Vector Analysis

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Outline

Chapter Overview

This appendix summarizes the vector identities and coordinate formulas used in the main text and shows the intermediate component steps that fix their signs and scale factors. Unless stated otherwise, the scalar fields are sufficiently differentiable and the vector components are resolved in a right-handed orthonormal basis. Second-derivative identities require the relevant mixed partial derivatives to commute.

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Triple Products

The scalar triple product of three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ is given by:

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right). \tag{A.1.1}$$

Expanding the cross product in Cartesian components gives

$$\begin{aligned} \vec{A} \cdot (\vec{B} \times \vec{C}) = A_x(B_y C_z - B_z C_y) - A_y(B_x C_z - B_z C_x) \\ + A_z(B_x C_y - B_y C_x), \end{aligned} \tag{A.1.2}$$

which is the cofactor expansion of the determinant

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{A.1.3}$$

A cyclic permutation of the three rows is an even permutation and leaves the determinant unchanged, which proves the three-way cyclic equality above.

Triple Products (cont.)

For a compact verification of the vector triple product, introduce the Levi–Civita symbol ε_{ijk} and Kronecker delta δ_{ij}. Repeated Cartesian indices in this appendix are summed from 1 to 3. Using $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$,

$$\begin{aligned}\left[\vec{A} \times (\vec{B} \times \vec{C})\right]_i &= \varepsilon_{ijk} A_j \varepsilon_{klm} B_l C_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m \\ &= B_i(A_j C_j) - C_i(A_j B_j).\end{aligned} \tag{A.1.4}$$

Restoring vector notation gives the “BAC–CAB” rule

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right). \tag{A.1.5}$$

Gradient Identities

For scalar fields f and g, and vector fields $\vec{A}$ and $\vec{B}$:

$$\vec{\nabla}(fg) = f\,\vec{\nabla}g + g\,\vec{\nabla}f, \tag{A.1.6}$$

$$\begin{aligned}\vec{\nabla}\left(\vec{A}\cdot\vec{B}\right) &= \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B} + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} \\ &\quad + \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right).\end{aligned} \tag{A.1.7}$$

For example, the first identity is simply $\partial_i(fg) = f\,\partial_i g + g\,\partial_i f$ in each component. For the second, expanding the right-hand side in Cartesian components and using the epsilon–delta contraction gives

$$\begin{aligned}\left[\vec{A}\times(\vec{\nabla}\times\vec{B})\right]_i &= \varepsilon_{ijk}A_j\varepsilon_{klm}\partial_l B_m = A_m\partial_i B_m - A_j\partial_j B_i, \\ \left[\vec{B}\times(\vec{\nabla}\times\vec{A})\right]_i &= B_m\partial_i A_m - B_j\partial_j A_i.\end{aligned} \tag{A.1.8}$$

Adding $(\vec{A}\cdot\vec{\nabla})B_i$ and $(\vec{B}\cdot\vec{\nabla})A_i$ cancels the last terms and leaves $A_m\partial_i B_m + B_m\partial_i A_m = \partial_i(A_m B_m)$, the ith component of $\vec{\nabla}(\vec{A}\cdot\vec{B})$.

Divergence Identities

For a scalar field f and vector fields $\vec{A}$ and $\vec{B}$:

$$\vec{\nabla} \cdot \left(f\vec{A}\right) = f\left(\vec{\nabla} \cdot \vec{A}\right) + \vec{A} \cdot \left(\vec{\nabla} f\right), \tag{A.1.9}$$

$$\vec{\nabla} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) - \vec{A} \cdot \left(\vec{\nabla} \times \vec{B}\right). \tag{A.1.10}$$

Both signs in the cross-product identity follow directly from the component product rule:

$$\begin{aligned} \partial_i(\varepsilon_{ijk} A_j B_k) &= \varepsilon_{ijk}(\partial_i A_j) B_k + \varepsilon_{ijk} A_j (\partial_i B_k) \\ &= \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}). \end{aligned} \tag{A.1.11}$$

Curl Identities

For a scalar field f and vector fields $\vec{A}$ and $\vec{B}$:

$$\vec{\nabla} \times \left(f\vec{A}\right) = f\left(\vec{\nabla} \times \vec{A}\right) + \left(\vec{\nabla} f\right) \times \vec{A}, \tag{A.1.12}$$

Indeed, its ith component is

$$\varepsilon_{ijk}\partial_j(fA_k) = f\,\varepsilon_{ijk}\partial_j A_k + \varepsilon_{ijk}(\partial_j f)A_k. \tag{A.1.13}$$

$$\begin{aligned} \vec{\nabla} \times \left(\vec{A} \times \vec{B}\right) = \vec{A}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{B}\left(\vec{\nabla} \cdot \vec{A}\right) \\ + \left(\vec{B} \cdot \vec{\nabla}\right)\vec{A} - \left(\vec{A} \cdot \vec{\nabla}\right)\vec{B}. \end{aligned} \tag{A.1.14}$$

To verify all four terms and their signs at once,

$$\begin{aligned} \left[\vec{\nabla} \times (\vec{A} \times \vec{B})\right]_i &= \varepsilon_{ijk}\partial_j(\varepsilon_{klm}A_l B_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j(A_l B_m) \\ &= A_i\partial_m B_m - B_i\partial_l A_l + B_m\partial_m A_i - A_l\partial_l B_i, \end{aligned} \tag{A.1.15}$$

which is the displayed vector identity component by component.

Second Derivative Identities

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0, \tag{A.1.16}$$

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) = 0, \tag{A.1.17}$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} f\right) = \nabla^2 f \quad \text{(Laplacian)}, \tag{A.1.18}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A}. \tag{A.1.19}$$

The first two zeros result from contracting the symmetric second derivative $\partial_i \partial_j$ with the antisymmetric ε_{ijk}. The final identity follows explicitly from

$$\begin{aligned} \left[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})\right]_i &= \varepsilon_{ijk} \partial_j (\varepsilon_{klm} \partial_l A_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j \partial_l A_m \\ &= \partial_i (\partial_m A_m) - \partial_j \partial_j A_i. \end{aligned} \tag{A.1.20}$$

Here $\nabla^2 \vec{A}$ denotes the Cartesian componentwise vector Laplacian; it is not obtained by applying the scalar curvilinear-coordinate formula separately to position-dependent physical components.

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Orthogonal Coordinate Systems

Let (q_1, q_2, q_3) be right-handed orthogonal coordinates with position vector $\vec{r}(q_1, q_2, q_3)$. Define the scale factors and unit vectors by

$$h_i = \left|\frac{\partial \vec{r}}{\partial q_i}\right|, \qquad \hat{e}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial q_i}, \qquad d\vec{r} = \sum_{i=1}^{3} h_i \hat{e}_i \, dq_i. \tag{A.2.1}$$

Thus the differential volume is $dv = h_1 h_2 h_3 \, dq_1 dq_2 dq_3$. The gradient follows by writing the same differential in two ways:

$$df = \sum_i \frac{\partial f}{\partial q_i} \, dq_i = \vec{\nabla} f \cdot d\vec{r} = \sum_i h_i (\vec{\nabla} f \cdot \hat{e}_i) \, dq_i, \tag{A.2.2}$$

so coefficient matching gives $\vec{\nabla} f \cdot \hat{e}_i = h_i^{-1} \partial_{q_i} f$.
For the divergence, the two faces normal to $\hat{e}_i$ have area

$$dS_i = \frac{h_1 h_2 h_3}{h_i} \prod_{j \neq i} dq_j. \tag{A.2.3}$$

Orthogonal Coordinate Systems (cont.)

Write $A_i = \vec{A} \cdot \hat{e}_i$ for the physical components of $\vec{A}$ along the local unit vectors. Their outward-flux difference, to first order in dq_i, is

$$\frac{\partial}{\partial q_i} \left(\frac{h_1 h_2 h_3}{h_i} A_i \right) dq_1 dq_2 dq_3. \tag{A.2.4}$$

Summing the three face pairs and dividing by dv gives the divergence formula below. Likewise, circulation around the $q_2 q_3$ face gives

$$(\vec{\nabla} \times \vec{A}) \cdot \hat{e}_1 = \frac{1}{h_2 h_3} \left[\frac{\partial(h_3 A_3)}{\partial q_2} - \frac{\partial(h_2 A_2)}{\partial q_3} \right], \tag{A.2.5}$$

with the other two components obtained by cyclic permutation. These three components form the curl determinant below, and applying the divergence formula to $\vec{\nabla} f$ gives the scalar Laplacian. The resulting master formulas are

$$\vec{\nabla} f = \sum_{i=1}^{3} \frac{\hat{e}_i}{h_i} \frac{\partial f}{\partial q_i}, \tag{A.2.6}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial q_i} \left(\frac{h_1 h_2 h_3}{h_i} A_i \right), \tag{A.2.7}$$

Orthogonal Coordinate Systems (cont.)

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}, \tag{A.2.8}$$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial q_i} \left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial f}{\partial q_i} \right). \tag{A.2.9}$$

The formulas below follow by deriving h_i and substituting them into these four master expressions.

Cartesian Coordinates (x, y, z)

The position vector and its coordinate derivatives are

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}, \qquad \partial_x \vec{r} = \hat{x}, \quad \partial_y \vec{r} = \hat{y}, \quad \partial_z \vec{r} = \hat{z}. \tag{A.2.10}$$

Hence $(h_x, h_y, h_z) = (1, 1, 1)$. Substitution into the master formulas gives the following component expressions.

Vector Operations

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}, \tag{A.2.11}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \tag{A.2.12}$$

$$\begin{aligned} \vec{\nabla} \times \vec{A} = {} & \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \\ & + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\ & + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right), \end{aligned} \tag{A.2.13}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{A.2.14}$$

Coordinate Transformation

$$\begin{aligned} x &= \rho\cos\phi, \\ y &= \rho\sin\phi, \\ z &= z, \qquad \rho \geq 0, \qquad 0 \leq \phi < 2\pi. \end{aligned} \tag{A.2.15}$$

Away from the coordinate axis, the inverse relations are

$$\rho = \sqrt{x^2 + y^2}, \qquad \phi = \mathrm{atan2}(y, x). \tag{A.2.16}$$

The azimuth returned by atan2 is reduced modulo 2π to the stated range. Writing the position vector in Cartesian basis and differentiating,

$$\begin{aligned} \vec{r} &= \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\hat{z}, \\ \frac{\partial\vec{r}}{\partial\rho} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} = \hat{\rho}, \\ \frac{\partial\vec{r}}{\partial\phi} &= \rho(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = \rho\hat{\phi}, \\ \frac{\partial\vec{r}}{\partial z} &= \hat{z}. \end{aligned} \tag{A.2.17}$$

Coordinate Transformation (cont.)

Therefore

$$(h_\rho, h_\phi, h_z) = (1, \rho, 1), \qquad dv = \rho\, d\rho\, d\phi\, dz. \tag{A.2.18}$$

Substituting these scale factors into the master formulas and expanding the curl determinant yields the expressions below.

Vector Operations

$$\vec{\nabla} f = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial f}{\partial z}, \tag{A.2.19}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}, \tag{A.2.20}$$

$$\begin{aligned} \vec{\nabla} \times \vec{A} = {} & \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) \\ & + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) \\ & + \hat{z}\,\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right], \end{aligned} \tag{A.2.21}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \tag{A.2.22}$$

Coordinate Transformation

$$
\begin{aligned}
x &= r\sin\theta\cos\phi, & y &= r\sin\theta\sin\phi, \\
z &= r\cos\theta, & r &\geq 0, \\
0 &\leq \theta \leq \pi, & 0 &\leq \phi < 2\pi.
\end{aligned}
\tag{A.2.23}
$$

For $r > 0$ and away from the polar axis, the inverse relations are

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2}, \\
\theta &= \arccos\left(\frac{z}{r}\right), \\
\phi &= \operatorname{atan2}(y, x).
\end{aligned}
\tag{A.2.24}
$$

Again, the azimuth is reduced modulo 2π to the stated range. Differentiating the position vector gives the local orthonormal basis and scale factors without guessing:

$$
\begin{aligned}
\frac{\partial \vec{r}}{\partial r} &= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} = \hat{r}, \\
\frac{\partial \vec{r}}{\partial \theta} &= r\,(\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}) = r\hat{\theta}, \\
\frac{\partial \vec{r}}{\partial \phi} &= r\sin\theta\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = r\sin\theta\,\hat{\phi}.
\end{aligned}
\tag{A.2.25}
$$

Coordinate Transformation (cont.)

Thus

$$(h_r, h_\theta, h_\phi) = (1, r, r\sin\theta), \qquad dv = r^2 \sin\theta \, dr \, d\theta \, d\phi. \tag{A.2.26}$$

Substitution into the master formulas gives the following results.

Vector Operations

$$\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}, \tag{A.2.27}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}, \tag{A.2.28}$$

$$\begin{aligned}\vec{\nabla}\times\vec{A} = {} & \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] \\ & + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] \\ & + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right],\end{aligned} \tag{A.2.29}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}. \tag{A.2.30}$$

The factors $1/\rho$, $1/r$, and $1/\sin\theta$ signal coordinate singularities. Cylindrical components at $\rho = 0$ and spherical components at $r = 0$ or $\theta = 0, \pi$ must be understood by taking limits of the underlying Cartesian field; apparent component divergences at those points need not be physical singularities.