Bessel Functions

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Outline

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Introduction

Bessel functions arise naturally when separation of variables is applied in cylindrical coordinates, for example in circular waveguides, circular membranes, and diffraction by circular apertures. Their dimensionless argument will be denoted by x. Unless otherwise stated, the limiting formulas below assume $x > 0$ and fixed real order $\nu \geq 0$. Bessel's differential equation is

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \tag{B.1.1}$$

On any interval that excludes the singular point $x = 0$, two linearly independent solutions are the Bessel functions of the first and second kinds, $J_\nu(x)$ and $Y_\nu(x)$.

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Bessel Functions of the First and Second Kind

For $\nu \notin \{-1, -2, \ldots\}$, the Bessel function of the first kind is represented, with a consistent branch for x^ν, by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{B.2.1}$$

Termwise differentiation verifies the definition directly. If a_n denotes the nth term, then

$$x^2 a_n'' + x a_n' - \nu^2 a_n = \left[(2n+\nu)^2 - \nu^2\right] a_n = 4n(n+\nu) a_n,$$

whereas, for $n \geq 1$,

$$x^2 a_{n-1} = -4n(n+\nu) a_n.$$

Thus the $x^2 a_{n-1}$ term cancels the differential contribution of a_n term by term, leaving $x^2 J_\nu'' + x J_\nu' + (x^2 - \nu^2) J_\nu = 0$.
The Gamma function appearing here is defined for $\Re z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

and integration by parts gives

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty t^z e^{-t}\,dt \\ &= \left[-t^z e^{-t}\right]_0^\infty + z\int_0^\infty t^{z-1}e^{-t}\,dt = z\Gamma(z),\end{aligned}$$

where the boundary term vanishes for $\Re z > 0$. Hence $\Gamma(n+1) = n!$ for integers $n \geq 0$. For the nonnegative orders used in regular cylindrical-field problems, $J_0(0) = 1$ and $J_\nu(x) \to 0$ as $x \to 0^+$ when $\nu > 0$. The often-repeated statement that J_ν is finite at the origin requires this order restriction: for a negative noninteger order it generally diverges. Negative integer orders satisfy $J_{-n}(x) = (-1)^n J_n(x)$.

For noninteger ν, the Bessel function of the second kind, or Neumann function, is defined by

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{B.2.2}$$

Bessel Functions of the First and Second Kind (cont.)

For an integer order m, both numerator and denominator vanish, and the definition is continued by the limit

$$Y_m(x) = \lim_{\nu \to m} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{B.2.3}$$

The nonzero Wronskian derived below proves that J_ν and Y_ν are linearly independent. Therefore, on an interval with $x > 0$, the general solution is

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x). \tag{B.2.4}$$

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Small- and Large-Argument Behavior

For fixed $\nu > 0$, the leading $n = 0$ term of (B.2.1) gives

$$J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \qquad x \to 0^+. \tag{B.3.1}$$

For positive noninteger ν, $J_{-\nu}$ dominates J_ν as $x \to 0^+$. Inserting its leading term into (B.2.2) and then using $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin(\pi\nu)$ gives the intermediate steps

$$Y_\nu(x) \sim -\frac{1}{\sin(\pi\nu)\,\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu},$$
$$\frac{1}{\sin(\pi\nu)\Gamma(1-\nu)} = \frac{\Gamma(\nu)}{\pi}.$$

Continuation to positive integer order gives the same leading result:

$$Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu, \qquad x \to 0^+, \quad \nu > 0. \tag{B.3.2}$$

Small- and Large-Argument Behavior (cont.)

Let γ_E denote Euler's constant. The order-zero case is different and must not be obtained by substituting $\nu = 0$ into (B.3.2):

$$\begin{aligned} J_0(x) &= 1 - \frac{x^2}{4} + O(x^4), \\ Y_0(x) &= \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma_E\right] + O(x^2 \ln x). \end{aligned} \tag{B.3.3}$$

Thus Y_ν is singular at the origin for every nonnegative real order and is excluded when a physical field must remain bounded at the cylinder axis.

For fixed ν and real $x \to +\infty$, define

$$\chi_\nu = x - \frac{\nu\pi}{2} - \frac{\pi}{4}.$$

The leading large-argument asymptotic forms are

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos \chi_\nu, \tag{B.3.4a}$$

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin \chi_\nu. \tag{B.3.4b}$$

Small- and Large-Argument Behavior (cont.)

When $x = k\rho$ with real $k > 0$, the factor $x^{-1/2}$ is the $\rho^{-1/2}$ amplitude spreading of a cylindrical wave. Spherical waves instead have amplitude proportional to r^{-1}; the comparison is about radial distance, not about an arbitrary dimensionless argument. The Hankel functions are defined with correlated upper and lower signs:

$$\begin{aligned} H_\nu^{(1,2)}(x) &= J_\nu(x) \pm iY_\nu(x), \\ H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x), \\ H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x). \end{aligned} \tag{B.3.5}$$

For $\nu > 0$, the dominant small-argument terms are therefore

$$H_\nu^{(1,2)}(x) \sim \mp i \frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu, \qquad x \to 0^+. \tag{B.3.6}$$

Keeping only this leading asymptotic is deliberate: for noninteger order, the subleading terms of Y_ν mean that simply appending the leading term of J_ν does not give a generally valid two-term expansion. At order zero,

$$H_0^{(1,2)}(x) = 1 \pm \frac{2i}{\pi} \left[\ln\left(\frac{x}{2}\right) + \gamma_E\right] + O(x^2 \ln x).$$

Small- and Large-Argument Behavior (cont.)

Combining (B.3.4a) and (B.3.4b) with $\cos\chi \pm i\sin\chi = e^{\pm i\chi}$ gives

$$H_\nu^{(1,2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{\pm i\chi_\nu}, \qquad x \to +\infty. \tag{B.3.7}$$

The propagation labels depend on the time convention. With $e^{i\omega t}$ and $x = k\rho$,

$$\Re\left\{e^{i\omega t} H_\nu^{(1)}(k\rho)\right\} \propto \cos(\omega t + k\rho - \text{constant}), \qquad \frac{d\rho}{dt} = -\frac{\omega}{k},$$

so $H_\nu^{(1)}$ is inward-propagating, while

$$\Re\left\{e^{i\omega t} H_\nu^{(2)}(k\rho)\right\} \propto \cos(\omega t - k\rho - \text{constant}), \qquad \frac{d\rho}{dt} = +\frac{\omega}{k},$$

so $H_\nu^{(2)}$ is outward-propagating. The roles reverse if the $e^{-i\omega t}$ convention is used.

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Derivation of Important Relations

The following identities are stated with their hypotheses and derivation steps so that signs and normalizations can be checked directly.

Recurrence and derivative relations. For any cylinder function $Z_\nu \in \{J_\nu, Y_\nu, H_\nu^{(1)}, H_\nu^{(2)}\}$, differentiation of its series or connection formula gives

$$\frac{d}{dx}\left[x^\nu Z_\nu(x)\right] = x^\nu Z_{\nu-1}(x), \tag{B.4.1a}$$

$$\frac{d}{dx}\left[x^{-\nu} Z_\nu(x)\right] = -x^{-\nu} Z_{\nu+1}(x). \tag{B.4.1b}$$

Applying the product rule and dividing by $x^{\pm\nu}$ gives

$$Z_\nu' + \frac{\nu}{x} Z_\nu = Z_{\nu-1}, \qquad Z_\nu' - \frac{\nu}{x} Z_\nu = -Z_{\nu+1}.$$

Adding and subtracting these equations yields

$$Z_{\nu-1}(x) + Z_{\nu+1}(x) = \frac{2\nu}{x} Z_\nu(x), \tag{B.4.2a}$$

$$Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2Z_\nu'(x). \tag{B.4.2b}$$

Derivation of Important Relations (cont.)

Wronskians. After division by x^2, Bessel's equation has the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

For two solutions y_1, y_2, define $W = y_1 y_2' - y_1' y_2$. Direct differentiation and substitution of the differential equation give

$$W' = y_1 y_2'' - y_1'' y_2 = -\frac{1}{x} W \quad \Longrightarrow \quad W = \frac{C}{x}.$$

Using the small-x forms (B.3.1)–(B.3.2) for $\nu > 0$ fixes C:

$$J_\nu Y_\nu' - J_\nu' Y_\nu \sim \frac{2\nu}{x} \frac{1}{2^\nu \Gamma(\nu+1)} \frac{2^\nu \Gamma(\nu)}{\pi} = \frac{2}{\pi x}.$$

Continuation in order includes $\nu = 0$, so

$$J_\nu(x) Y_\nu'(x) - J_\nu'(x) Y_\nu(x) = \frac{2}{\pi x}, \tag{B.4.3a}$$

$$J_\nu(x) H_\nu^{(1,2)\prime}(x) - J_\nu'(x) H_\nu^{(1,2)}(x) = \pm\frac{2i}{\pi x}. \tag{B.4.3b}$$

Derivation of Important Relations (cont.)

The second identity follows immediately by inserting $H_\nu^{(1,2)} = J_\nu \pm iY_\nu$; the J_ν–J_ν Wronskian is zero.

Generating function and integer-order integrals. For $t \neq 0$,

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \tag{B.4.4}$$

To verify the coefficient, expand the two exponentials:

$$e^{xt/2}e^{-x/(2t)} = \sum_{p=0}^{\infty} \frac{(xt/2)^p}{p!} \sum_{q=0}^{\infty} \frac{(-x/(2t))^q}{q!}.$$

For $n \geq 0$, the coefficient of t^n has $p = q + n$ and is

$$\sum_{q=0}^{\infty} \frac{(-1)^q}{q!(q+n)!} \left(\frac{x}{2}\right)^{2q+n} = J_n(x);$$

negative n follows from $J_{-n} = (-1)^n J_n$.

Derivation of Important Relations (cont.)

Setting $t = e^{i\theta}$ in (B.4.4) gives

$$e^{ix \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta}.$$

Fourier coefficient extraction therefore yields, for integer n,

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x \sin \theta - n\theta)}\, d\theta \tag{B.4.5a}$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin \theta)\, d\theta \tag{B.4.5b}$$

$$= \frac{1}{2\pi i^n} \int_0^{2\pi} e^{i(x \cos \theta + n\theta)}\, d\theta. \tag{B.4.5c}$$

The middle formula is not valid for arbitrary noninteger ν without an additional integral term; restricting it to $n \in \mathbb{Z}$ is essential.

Derivation of Important Relations (cont.)

Orthogonality and normalization at zeros. Let $a > 0$, let $\alpha_{\nu m} > 0$ be the mth positive zero of J_ν for $m = 1, 2, \ldots$, and define

$$R_m(\rho) = J_\nu\left(\frac{\alpha_{\nu m}\rho}{a}\right), \qquad 0 \le \rho \le a.$$

The radial Bessel equation is in Sturm–Liouville form,

$$\frac{d}{d\rho}\left(\rho\frac{dR_m}{d\rho}\right) + \left(\frac{\alpha_{\nu m}^2}{a^2}\rho - \frac{\nu^2}{\rho}\right)R_m = 0.$$

Multiply the m equation by R_n, multiply the n equation by R_m, subtract, and integrate. The result is

$$\left(\frac{\alpha_{\nu m}^2 - \alpha_{\nu n}^2}{a^2}\right)\int_0^a \rho R_m R_n \, d\rho = \left[\rho\left(R_m R_n' - R_n R_m'\right)\right]_0^a.$$

The endpoint at a vanishes because $R_m(a) = R_n(a) = 0$, and regularity makes the endpoint at 0 vanish. Hence the integral is zero for $m \neq n$.

Derivation of Important Relations (cont.)

For $m = n$, set $u = \alpha_{\nu n}\rho/a$ and use the directly differentiable identity

$$\frac{d}{du}\left\{\frac{u^2}{2}\left[J_\nu^2(u) - J_{\nu-1}(u)J_{\nu+1}(u)\right]\right\} = uJ_\nu^2(u).$$

It gives

$$\begin{aligned}
\int_0^a \rho R_n^2\, d\rho &= \frac{a^2}{\alpha_{\nu n}^2}\int_0^{\alpha_{\nu n}} uJ_\nu^2(u)\, du \\
&= \frac{a^2}{2}\left[J_\nu^2(\alpha_{\nu n}) - J_{\nu-1}(\alpha_{\nu n})J_{\nu+1}(\alpha_{\nu n})\right] \\
&= \frac{a^2}{2}J_{\nu+1}^2(\alpha_{\nu n}),
\end{aligned}$$

because $J_\nu(\alpha_{\nu n}) = 0$ and the recurrence relation gives $J_{\nu-1}(\alpha_{\nu n}) = -J_{\nu+1}(\alpha_{\nu n})$. Combining both cases,

$$\int_0^a J_\nu\left(\frac{\alpha_{\nu m}\rho}{a}\right) J_\nu\left(\frac{\alpha_{\nu n}\rho}{a}\right)\rho\, d\rho = \frac{a^2}{2}J_{\nu+1}^2(\alpha_{\nu n})\delta_{mn}. \tag{B.4.6}$$

Derivation of Important Relations (cont.)

Circular-waveguide TE modes instead use positive zeros $\alpha'_{\nu n}$ of J'_ν. The same Sturm–Liouville subtraction proves orthogonality: at $\rho = a$ both radial derivatives vanish, while regularity again makes the $\rho = 0$ endpoint term vanish. At a derivative zero, the recurrence equations give $J_{\nu-1}(\alpha') = J_{\nu+1}(\alpha') = (\nu/\alpha')J_\nu(\alpha')$, so the normalization is

$$\begin{aligned}\int_0^a J_\nu\left(\frac{\alpha'_{\nu m}\rho}{a}\right) J_\nu\left(\frac{\alpha'_{\nu n}\rho}{a}\right)\rho\,d\rho \\ = \frac{a^2}{2}\left(1 - \frac{\nu^2}{(\alpha'_{\nu n})^2}\right) J_\nu^2(\alpha'_{\nu n})\delta_{mn}.\end{aligned} \tag{B.4.7}$$

Only positive derivative zeros are included; the formal zero of J'_0 at the origin is not a nontrivial TE_{00} waveguide mode.

Derivation of Important Relations

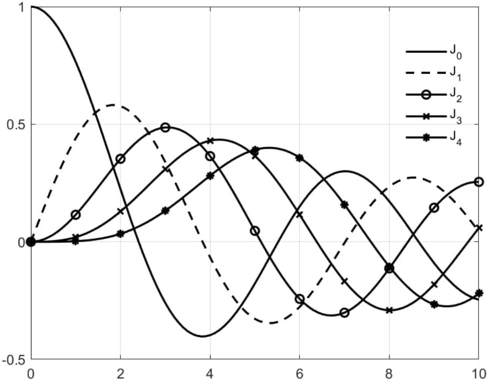

Figure B.1: Bessel functions of the first kind for $\nu = 0, 1, 2, 3, 4$.

Derivation of Important Relations

Derivation of Important Relations

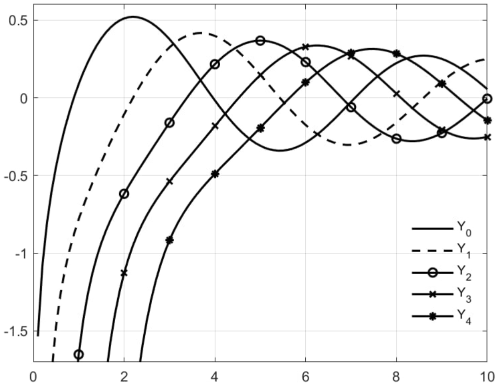

Figure B.2: Bessel functions of the second kind for $\nu = 0, 1, 2, 3, 4$.