Scattering from a PEC Wedge

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Reading: Ufimtsev. *Fundamentals of the Physical Theory of Scattering*.
John Wiley & Sons, 2014.
Section 2.1.

Scattering from a PEC Wedge

2-D wave equation with current sources ($e^{i\omega t}$ convention):

$$\left(\nabla^2 + k^2\right) E_z = i\omega\mu\, Jz, \tag{1}$$

$$\left(\nabla^2 + k^2\right) H_z = -i\omega\epsilon\, Mz. \tag{2}$$

Define the Green function:

$$\left(\nabla^2 + k^2\right) G = -\delta(\rho - \rho', \phi - \phi'), \tag{3}$$

then

$$E_z = \int_S -i\omega\mu J_z(\rho', \phi') G\, ds', \tag{4}$$

$$H_z = \int_S i\omega\epsilon M_z(\rho', \phi') G\, ds'. \tag{5}$$

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Geometry: Infinite wedge with faces $\phi = 0$ and $\phi = n\pi$, line source I_0 at (ρ', ϕ'), 2-D $(\partial/\partial z \equiv 0)$.

Boundary conditions (PEC):

$$G_s = 0 \quad \text{(soft / TM: } E_z\text{)}, \tag{6}$$

$$\frac{\partial G_h}{\partial n} = 0 \quad \text{(hard / TE: } H_z\text{)}. \tag{7}$$

Solution:

$$E_z = -i\omega\mu I_0 G_s, \tag{8}$$

$$H_z = i\omega\epsilon I_0 G_h. \tag{9}$$

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Laplacian in polar coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}. \tag{10}$$

Use $G = R(\rho)\Phi(\phi)$:

$$\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0, \quad x = k\rho, \tag{11}$$

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0. \tag{12}$$

Soft (Dirichlet) faces $\phi = 0, n\pi$:

$$\Phi = \sin(\nu\phi), \quad \nu = \frac{m}{n},\ m = 1, 2, \dots \tag{13}$$

Hard (Neumann) faces:

$$\Phi = \cos(\nu\phi), \quad \nu = \frac{m}{n},\ m = 0, 1, 2, \dots \tag{14}$$

Radial solutions of (11):

$$R = \begin{cases} J_\nu(k\rho), \\ H_\nu^{(2)}(k\rho) \end{cases} \tag{15}$$

Only Bessel functions $J_\nu(k\rho)$ with positive indices $\nu \geq 0$ remain finite at $\rho = 0$, and are thus suitable for $\rho \leq \rho'$. Hankel functions are used for $\rho \geq \rho'$, as they satisfy Sommerfeld's radiation condition at infinity ($\rho \to \infty$):

$$\lim_{\rho \to \infty} \sqrt{\rho} \left(\frac{dG}{d\rho} + ik\, G \right) = 0. \tag{16}$$

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The solution can be written as

$$G_s = \sum_{m=1}^{\infty} a_m \begin{cases} J_\nu(k\rho)\, H_\nu^{(2)}(k\rho') \sin\nu\phi' \sin\nu\phi, & \rho \le \rho', \\ J_\nu(k\rho')\, H_\nu^{(2)}(k\rho) \sin\nu\phi' \sin\nu\phi, & \rho \ge \rho'. \end{cases} \tag{17}$$

$$G_h = \sum_{m=0}^{\infty} b_m \begin{cases} J_\nu(k\rho)\, H_\nu^{(2)}(k\rho') \cos\nu\phi' \cos\nu\phi, & \rho \le \rho', \\ J_\nu(k\rho')\, H_\nu^{(2)}(k\rho) \cos\nu\phi' \cos\nu\phi, & \rho \ge \rho', \end{cases} \tag{18}$$

The solutions are constructed so that they satisfy the boundary conditions and are reciprocal under $(\rho, \phi) \leftrightarrow (\rho', \phi')$ (interchanging source and observation).

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To find the coefficients a_m and b_m, Green's identity over region S bounded by L is applied:

$$\oint_L \partial_n G\, dl = \int_S \nabla^2 G\, ds, \quad ds = \rho\, d\rho\, d\phi. \tag{19}$$

Using

$$\nabla^2 G = -k^2 G - \delta(\rho - \rho', \phi - \phi'), \tag{20}$$

and shrinking the contour around (ρ', ϕ') gives

$$\begin{aligned} &\int_{\phi'-\psi}^{\phi'+\psi} \left[\partial_\rho G\big|_{\rho'+0} - \partial_\rho G\big|_{\rho'-0} \right] \rho'\, d\phi \\ &= -\int_{\phi'-\psi}^{\phi'+\psi} d\phi \lim_{\varepsilon \to 0} \int_{\rho'-\varepsilon}^{\rho'+\varepsilon} \delta(\rho - \rho', \phi - \phi') \rho d\rho. \end{aligned} \tag{21}$$

Polar delta:

$$\delta(\rho - \rho', \phi - \phi') = \frac{1}{\rho}\,\delta(\rho - \rho')\,\delta(\phi - \phi'). \tag{22}$$

Hence

$$\int_{\phi'-\psi}^{\phi'+\psi} \left[\partial_\rho G\big|_{\rho'+0} - \partial_\rho G\big|_{\rho'-0}\right] \rho'\, d\phi = -\int_{\phi'-\psi}^{\phi'+\psi} \delta(\phi - \phi') d\phi. \tag{23}$$

The above is valid for arbitrary ψ. This is possible if the integrands of both side are equal:

$$\partial_\rho G\big|_{\rho'+0} - \partial_\rho G\big|_{\rho'-0} = \frac{-1}{\rho'}\,\delta(\phi - \phi'). \tag{24}$$

Sine orthogonality (similar for cos):

$$\int_0^{n\pi} \sin \nu\phi \, \sin \nu_t\phi \, d\phi = \begin{cases} \frac{1}{2} n\pi, & m = t, \\ 0, & m \neq t, \end{cases} \tag{25}$$

Consider the soft case first. Apply (24) to the (17), multiply $\sin \nu_t \phi$ ($\nu_t = t/n$), and integrate over ϕ from 0 to $n\pi$ yields:

$$k\rho' \, \frac{n\pi}{2} a_m \left[J_\nu(k\rho') \frac{d}{d(k\rho')} H_\nu^{(2)}(k\rho') - H_\nu^{(2)}(k\rho') \frac{d}{d(k\rho')} J_\nu(k\rho') \right] = -1. \tag{26}$$

Wronskian:

$$W\Big[J_\nu(x), H_\nu^{(2)}(x) \Big] = J_\nu(x) \, H_\nu^{(2)\prime}(x) - H_\nu^{(2)}(x) \, J'_\nu(x) = -\frac{2i}{\pi x}. \tag{27}$$

Therefore

$$a_m = -\frac{i}{n}. \tag{28}$$

Similarly, for the hard case:

$$b_m = -\varepsilon_m \frac{i}{n}, \qquad \varepsilon_0 = \tfrac{1}{2},\ \varepsilon_{m\geq 1} = 1. \tag{29}$$

Thus, using a more compact notation, to solutions are written as

$$G_s = -\frac{i}{n} \sum_{m=1}^{\infty} J_\nu(k\rho_<)\, H_\nu^{(2)}(k\rho_>)\, \sin\nu\phi' \,\sin\nu\phi. \tag{30}$$

$$G_h = -\varepsilon_m \frac{i}{n} \sum_{m=0}^{\infty} J_\nu(k\rho_<)\, H_\nu^{(2)}(k\rho_>)\, \cos\nu\phi' \,\cos\nu\phi. \tag{31}$$

where $\rho_< = \min(\rho, \rho')$ and $\rho_> = \max(\rho, \rho')$.

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Since

$$\sin\nu\phi' \, \sin\nu\phi = \frac{1}{2}\left[\cos\nu(\phi-\phi') - \cos\nu(\phi+\phi')\right]$$

and

$$\cos\nu\phi' \, \cos\nu\phi = \frac{1}{2}\left[\cos\nu(\phi-\phi') + \cos\nu(\phi+\phi')\right]$$

and noted that when $m = 0 \implies \sin\nu\phi' \, \sin\nu\phi = 0$, thus,

$$G_{s,h} = -\varepsilon_m \, \frac{i}{2n} \sum_{m=0}^{\infty} J_\nu(k\rho_<) \, H_\nu^{(2)}(k\rho_>) \, \left[\cos\nu(\phi-\phi') \mp \cos\nu(\phi+\phi')\right]. \tag{32}$$

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Transition to plane-wave excitation ($k\rho' \to \infty$):

$$H_\nu^{(2)}(k\rho') \approx \sqrt{\frac{2}{\pi k\rho'}}\, e^{-i(k\rho' - \pi/4 - \nu\pi/2)} \tag{33}$$

Then

$$\begin{aligned} G_{s,h} &= -\varepsilon_m \frac{i}{n}\sqrt{\frac{1}{2\pi k\rho'}} e^{-i(k\rho' - \pi/4)} \\ &\qquad \sum_{m=0}^{\infty} J_\nu(k\rho) i^\nu \left[\cos\nu(\phi - \phi') \mp \cos\nu(\phi + \phi')\right] \\ &= -i\sqrt{\frac{i}{2\pi k\rho'}} e^{-ik\rho'} \Psi_{s,h}(k\rho), \end{aligned} \tag{34}$$

where

$$\Psi_{s,h}(k\rho) = \frac{1}{n}\sum_{m=0}^{\infty} \varepsilon_m J_\nu(k\rho) i^\nu \left[\cos\nu(\phi - \phi') \mp \cos\nu(\phi + \phi')\right]. \tag{35}$$