this identity yields the differential form of the instantaneous Poynting theorem:

$$\vec{\nabla} \cdot \vec{S} = -\partial_t \left(\frac{1}{2}\vec{E} \cdot \vec{D} + \frac{1}{2}\vec{H} \cdot \vec{B}\right) - \vec{H} \cdot \vec{M} - \vec{E} \cdot \vec{J}, \tag{1.5.2}$$

or equivalently,

$$\vec{\nabla} \cdot \vec{S} = -\partial_t (w_e + w_m) - p_l, \tag{1.5.3}$$

where $\vec{S} = \vec{E} \times \vec{H}$ is the instantaneous Poynting vector, representing power flux density ($\mathrm{W/m^2}$); $w_e = \frac{1}{2}\vec{E} \cdot \vec{D}$ is the instantaneous electric energy density; $w_m = \frac{1}{2}\vec{H} \cdot \vec{B}$ is the instantaneous magnetic energy density; and $p_l = \vec{H} \cdot \vec{M} + \vec{E} \cdot \vec{J}$ denotes the instantaneous power loss (or gain) per unit volume.

Integrating this expression over a volume V and applying the divergence theorem yields the integral form:

Poynting Theorem (cont.)

$$\oint_S \vec{S} \cdot d\vec{s} = -\partial_t \int_V (w_e + w_m) dv - \int_V p_l \, dv. \tag{1.5.4}$$

In the case of time-harmonic fields, it is more practical to work with time-averaged quantities. Assuming fields of the form $\vec{E}(\vec{r}, t) = \Re\{\vec{E}(\vec{r}) e^{i\omega t}\} = [\vec{E}(\vec{r}) e^{i\omega t} + \vec{E}^*(\vec{r}) e^{-i\omega t}]/2$, the time-dependent Poynting vector becomes:

$$\vec{S}(\vec{r}, t) = \frac{1}{2} \Re \left\{ \vec{E} \times \vec{H}^* \right\} + \frac{1}{2} \Re \left\{ \vec{E} \times \vec{H} e^{2i\omega t} \right\}. \tag{1.5.5}$$

Here, the first term represents a steady (time-invariant) component, while the second term oscillates at frequency 2ω. The 2ω term does not contribute to the net energy transfer over a full period and hence averages to zero. Thus, the time-averaged Poynting vector over one period $T = 2\pi/\omega$ is:

$$\begin{aligned} \vec{S}_{\text{avg}} &= \frac{1}{T} \int_0^T \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t) \, dt \\ &= \frac{1}{2} \Re \left\{ \vec{E} \times \vec{H}^* \right\} = \Re \left\{ \vec{S}_c \right\}, \end{aligned} \tag{1.5.6}$$

Poynting Theorem (cont.)

where $\vec{S}_{\text{avg}}$ is the time-averaged Poynting vector and

$$\vec{S}_c = \frac{1}{2}\vec{E} \times \vec{H}^* \tag{1.5.7}$$

is defined as the complex Poynting vector. Applying

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}^*) = \vec{H}^* \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}^*)$$

and substituting

$$\vec{\nabla} \times \vec{E} = -i\omega\mu\vec{H} - \vec{M}, \qquad \vec{\nabla} \times \vec{H}^* = -i\omega\epsilon^*\vec{E}^* + \vec{J}^*$$

gives the complex Poynting theorem:

$$\begin{aligned} \vec{\nabla} \cdot \vec{S}_c &= -i2\omega \left(\frac{\mu|\vec{H}|^2}{4} - \frac{\epsilon^*|\vec{E}|^2}{4} \right) - \frac{\vec{E} \cdot \vec{J}^*}{2} - \frac{\vec{H}^* \cdot \vec{M}}{2} \\ &= -i2\omega \left(w_m^{\text{avg}} - w_e^{\text{avg}} \right) - p_l^{\text{avg}}, \end{aligned} \tag{1.5.8}$$

where $w_e^{\text{avg}} = \frac{1}{4}\vec{E} \cdot \vec{D}^*$, $w_m^{\text{avg}} = \frac{1}{4}\vec{H} \cdot \vec{B}^*$, $p_l^{\text{avg}} = \frac{1}{2}(\vec{E} \cdot \vec{J}^* + \vec{H}^* \cdot \vec{M})$ are the time-averaged electric energy density, magnetic energy density, and power loss/gain

Poynting Theorem (cont.)

density, respectively. The use of time-averaged quantities w_e^{avg} and w_m^{avg} arises naturally from averaging over a complete period of oscillation. These represent the average energy stored in the electric and magnetic fields over time.

The Uniqueness Theorem

The uniqueness theorem establishes the conditions under which a given set of sources within a specified volume V produces a unique electromagnetic field solution. To demonstrate this, let us assume that two distinct solutions, $(\vec{E}_1, \vec{H}_1)$ and $(\vec{E}_2, \vec{H}_2)$, can exist for the same set of sources, $\vec{J}$ and $\vec{M}$, in a volume V. From (1.4.5) and (1.4.6), we have:

$$\begin{cases} \vec{\nabla} \times \vec{E}_1 = -i\omega\mu\vec{H}_1 - \vec{M} \\ \vec{\nabla} \times \vec{H}_1 = i\omega\epsilon\vec{E}_1 + \vec{J} \end{cases} , \quad \begin{cases} \vec{\nabla} \times \vec{E}_2 = -i\omega\mu\vec{H}_2 - \vec{M} \\ \vec{\nabla} \times \vec{H}_2 = i\omega\epsilon\vec{E}_2 + \vec{J} \end{cases} . \tag{1.5.9}$$

We then define the difference fields as $\delta\vec{E} = \vec{E}_1 - \vec{E}_2$ and $\delta\vec{H} = \vec{H}_1 - \vec{H}_2$. These difference fields must satisfy the source-free (homogeneous) Maxwell equations:

$$\begin{cases} \vec{\nabla} \times \delta\vec{E} = -i\omega\mu\delta\vec{H} \\ \vec{\nabla} \times \delta\vec{H} = i\omega\epsilon\delta\vec{E} \end{cases} . \tag{1.5.10}$$

Now, let us dot the first equation in (1.5.10) with $\delta\vec{H}^*$ and the conjugate of the second with $\delta\vec{E}$, and subtract the two:

$$\begin{aligned} &\vec{\nabla}\times\delta\vec{E} \cdot \delta\vec{H}^* - \vec{\nabla} \times \delta\vec{H}^* \cdot \delta\vec{E} \\ &\quad = \vec{\nabla} \cdot \left(\delta\vec{E} \times \delta\vec{H}^*\right) = -i\omega\left(\mu|\delta\vec{H}|^2 - \epsilon^*|\delta\vec{E}|^2\right). \end{aligned} \tag{1.5.11}$$

The Uniqueness Theorem (cont.)

By integrating over the volume V and applying the divergence theorem (1.1.9), we derive the following integral relation:

$$\oint_S \left(\delta\vec{E} \times \delta\vec{H}^*\right) \cdot d\vec{s} = -i\omega \int_V \left(\mu|\delta\vec{H}|^2 - \epsilon^*|\delta\vec{E}|^2\right) dv. \tag{1.5.12}$$

The key insight is that if the surface integral in (1.5.12) is zero, then the volume integral must also be zero:

$$-i\omega \int_V \left(\mu|\delta\vec{H}|^2 - \epsilon^*|\delta\vec{E}|^2\right) dv = 0.$$

Writing $\mu = \Re\{\mu\} + i\Im\{\mu\}$ and $\epsilon^* = \Re\{\epsilon\} - i\Im\{\epsilon\}$, the integrand becomes

$$\begin{aligned}\mu|\delta\vec{H}|^2 - \epsilon^*|\delta\vec{E}|^2 = &\Re\{\mu\}|\delta\vec{H}|^2 - \Re\{\epsilon\}|\delta\vec{E}|^2 \\ &+ i\left(\Im\{\mu\}|\delta\vec{H}|^2 + \Im\{\epsilon\}|\delta\vec{E}|^2\right).\end{aligned}$$

Since $-i(A + iB) = B - iA$, the real and imaginary parts must vanish separately:

$$\begin{cases} \displaystyle\int_V \left(\Re\{\mu\}|\delta\vec{H}|^2 - \Re\{\epsilon\}|\delta\vec{E}|^2\right) dv = 0 \\ \displaystyle\int_V \left(\Im\{\mu\}|\delta\vec{H}|^2 + \Im\{\epsilon\}|\delta\vec{E}|^2\right) dv = 0 \end{cases}. \tag{1.5.13}$$

The Uniqueness Theorem (cont.)

For dissipative media, both $\Im\{\epsilon\}$ and $\Im\{\mu\}$ are negative (with $e^{i\omega t}$ convention), implying that $\delta\vec{E} = \delta\vec{H} = 0$ everywhere inside V, which ensures a unique solution. For the surface integral on the left-hand side of (1.5.12) to be zero, vector identities yield the following equation:

$$\begin{aligned}\oint_S (\delta\vec{E} \times \delta\vec{H}^*) \cdot d\vec{s} &= \oint_S \hat{n} \times \delta\vec{E} \cdot \delta\vec{H}^* ds \\ &= \oint_S \delta\vec{H}^* \times \hat{n} \cdot \delta\vec{E} ds = 0.\end{aligned} \tag{1.5.14}$$

This condition is satisfied if any of the following boundary conditions are specified on the surface S enclosing the volume V:

1. The tangential component of $\vec{E}$ is specified over the entire surface S (i.e., $\hat{n} \times \delta\vec{E} = 0$).
2. The tangential component of $\vec{H}$ is specified over the entire surface S (i.e., $\hat{n} \times \delta\vec{H} = 0$).

The Uniqueness Theorem (cont.)

3. The tangential component of $\vec{E}$ is specified over a portion of S, and the tangential component of $\vec{H}$ is specified over the remainder of the surface.

Uniqueness is thus established for strictly dissipative media. For lossless media at real frequency, the limiting-loss argument must be understood as the limiting absorption principle: uniqueness holds when the limiting solution satisfies the proper radiation condition and the frequency is not an interior resonance of the boundary-value problem. At resonant frequencies, nontrivial source-free fields may satisfy the same homogeneous boundary data, so uniqueness can fail unless an additional radiation, causality, or normalization condition removes the resonant solution. The uniqueness theorem provides the theoretical basis for the equivalence principle in subsection 4 and the image theory discussed in Chapter 3.

The Reciprocity Theorem

The reciprocity theorem is a powerful principle that relates the fields produced by two different sets of source distributions within the same simple medium. Let us consider two sets of sources, $(\vec{J}_1, \vec{M}_1)$ and $(\vec{J}_2, \vec{M}_2)$, which generate the fields $(\vec{E}_1, \vec{H}_1)$ and $(\vec{E}_2, \vec{H}_2)$, respectively. We can define the *reaction* of field 1 on source 2, and vice versa, as:

$$\langle 1, 2 \rangle = \int_V \left(\vec{E}_1 \cdot \vec{J}_2 - \vec{H}_1 \cdot \vec{M}_2 \right) dv, \tag{1.5.15a}$$

$$\langle 2, 1 \rangle = \int_V \left(\vec{E}_2 \cdot \vec{J}_1 - \vec{H}_2 \cdot \vec{M}_1 \right) dv. \tag{1.5.15b}$$

From (1.4.12) and (1.4.13), we have the following set of equations:

$$\vec{\nabla} \times \vec{E}_1 = -i\omega\mu\vec{H}_1 - \vec{M}_1, \tag{1.5.16a}$$

$$\vec{\nabla} \times \vec{H}_1 = i\omega\epsilon_c\vec{E}_1 + \vec{J}_1, \tag{1.5.16b}$$

$$\vec{\nabla} \times \vec{E}_2 = -i\omega\mu\vec{H}_2 - \vec{M}_2, \tag{1.5.16c}$$

$$\vec{\nabla} \times \vec{H}_2 = i\omega\epsilon_c\vec{E}_2 + \vec{J}_2. \tag{1.5.16d}$$

The Reciprocity Theorem (cont.)

By dotting (1.5.16a) with $\vec{H}_2$ and (1.5.16d) with $\vec{E}_1$ and then subtracting the two, we get:

$$-\vec{\nabla} \cdot \left(\vec{E}_1 \times \vec{H}_2\right) = i\omega\mu\vec{H}_1 \cdot \vec{H}_2 + \vec{H}_2 \cdot \vec{M}_1 + i\omega\epsilon_c\vec{E}_1 \cdot \vec{E}_2 + \vec{E}_1 \cdot \vec{J}_2. \tag{1.5.17a}$$

Then, by interchanging the subscripts 1 and 2, we get:

$$-\vec{\nabla} \cdot \left(\vec{E}_2 \times \vec{H}_1\right) = i\omega\mu\vec{H}_2 \cdot \vec{H}_1 + \vec{H}_1 \cdot \vec{M}_2 + i\omega\epsilon_c\vec{E}_2 \cdot \vec{E}_1 + \vec{E}_2 \cdot \vec{J}_1. \tag{1.5.17b}$$

Subtracting the two equations allows us to derive the Lorentz reciprocity theorem in its differential form:

$$\begin{aligned} &\vec{\nabla} \cdot \left(\vec{E}_2 \times \vec{H}_1 - \vec{E}_1 \times \vec{H}_2\right) \\ &= \left(\vec{E}_1 \cdot \vec{J}_2 - \vec{H}_1 \cdot \vec{M}_2\right) - \left(\vec{E}_2 \cdot \vec{J}_1 - \vec{H}_2 \cdot \vec{M}_1\right). \end{aligned} \tag{1.5.18}$$

Integrating this equation over a volume V and applying the divergence theorem, we obtain the integral form of the theorem:

$$\oint_S \left(\vec{E}_2 \times \vec{H}_1 - \vec{E}_1 \times \vec{H}_2\right) \cdot d\vec{s} = \langle 1, 2\rangle - \langle 2, 1\rangle. \tag{1.5.19}$$

The Reciprocity Theorem (cont.)

If the surface S is taken to extend to infinity and both fields satisfy the outgoing Sommerfeld radiation condition, the surface integral becomes zero. The cancellation is not merely because the fields vanish: radiating fields fall as $1/r$, while the area grows as r^2; the outgoing-wave relation between $\vec{E}$ and $\vec{H}$ makes the two cross terms cancel. This leaves the Rayleigh-Carson reciprocity theorem:

$$\langle 1, 2 \rangle = \langle 2, 1 \rangle. \tag{1.5.20}$$

This theorem expresses a fundamental symmetry: the reaction of field 1 on source 2 is identical to the reaction of field 2 on source 1. In a source-free region, we can further simplify (1.5.19) to

$$\oint_S \left(\vec{E}_2 \times \vec{H}_1 - \vec{E}_1 \times \vec{H}_2 \right) \cdot d\vec{s} = 0. \tag{1.5.21}$$

As an example, if $(\vec{E}_1, \vec{H}_1)$ and $(\vec{E}_2, \vec{H}_2)$ represent two different modes in a section of a hollow waveguide, these electromagnetic field pairs must satisfy (1.5.21).
As a corollary of (1.5.19), we can prove that an impressed tangential electric current source placed on a PEC will not radiate, and similarly, an impressed tangential magnetic current source on a PMC will not radiate. Referring to the illustration shown in Fig. 3,

The Reciprocity Theorem (cont.)

the proof is briefly outlined: Assume a tangential source current $\vec{J}_1$ is placed on a PEC surface, and an arbitrary source $\vec{J}_2 = Idl\hat{a}$ exists outside of the PEC. From the reciprocity theorem, we have:

$$\begin{aligned} \langle 1, 2 \rangle &= \int_V \vec{E}_1 \cdot \vec{J}_2 \, dv = Idl\vec{E}_1(\vec{r}) \cdot \hat{a} \\ &= \langle 2, 1 \rangle = \int_V \vec{E}_2 \cdot \vec{J}_1 \, dv = 0. \end{aligned} \tag{1.5.22}$$

The last equality holds because $\vec{E}_2$ generated by $\vec{J}_2$ must satisfy the matching condition that the tangential component of $\vec{E}_2$ on the PEC is zero. From (1.5.22), we have $\vec{E}_1(\vec{r}) \cdot \hat{a} = 0$. Since $\vec{r}$ and $\hat{a}$ are arbitrary, we can deduce that $\vec{E}_1 = 0$ everywhere, indicating that an electric current element placed tangentially on a PEC surface does not radiate. A similar proof applies when an impressed magnetic current is placed on a PMC, showing that it also does not radiate.

The Reciprocity Theorem (cont.)

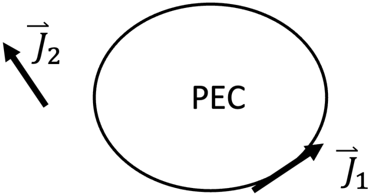

Figure: Illustration of (1.5.22).

Notice that our derivation assumes a simple medium but applies to general nonhomogeneous media, excluding nonreciprocal anisotropic materials with a nonsymmetric tensor.

Equivalence Principle

The equivalence principle is a conceptual framework that allows the replacement of the actual sources within a region by an equivalent distribution of sources on its boundary, without altering the electromagnetic fields in the exterior region. Its validity follows directly from the uniqueness theorem: once the fields on a closed surface are specified, the solution to Maxwell equations outside that surface is uniquely determined. This principle is foundational in electromagnetic theory and exists in multiple forms, most notably the volume equivalence principle and the surface equivalence principle.

A. Volume Equivalence Principle.

Consider a distribution of sources $(\vec{J}, \vec{M})$ radiating in free space, characterized by permittivity ϵ_0 and permeability μ_0:

$$\begin{cases} \vec{\nabla} \times \vec{E}_0 = -i\omega\mu_0 \vec{H}_0 - \vec{M}, \\ \vec{\nabla} \times \vec{H}_0 = i\omega\epsilon_0 \vec{E}_0 + \vec{J}. \end{cases} \tag{1.5.23}$$

Now consider the presence of a material object with permittivity ϵ and permeability μ, embedded in the environment. The total fields in this configuration satisfy:

$$\begin{cases} \vec{\nabla} \times \vec{E} = -i\omega\mu \vec{H} - \vec{M}, \\ \vec{\nabla} \times \vec{H} = i\omega\epsilon \vec{E} + \vec{J}. \end{cases} \tag{1.5.24}$$

Equivalence Principle (cont.)

Subtracting the vacuum field equations from those with the material yields the equations governing the scattered fields ($\vec{E}_s = \vec{E} - \vec{E}_0, \vec{H}_s = \vec{H} - \vec{H}_0$):

$$\begin{cases} \vec{\nabla} \times \vec{E}_s = -i\omega(\mu\vec{H} - \mu_0\vec{H}_0), \\ \vec{\nabla} \times \vec{H}_s = i\omega(\epsilon\vec{E} - \epsilon_0\vec{E}_0). \end{cases} \tag{1.5.25}$$

These can be rewritten in the form:

$$\begin{cases} \vec{\nabla} \times \vec{E}_s = -i\omega\mu_0\vec{H}_s - \vec{M}_{eq}, \\ \vec{\nabla} \times \vec{H}_s = i\omega\epsilon_0\vec{E}_s + \vec{J}_{eq}, \end{cases} \tag{1.5.26}$$

where the equivalent volume sources are defined as:

$$\begin{cases} \vec{M}_{eq} = i\omega\,(\mu - \mu_0)\,\vec{H}, \\ \vec{J}_{eq} = i\omega\,(\epsilon - \epsilon_0)\,\vec{E}. \end{cases} \tag{1.5.27}$$

These equivalent currents are non-zero only within the volume of the inhomogeneity and generate the same scattered fields as those produced by the original material object.

B. Surface Equivalence Principle.

Equivalence Principle (cont.)

The surface equivalence principle states that the electromagnetic effect of sources inside a closed surface can be exactly replicated outside that surface by an appropriate distribution of equivalent surface currents.

Referring to the configuration on the left-hand side of Fig. 4, consider a region V_2 containing sources or material inhomogeneities. If interest is restricted to the fields external to V_2, one may replace all internal sources with surface-equivalent currents defined on the boundary of V_2. Based on the tangential matching conditions (1.2.3) and (1.2.5), the equivalent surface currents are given by:

$$\begin{cases} \vec{J}_s = \hat{n} \times \left(\vec{H}_1 - \vec{H}_2' \right) , \\ \vec{M}_s = -\hat{n} \times \left(\vec{E}_1 - \vec{E}_2' \right) , \end{cases} \tag{1.5.28}$$

where $\hat{n}$ is the outward normal to the surface of V_2, $(\vec{E}_1, \vec{H}_1)$ are the fields in the original configuration, and $(\vec{E}_2', \vec{H}_2')$ are the fields inside V_2 when only the equivalent surface currents are present.

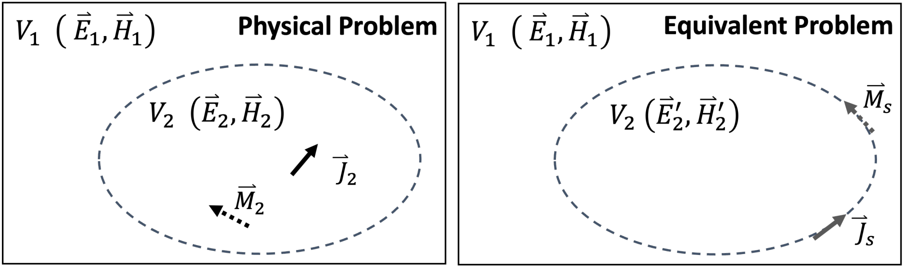

Figure: Illustration of the surface equivalence principle.

Three notable special cases are commonly employed in practical applications:

1. *Love Equivalence Principle*: If the interior fields are assumed to vanish, the equivalent currents reduce to:

$$\begin{cases} \vec{J}_s = \hat{n} \times \vec{H}_1, \\ \vec{M}_s = -\hat{n} \times \vec{E}_1. \end{cases} \tag{1.5.29a}$$

Equivalence Principle (cont.)

2. *PEC Replacement*: If the region V_2 is replaced by a PEC, the equivalent currents are:

$$\begin{cases} \vec{J}_s = 0, \\ \vec{M}_s = -\hat{n} \times \vec{E}_1. \end{cases} \tag{1.5.29b}$$

3. *PMC Replacement*: If the region V_2 is replaced by a PMC, the equivalent currents are:

$$\begin{cases} \vec{J}_s = \hat{n} \times \vec{H}_1, \\ \vec{M}_s = 0. \end{cases} \tag{1.5.29c}$$

The equivalence principle allows replacing actual sources or material variations within a region by equivalent electric and magnetic currents on its boundary, without altering the exterior fields. This follows directly from the uniqueness theorem and provides a powerful simplification for radiation and scattering problems. Both volume and surface forms exist, with the surface formulation commonly preferred for its reduced dimensionality in practical computations.

Further Reading

Numerous established texts provide clear summaries of Maxwell equations. This chapter follows the presentation of Barkeshli K. Barkeshli. *Advanced Electromagnetics and Scattering Theory*. Springer, 2015, with the treatment of fundamental theorems aligning with standard references such as Harrington Roger F Harrington. *Time-Harmonic Electromagnetic Fields*. IEEE Press, 2001, Balanis Constantine A. Balanis. *Advanced Engineering Electromagnetics*. 2nd. Wiley, 2012, and Jin Jian-Ming Jin. *Theory and Computation of Electromagnetic Fields*. John Wiley & Sons, 2015. Someda's text Carlo G Someda. *Electromagnetic Waves*. 2nd. CRC Press, 2006 offers additional perspective from optics, including a focused discussion on polarization.

Although vector algebra is the most familiar framework for expressing Maxwell equations, alternative formalisms provide more compact and geometric representations. The differential-form approach, introduced to electromagnetics by Deschamps Georges A Deschamps. "Electromagnetics and differential forms". In: *Proceedings of the IEEE* 69.6 (2005), pp. 676–696 and expanded by Lindell Ismo V Lindell. *Differential Forms in Electromagnetics*. John Wiley & Sons, 2004.

Problems

1. Show (1.1.20) by using the divergence theorem.
2. Show that $-\hat{r} \times (\hat{r} \times \vec{u}) = \hat{\theta}(\hat{\theta} \cdot \vec{u}) + \hat{\phi}(\hat{\phi} \cdot \vec{u}) = (\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}) \cdot \vec{u}$ where $\vec{u}$ is an arbitrary vector. We use the dyadic notation in the last expression.
3. Show that $\vec{\nabla}(1/R) = -\vec{\nabla}'(1/R) = -\vec{R}/R^3$ with $\vec{R} = \vec{r} - \vec{r}\,'$ and $R = |\vec{R}|$. $\vec{\nabla}$ denotes operation with respect to $\vec{r}$, and $\vec{\nabla}'$ denotes operation with respect to $\vec{r}\,'$.
4. Show that $\vec{\nabla} \cdot (\vec{R}/R^3) = -\nabla^2(1/R) = 4\pi\delta(\vec{R})$ with $\vec{R} = \vec{r} - \vec{r}\,'$.
5. Consider the vector Helmholtz equation $(\nabla^2 + k^2)\vec{E} = 0$. Suppose $\vec{\nabla} \cdot \vec{E} = 0$, show that $\vec{E} = \vec{\nabla} \times (\psi\vec{u})$ is a solution with $\vec{u}$ being a constant vector and ψ satisfies the scalar Helmholtz equation (1.4.16).
6. Show that with inhomogeneous permittivity $\epsilon(\vec{r})$, in a source-free region the electric field satisfies the equation $(\nabla^2 + k^2)\vec{E} = -\vec{\nabla}(\vec{E} \cdot \vec{\nabla} \ln \epsilon)$.