Legendre Polynomials and Functions

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July 12, 2026

Outline

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Introduction

Legendre's differential equation arises from separation of variables in spherical coordinates. For example, the azimuthally symmetric angular equation is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\lambda\Theta=0.$$

Set $x=\cos\theta$ and write $\Theta(\theta)=y(x)$. The chain rule gives every step in the change of variable:

$$\begin{aligned}
\frac{d\Theta}{d\theta} &= \frac{dy}{dx}\frac{dx}{d\theta} = -\sin\theta\, y',\\
\sin\theta\frac{d\Theta}{d\theta} &= -(1-x^2)y',\\
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= -\frac{d}{dx}\left[-(1-x^2)y'\right]\\
&= (1-x^2)y''-2xy'.
\end{aligned}$$

Introduction (cont.)

Requiring a nonzero solution that remains bounded at both poles $x = \pm 1$ turns this singular Sturm–Liouville problem into an eigenvalue problem. Its eigenvalues are $\lambda = n(n+1)$, $n = 0, 1, 2, \ldots$, and hence

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0. \tag{C.1.1}$$

On an open interval not containing the singular endpoints, two linearly independent solutions are the Legendre functions of the first and second kinds, denoted by $P_n(x)$ and $Q_n(x)$. For nonnegative integer n, P_n is the degree-n solution that is finite at both endpoints, whereas Q_n is singular there.

The polynomial $P_n(x)$ is defined for every real or complex x; the interval $[-1, 1]$ is its angular and orthogonality domain, not the domain of the polynomial itself. These ordinary Legendre polynomials express the polar-angle dependence of azimuthally symmetric separated solutions of Laplace's and Helmholtz equations; nonzero azimuthal order uses the associated functions introduced below. Exact spherical symmetry would retain only $n = 0$; general spherical-coordinate problems need the higher angular orders.

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Legendre Polynomials

The first few Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1), \\
P_3(x) &= \tfrac{1}{2}(5x^3 - 3x), \\
P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3), \\
P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x).
\end{aligned} \tag{C.2.1}$$

They follow from Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n, \qquad n = 0, 1, 2, \ldots . \tag{C.2.2}$$

Parity, endpoint values, and recurrence

The generating function is

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n, \qquad |t| < 1, \qquad -1 \le x \le 1. \tag{C.2.3}$$

It makes the parity and endpoint values transparent:

$$\begin{aligned}
g(-x,t) &= g(x,-t) \\
&= \sum_{n=0}^{\infty} (-1)^n P_n(x) t^n, && P_n(-x) = (-1)^n P_n(x), \\
g(1,t) &= (1-t)^{-1} = \sum_{n=0}^{\infty} t^n, && P_n(1) = 1, \\
g(-1,t) &= (1+t)^{-1} \\
&= \sum_{n=0}^{\infty} (-1)^n t^n, && P_n(-1) = (-1)^n.
\end{aligned}$$

Parity, endpoint values, and recurrence (cont.)

In particular, $P_n(0) = 0$ for odd n. Differentiating (C.2.3) with respect to t gives

$$(1 - 2xt + t^2)\frac{\partial g}{\partial t} = (x - t)g.$$

Substitution of $g = \sum_{n\geq 0} P_n t^n$ and comparison of the coefficient of t^n yields

$$(n+1)P_{n+1} - 2nxP_n + (n-1)P_{n-1} = xP_n - P_{n-1},$$

and therefore

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \qquad n \geq 1. \tag{C.2.4}$$

Orthogonality and normalization

Equation (C.1.1) has the self-adjoint form

$$\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] + n(n+1)P_n(x) = 0. \tag{C.2.5}$$

Write this equation once for P_n, once for P_m, multiply by P_m and P_n, respectively, and subtract. The product rule then gives

$$\frac{d}{dx}\left\{(1-x^2)\left[P_m P_n' - P_n P_m'\right]\right\} \\ + \left[n(n+1) - m(m+1)\right] P_m P_n = 0.$$

After integration from -1 to 1, the boundary term vanishes because P_n, P_m and their derivatives are finite while $1-x^2=0$ at both endpoints. For $m \neq n$, the eigenvalues are distinct, so

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0.$$

Orthogonality and normalization (cont.)

For $m = n$, Rodrigues' formula supplies the norm. Integrating by parts n times (all boundary terms vanish because $(x^2 - 1)^n$ has zeros of order n at $x = \pm 1$) gives

$$\begin{aligned}\int_{-1}^{1} P_n^2 \, dx &= \frac{(-1)^n}{2^n n!} \int_{-1}^{1} P_n^{(n)}(x)(x^2 - 1)^n \, dx \\ &= \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (1 - x^2)^n \, dx.\end{aligned}$$

Here Rodrigues' formula gives $P_n^{(n)} = (2n)!/(2^n n!)$. The beta integral evaluates to

$$\int_{-1}^{1} (1 - x^2)^n \, dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!},$$

so the complete orthogonality and normalization relation is

$$\int_{-1}^{1} P_m(x) P_n(x) \, dx = \frac{2}{2n+1} \, \delta_{mn}, \qquad m, n = 0, 1, 2, \ldots. \tag{C.2.6}$$

Orthogonality and normalization (cont.)

Consequently, if $f \in L^2[-1,1]$, its Legendre-series coefficients are obtained by multiplying the expansion by P_m and using (C.2.6):

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x), \qquad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{C.2.7}$$

Pointwise convergence requires the usual additional regularity conditions; square-integrability guarantees convergence in the L^2 sense.

Orthogonality and normalization

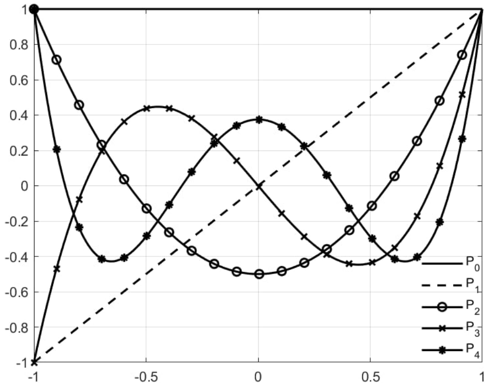

Figure C.1: Legendre polynomials plotted in the figure for $n = 0, 1, 2, 3, 4$.

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Associated Legendre Functions

For $-1 \leq x \leq 1$, this book uses the phase-free engineering convention

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}, \qquad 0 \leq m \leq n. \tag{C.3.1}$$

Many mathematical-physics references include the Condon–Shortley factor $(-1)^m$. Thus the convention here and that convention are related by

$$P^m_{n,\mathrm{here}}(x) = (-1)^m P^m_{n,\mathrm{CS}}(x).$$

No differential equation or squared normalization changes under this constant phase. In particular, the convention used here gives the identity employed in Chapter 4,

$$P_n^1(\cos\theta) = \sin\theta\, P_n'(\cos\theta) = -\frac{d}{d\theta} P_n(\cos\theta).$$

To derive the associated equation without skipping the differentiation step, let $u = d^m P_n/dx^m$. Differentiating (C.1.1) m times and applying the product rule gives

$$(1-x^2)u'' - 2(m+1)xu' + [n(n+1) - m(m+1)]\, u = 0. \tag{C.3.2}$$

Associated Legendre Functions (cont.)

Now put $s = 1 - x^2$ and $v = s^{m/2} u = P_n^m$, so that

$$\begin{aligned} u &= s^{-m/2} v, \\ u' &= s^{-m/2} \left(v' + \frac{mx}{s} v \right), \\ u'' &= s^{-m/2} \left[v'' + \frac{2mx}{s} v' + \frac{m\left(1 + (m+1)x^2\right)}{s^2} v \right]. \end{aligned}$$

Substitution into (C.3.2) and collection of the v' and v terms gives

$$(1 - x^2) v'' - 2x v' + \left[n(n+1) - \frac{m^2}{1 - x^2} \right] v = 0. \tag{C.3.3}$$

With $x = \cos\theta$, the regular polar factor is $P_n^m(\cos\theta)$. For $m > 0$, multiplication by either real factor $\cos(m\phi)$ or $\sin(m\phi)$, or equivalently use of the complex pair $e^{\pm im\phi}$, supplies the azimuthal dependence. For $m = 0$, the sole nonzero azimuthal factor is the constant 1; $\sin(0\phi)$ is identically zero and is not a second solution. The choice to include or omit the Condon–Shortley phase changes only an overall phase convention.

Associated Legendre Functions (cont.)

For fixed m, equation (C.3.3) has the self-adjoint form

$$\frac{d}{dx}\left[(1-x^2)v_n'\right]+\left[n(n+1)-\frac{m^2}{1-x^2}\right]v_n=0, \qquad v_n=P_n^m.$$

Write this equation for v_n and v_l, multiply by v_l and v_n, and subtract:

$$\frac{d}{dx}\left\{(1-x^2)\left[v_l v_n'-v_n v_l'\right]\right\}+\left[n(n+1)-l(l+1)\right]v_l v_n=0.$$

After integration on $[-1,1]$, the boundary term vanishes. For $m=0$ this is the endpoint argument already used for P_n; for $m\geq 1$, $v_n=(1-x^2)^{m/2}$ times a polynomial, so each boundary product is $O((1-x^2)^m)$. Thus distinct degrees are orthogonal for fixed m. The norm follows by integrating by parts m times:

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2dx=\int_{-1}^{1}(1-x^2)^m\left[P_n^{(m)}(x)\right]^2dx$$
$$=(-1)^m\int_{-1}^{1}P_n(x)$$

Associated Legendre Functions (cont.)

$$\times \frac{d^m}{dx^m}\left[(1-x^2)^m P_n^{(m)}(x)\right] dx.$$

Differentiating Rodrigues' formula gives

$$\frac{d^m}{dx^m}\left[(1-x^2)^m P_n^{(m)}(x)\right] = (-1)^m \frac{(n+m)!}{(n-m)!} P_n(x).$$

Using (C.2.6), one obtains

$$\int_{-1}^{1} P_l^m(x) P_n^m(x)\, dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}\, \delta_{ln}, \tag{C.3.4}$$

for $l, n \geq m \geq 0$. Equivalently, $x = \cos\theta$ and $dx = -\sin\theta\, d\theta$ give

$$\int_0^{\pi} [P_n^m(\cos\theta)]^2 \sin\theta\, d\theta = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}. \tag{C.3.5}$$

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Legendre Functions of the Second Kind

On $-1 < x < 1$, the real second solution (more precisely, the Ferrers function of the second kind) is

$$Q_n(x) = \frac{1}{2} P_n(x) \ln\left(\frac{1+x}{1-x}\right) - \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(x) P_{n-k}(x), \quad n = 0, 1, 2, \ldots. \tag{C.4.1}$$

where the sum is empty for $n = 0$. For example,

$$Q_0(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right),$$
$$Q_1(x) = \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1,$$
$$Q_2(x) = \frac{3x^2 - 1}{4} \ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}.$$

Legendre Functions of the Second Kind (cont.)

Direct substitution verifies (C.1.1); alternatively, these functions follow from Q_0, Q_1 using the same three-term recurrence (C.2.4). To retain the Wronskian sign, divide the differential equation by $1 - x^2$ and set $W = P_n Q_n' - P_n' Q_n$. Abel's identity gives

$$W' = \frac{2x}{1-x^2} W \quad \Longrightarrow \quad W = \frac{C}{1-x^2}.$$

As $x \to 1^-$, $P_n \to 1$ and the leading logarithmic term in Q_n gives $(1-x^2)Q_n' \to 1$, whereas $(1-x^2)P_n' Q_n \to 0$; hence $C = 1$. Therefore

$$P_n Q_n' - Q_n P_n' = \frac{1}{1-x^2},$$

so P_n and Q_n are linearly independent on $(-1,1)$. The logarithm makes Q_n singular at $x = \pm 1$.

The real Legendre function on either exterior interval $|x| > 1$ uses a different real logarithmic expression,

$$\begin{aligned} Q_n(x) &= \frac{1}{2} P_n(x) \ln\left(\frac{x+1}{x-1}\right) \\ &\quad - \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(x) P_{n-k}(x), \qquad |x| > 1. \end{aligned} \tag{C.4.2}$$

Legendre Functions of the Second Kind (cont.)

for which the logarithm's ratio is positive. Complex analytic continuation requires a specified branch and must not be obtained by naively inserting $|x| > 1$ into the interior principal-log formula. The exterior $Q_n(\xi)$ occurs, for example, as a radial factor in prolate-spheroidal separations. For a spherical exterior problem, by contrast, the radial Laplace factors are $r^{-(n+1)}$, not $Q_n(\cos\theta)$.

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Summary of Important Relations

For $n = 0, 1, 2, \ldots$, and $0 \le m \le n$, the principal relations are

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

$$|t| < 1, \qquad -1 \le x \le 1,$$

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn},$$

$$\frac{d}{dx}\left[(1 - x^2) P_n'(x)\right] + n(n+1) P_n(x) = 0,$$

$$\int_0^{\pi} [P_n^m(\cos\theta)]^2 \sin\theta\, d\theta = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}.$$