# Asymptotic Evaluation of Integrals: Introduction to the Steepest Descent Method

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**Reading**: Pathak and Burkholder. *Electromagnetic Radiation, Scattering, and Diffraction*. John Wiley & Sons, 2021. Section 14.1, 14.2.1, 14.2.2, 14.2.4.

# Outline


1. Steepest Descent for Radiation Integrals
2. Saddle Point Topology
3. Asymptotic Evaluation: First-Order Saddle Point (No Singularities)
4. Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM))
5. Summary

# Table of contents

1. Steepest Descent for Radiation Integrals
2. Saddle Point Topology
3. Asymptotic Evaluation: First-Order Saddle Point (No Singularities)
4. Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM))
5. Summary

# Steepest Descent for Radiation Integrals

EM radiation/scattering in exterior regions can be expressed as radiation integrals in the spatial or spectral domain. In general, these integrals cannot be evaluated exactly in closed form.

Approximate analytical methods are therefore used. In this section, the focus is on spectral-domain integrals of the form:

$$I(K) = \int_C g(\xi)\, e^{Kf(\xi)}\, d\xi,$$

where

- $K$ is a large positive parameter (increasing with frequency),
- $f(\xi), g(\xi)$ are analytic on the contour $C$,
- $C$ extends to infinity, where the integrand vanishes.

# Table of contents


1 Steepest Descent for Radiation Integrals

2 Saddle Point Topology

3 Asymptotic Evaluation: First-Order Saddle Point (No Singularities)

4 Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM))

5 Summary

# Saddle Point Topology

Asymptotic high-frequency (HF) solutions improve as $K \to \infty$. The method of steepest descent transforms $I(K)$ into a simpler canonical form that retains all essential features. We define a stationary point (saddle point) of the integrand in

$$I(K) = \int_C g(\xi) e^{Kf(\xi)}\, d\xi$$

by the condition

$$\left. \frac{df(\xi)}{d\xi} \right|_{\xi=\xi_s} = f'(\xi_s) = 0,$$

where $\xi_s$ is a first-order saddle point if $f''(\xi_s) \neq 0$.

Let $\xi = x + iy$, where $x, y \in \mathbb{R}$. Express $f(\xi)$ as:

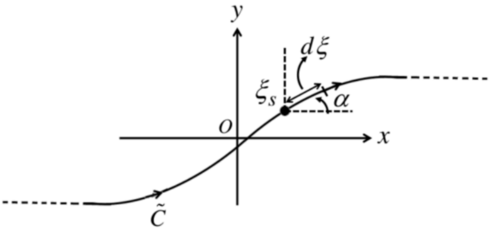
$$f(\xi) = u(x, y) + iv(x, y).$$

where $u, v \in \mathbb{R}$. Then

$$f'(\xi) = \frac{du}{d\xi} + i\frac{dv}{d\xi} = \left(\frac{\partial u}{\partial x}\frac{dx}{d\xi} + \frac{\partial u}{\partial y}\frac{dy}{d\xi}\right) + i\left(\frac{\partial v}{\partial x}\frac{dx}{d\xi} + \frac{\partial v}{\partial y}\frac{dy}{d\xi}\right).$$

At the stationary point $\xi = \xi_s$, let $\tilde{C}$ define a path with passes $\xi_s$, and

$$d\xi = dx + i\,dy, \qquad dx = d\xi \cos\alpha, \quad dy = d\xi \sin\alpha.$$




Reading, p. 754

# Saddle Point Topology

It follows that

$$\frac{du}{d\xi} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha; \quad \frac{dv}{d\xi} = \frac{\partial v}{\partial x}\cos\alpha + \frac{\partial v}{\partial y}\sin\alpha,$$

At $\xi = \xi_s$, we have the above both equal to zero. Since $\alpha$ can be arbitrary, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Since $f(\xi)$ is analytic at $\xi = \xi_s$, from the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

the Laplace equation is also satisfied:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Thus, $u$ and $v$ are harmonic functions that cannot have absolute maxima or minima.

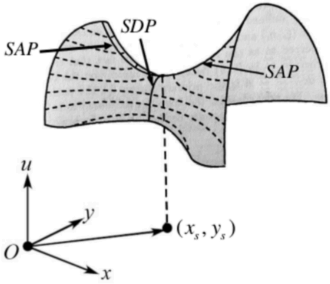
# Saddle Point Topology

In particular, attention is given to the topology of $u(\xi)$, since it determines

$$|e^{Kf(\xi)}| = e^{Ku(\xi)},$$

and thereby governs the behavior of the integral $I(K)$.

It is therefore desirable to deform the contour $C$ so that $e^{Ku(\xi)}$ varies most rapidly while ensuring that $I(K)$ remains bounded.




Reading, p. 755

# Saddle Point Topology

To examine the maximum rate of change of $\frac{du}{d\xi}$ as a function of $\alpha$ at $\xi = \xi_s$, the condition for a maximum is

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial u}{\partial\xi}\right) = \frac{\partial^2 u}{\partial\alpha\partial\xi} = 0 = -\frac{\partial u}{\partial x}\sin\alpha + \frac{\partial u}{\partial y}\cos\alpha.$$

From the Cauchy–Riemann equations, the above continues

$$\ldots = -\frac{\partial v}{\partial y}\sin\alpha - \frac{\partial v}{\partial x}\cos\alpha = -\frac{\partial v}{\partial y}\frac{dy}{d\xi} - \frac{\partial v}{\partial x}\frac{dx}{d\xi} = -\frac{dv}{d\xi},$$

or equivalently,

$$\frac{\partial^2 u}{\partial\alpha\partial\xi} = 0 \quad\Rightarrow\quad -\frac{dv}{d\xi} = 0.$$

Therefore, the steepest direction of $\tilde{C}$ through $\xi_s$ is along a path where $\frac{dv}{d\xi} = 0$, i.e., $v =$ constant. This is the path on which $e^{iKv(\xi)} =$ constant, corresponding to the constant phase path.

# Saddle Point Topology

Near $\xi = \xi_s$, expand

$$f(\xi) \approx f(\xi_s) + \frac{1}{2} f''(\xi_s)(\xi - \xi_s)^2$$

for small $|\xi - \xi_s|$. Thus,

$$e^{Kf(\xi)} \approx e^{Kf(\xi_s)} e^{Kf''(\xi_s)(\xi-\xi_s)^2/2}.$$

Let $\xi - \xi_s = |\xi - \xi_s| e^{i\theta}$, then

$$e^{Kf(\xi)} \approx e^{Kf(\xi_s)} e^{Kf''(\xi_s)|\xi-\xi_s|^2(\cos 2\theta + i \sin 2\theta)/2}.$$

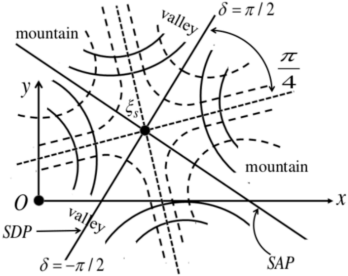
Let $\delta = \arg f''(\xi_s) + 2\theta$, then

$$e^{Kf(\xi)} \approx e^{Kf(\xi_s)} e^{K|f''(\xi_s)||\xi-\xi_s|^2(\cos 2\delta + i \sin 2\delta)/2}.$$

# Saddle Point Topology

The following are true:

- $\delta = 0, \pi$: steepest ascent path (SAP).
- $\delta = \pm\frac{\pi}{2}$: steepest descent path (SDP).
- $\delta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$: constant level path.




Reading, p. 755

# Table of contents


1. Steepest Descent for Radiation Integrals
2. Saddle Point Topology
3. Asymptotic Evaluation: First-Order Saddle Point (No Singularities)
4. Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM))
5. Summary

# Asymptotic Evaluation: First-Order Saddle Point (No Singularities)

Since $I(K)$ cannot usually be solved in closed form, the **steepest descent method** is used to obtain an approximate solution valid for moderate-to-large $K$. COnsider a first-order saddle point
$f'(\xi_s) = 0, \quad f''(\xi_s) \neq 0.$

The original contour $C$ is deformed into the SDP. If $g(\xi)$ is free of singularities, then $C$ can be transformed continuously into the SDP. Since $|e^{Kf}|$ decays rapidly on either side of $\xi = \xi_s$, it is sufficient to deform $C$ into the SDP only in the region near the saddle point $\xi_s$.

Elsewhere, contributions from the SDP are exponentially small or negligible, and the SDP can merge back into $C$.

# Asymptotic Evaluation: First-Order Saddle Point (No Singularities)

Along the SDP:

$$|e^{Kf(\xi)}| \leq |e^{Kf(\xi_s)}|, \qquad \Re f(\xi) \leq \Re f(\xi_s), \quad \Im f(\xi) = \Im f(\xi_s).$$

Main contribution to $I(K)$ occurs near $\xi = \xi_s$.

Mapping to canonical form:

$$f(\xi) - f(\xi_s) = -\alpha^2, \qquad -\infty < \alpha < \infty,$$

with saddle point at $\alpha = 0$. Thus,

$$I(K) = \int_{\text{SDP}} g(\xi) e^{Kf(\xi)} d\xi = e^{Kf(\xi_s)} \int_{-\infty}^{\infty} g(\xi) \frac{d\xi}{d\alpha} e^{-K\alpha^2} d\alpha$$
$$= e^{Kf(\xi_s)} \int_{-\infty}^{\infty} G(\alpha) e^{-K\alpha^2} d\alpha,$$

where

$$G(\alpha) = g(\xi) \frac{d\xi}{d\alpha}.$$

# Asymptotic Evaluation: First-Order Saddle Point (No Singularities)

The term $e^{-K\alpha^2}$ decays rapidly away from $\alpha = 0$ on the SDP. For slowly varying $G(\alpha)$, one may approximate

$$I(K) = e^{Kf(\xi_s)} \int_{-\infty}^{\infty} \left[ \sum_{n=0}^{\infty} c_n \alpha^n \right] e^{-K\alpha^2} \, d\alpha,$$

where

$$G(\alpha) = \sum_{n=0}^{\infty} c_n \alpha^n, \qquad c_n = \frac{1}{n!} \left. \frac{d^n G(\alpha)}{d\alpha^n} \right|_{\alpha=0}.$$

Interchanging sum and integration:

$$I(K) = e^{Kf(\xi_s)} \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n G(\alpha)}{d\alpha^n} \right|_{\alpha=0} \int_{-\infty}^{\infty} \alpha^n e^{-K\alpha^2} d\alpha.$$

# Asymptotic Evaluation: First-Order Saddle Point (No Singularities)

The Gaussian integral

$$\int_{-\infty}^{\infty} \alpha^n e^{-K\alpha^2}\, d\alpha = \begin{cases} \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{K^{(n+1)/2}}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

where $\Gamma(z)$ is the Gamma function. For $n = 2m$, this reduces to

$$\int_{-\infty}^{\infty} \alpha^{2m} e^{-K\alpha^2}\, d\alpha = \frac{\Gamma\left(m + \frac{1}{2}\right)}{K^{m+1/2}}, \quad m = 0, 1, 2, \ldots$$

Hence, the asymptotic expansion of $I(K)$ in inverse powers of large $K$ is

$$I(K) = e^{Kf(\xi_s)} \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left. \frac{d^{2m} G(\alpha)}{d\alpha^{2m}} \right|_{\alpha=0} \frac{\Gamma\left(m + \frac{1}{2}\right)}{K^{m+1/2}}.$$

# Asymptotic Evaluation: First-Order Saddle Point (No Singularities)

For large $K$, it suffices to retain only the leading term ($m = 0$):

$$I(K) \sim G(0)\, e^{Kf(\xi_s)} \sqrt{\tfrac{\pi}{K}}.$$

Noted that

$$f'(\xi)\,\frac{d\xi}{d\alpha} = -2\alpha, \qquad f'(\xi)\frac{d^2\xi}{d\alpha^2} + f''(\xi)\left(\frac{d\xi}{d\alpha}\right)^2 = -2.$$

Hence,

$$\left.\frac{d\xi}{d\alpha}\right|_{\alpha=0} = \sqrt{\frac{-2}{f''(\xi_s)}}.$$

We have

$$I(K) \sim g(\xi_s)\left[\frac{d\xi}{d\alpha}\right]_{\alpha=0} e^{Kf(\xi_s)}\sqrt{\tfrac{\pi}{K}} = g(\xi_s)\sqrt{\frac{-2\pi}{Kf''(\xi_s)}}\,e^{Kf(\xi_s)}.$$

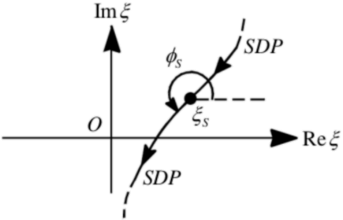
# Asymptotic Evaluation: First-Order Saddle Point (No Singularities)

Along the steepest descent path (SDP), let

$$\left.\frac{d\xi}{d\alpha}\right|_{\alpha=0} = \left|\sqrt{\frac{-2}{f''(\xi_s)}}\right| e^{i\phi_s},$$

where $\phi_s = \arg(d\xi)$. Thus,

$$I(K) \sim g(\xi_s)\left|\sqrt{\frac{-2}{f''(\xi_s)}}\right| e^{i\phi_s} e^{Kf(\xi_s)}.$$




Reading, p. 760

# Table of contents


1 Steepest Descent for Radiation Integrals

2 Saddle Point Topology

3 Asymptotic Evaluation: First-Order Saddle Point (No Singularities)

4 Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM))

5 Summary

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

Consider

$$I(K) = \int_C g(\xi) e^{Kf(\xi)} d\xi.$$

- $f(\xi)$ has a first-order saddle point at $\xi = \xi_s$, so $f'(\xi_s) = 0$, $f''(\xi_s) \neq 0$.
- $g(\xi)$ has a simple pole at $\xi = \xi_p$.

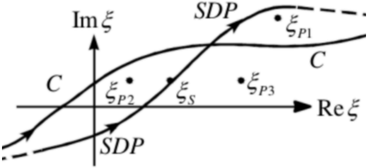
Deforming the contour $C$ into the SDP and applying the residue theorem:

$$I(K) = \int_{\text{SDP}} g(\xi) e^{Kf(\xi)} d\xi + (2\pi i R_p) Q,$$

where $R_p$ is the residue at $\xi = \xi_p$.

$$Q = \begin{cases} +1, & \text{SDP encloses } \xi_p = \xi_{p1} \text{ (CCW)}, \\ -1, & \text{SDP encloses } \xi_p = \xi_{p2} \text{ (CW)}, \\ 0, & \text{SDP encloses no pole, } \xi_p = \xi_{p3}. \end{cases}$$

Reading, p. 757



# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

The mapping

$$f(\xi) = f(\xi_s) - \alpha^2$$

transforms the SDP from the $\xi$-plane to the real $\alpha$ axis. The residue $R_p$ is

$$R_p = \lim_{\xi \to \xi_p} (\xi - \xi_p) g(\xi) e^{Kf(\xi)},$$

or equivalently

$$R_p e^{-Kf(\xi_p)} = \lim_{\xi \to \xi_p} (\xi - \xi_p) g(\xi) = \lim_{\alpha \to \alpha_p} (\alpha - \alpha_p) G(\alpha).$$

To measure the separation between $\xi_s$ and $\xi_p$:

$$f(\xi_s) - f(\xi_p) = \alpha_p^2 \equiv -ia, \quad \alpha_p = \sqrt{f(\xi_s) - f(\xi_p)}.$$

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

If $\xi_p$ is close to $\xi_s$, then

$$f(\xi_p) \approx f(\xi_s) + \tfrac{1}{2} f''(\xi_s)(\xi_p - \xi_s)^2, \quad \alpha_p^2 \approx -\tfrac{f''(\xi_s)}{2}(\xi_p - \xi_s)^2.$$

Let $\xi_p - \xi_s = |\xi_p - \xi_s| e^{i\phi_p}$, where $\phi_p$ is the angle of the vector from $\xi_s$ to $\xi_p$. Thus,

$$\alpha_p \approx \frac{|\xi_p - \xi_s| e^{i\phi_p}}{\sqrt{-2/f''(\xi_s)}}.$$

Since

$$\sqrt{\tfrac{-2}{f''(\xi_s)}} = \left| \sqrt{\tfrac{-2}{f''(\xi_s)}} \right| e^{i\phi_s},$$

the argument of $\alpha_p$ is

$$\alpha_p \equiv |\alpha_p| e^{i\psi} \approx \frac{|\xi_p - \xi_s| e^{i(\phi_p - \phi_s)}}{\left| \sqrt{-2/f''(\xi_s)} \right|}, \quad \xi_p \text{ near } \xi_s.$$

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

The pole of $g(\xi)$ at $\xi = \xi_p$ strongly influences the saddle point contribution from $\xi = \xi_s$ when $\xi_p$ is near $\xi_s$ (i.e., small *ia*). If *ia* is large, $\xi_p$ is far from $\xi_s$, and the asymptotic solution is a superposition of:
- isolated pole contribution (residue term),
- isolated saddle point contribution (SDP term).

As $\xi_p \to \xi_s$, $g(\xi_s)$ becomes singular. Thus, the Pauli–Clemmow method (PCM) accounts for the pole at $\alpha = \alpha_p$ by splitting

$$G(\alpha) \equiv \frac{G_a(\alpha)}{\alpha - \alpha_p},$$

where $G_a(\alpha)$ is analytic in the $\alpha$-plane.

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

The asymptotic evaluation can now be expressed as

$$I^{\mathrm{PCM}}(K) = I^{\mathrm{PCM}}_{\mathrm{SDP}}(K) + 2\pi i R_p Q,$$

with

$$I^{\mathrm{PCM}}_{\mathrm{SDP}}(K) = e^{Kf(\xi_s)} \int_{-\infty}^{\infty} \frac{G_a(\alpha)}{\alpha - \alpha_p}\, e^{-K\alpha^2}\, d\alpha.$$

Since $G_a(\alpha)$ is analytic in the $\alpha$-plane (near $\alpha = 0$ and $\alpha = \alpha_p$), it can be expanded as a power series:

$$G_a(\alpha) = \sum_{n=0}^{\infty} c_n \alpha^n.$$

Retaining only the leading term ($n = 0$) yields

$$G_a(\alpha) \approx c_0 = G_a(0).$$

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

Thus

$$I_{\text{SDP}}^{\text{PCM}}(K) \sim e^{Kf(\xi_s)} G_a(0) \int_{-\infty}^{\infty} \frac{(\alpha + \alpha_p)}{(\alpha - \alpha_p)(\alpha + \alpha_p)} e^{-K\alpha^2}\, d\alpha$$
$$\sim e^{Kf(\xi_s)} G_a(0)\alpha_p \int_{-\infty}^{\infty} \frac{e^{-K\alpha^2}}{\alpha^2 - \alpha_p^2}\, d\alpha,$$

since $\frac{\alpha}{\alpha^2-\alpha_p^2} e^{-K\alpha^2}$ is odd. With $G_a(0) = -\alpha_p G(0)$, $\alpha_p^2 = -ia$, and $G(0) = g(\xi_s)\left[\frac{d\xi}{d\alpha}\right]_{\alpha=0} = g(\xi_s)\sqrt{\frac{-2}{f''(\xi_s)}}\, e^{i\phi_s}$ :

$$I_{\text{SDP}}^{\text{PCM}}(K) \sim e^{Kf(\xi_s)}(ia)G(0) \int_{-\infty}^{\infty} \frac{e^{-K\alpha^2}}{\alpha^2 + ia}\, d\alpha$$
$$\sim g(\xi_s)\sqrt{\frac{-2\pi}{Kf''(\xi_s)}}\, e^{i\phi_s} e^{Kf(\xi_s)} \left[\sqrt{\frac{K}{\pi}}(ia) \int_{-\infty}^{\infty} \frac{e^{-K\alpha^2}}{\alpha^2 + ia}\, d\alpha\right].$$

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

The bracketed term can be expressed as a Fresnel-type integral. Let $\sqrt{Ka} > 0$ (so $\sqrt{a} > 0$ initially), and define

$$F(\sqrt{Ka}) \equiv \sqrt{\tfrac{K}{\pi}}(ia) \int_{-\infty}^{\infty} \frac{e^{-K\alpha^2}}{\alpha^2 + ia}\, d\alpha = \sqrt{\tfrac{K}{\pi}}(ia) I_0(K).$$

From

$$I_0(K) e^{-iKa} = \int_{-\infty}^{\infty} \frac{e^{-K(\alpha^2 + ia)}}{\alpha^2 + ia}\, d\alpha,$$

It follows

$$\begin{aligned} \frac{d}{dK}\Big(I_0(K) e^{-iKa}\Big) &= -\int_{-\infty}^{\infty} e^{-K(\alpha^2 + ia)}\, d\alpha \\ &= -2e^{-iKa} \int_0^{\infty} e^{-K\alpha^2}\, d\alpha = -\sqrt{\tfrac{\pi}{K}}\, e^{-iKa} \end{aligned}$$

Since $\int_0^\infty e^{-K\alpha^2}\, d\alpha = \frac{1}{2}\sqrt{\frac{\pi}{K}}$.

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

Integrating the above both sides with:

$$\int_K^{\infty} du \, \frac{d}{du}\Big(I_0(u)e^{-iua}\Big) = -\sqrt{\pi}\int_K^{\infty} \frac{e^{-iua}}{\sqrt{u}}\, du.$$

Thus,

$$I_0(u)e^{-iua}\Big|_K^{\infty} = -\sqrt{\pi}\int_K^{\infty} \frac{e^{-iua}}{\sqrt{u}}\, du.$$

Since $I_0(u) \to 0$ as $u \to \infty$, and let $\tau = \sqrt{ua}$, we get

$$I_0(K)e^{-iKa} = 2\sqrt{\frac{\pi}{a}}\int_{\sqrt{Ka}}^{\infty} e^{-i\tau^2}\, d\tau.$$

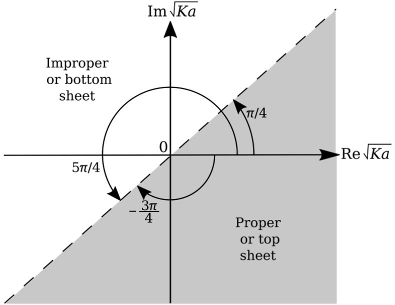
Finally:

$$F(\sqrt{Ka}) = 2i\sqrt{Ka}\, e^{iKa}\int_{\sqrt{Ka}}^{\infty} e^{-i\tau^2}\, d\tau, \quad \sqrt{Ka} > 0.$$

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

The derivation above is defined for $\sqrt{Ka} > 0$. It exhibits a branch cut.

- If $\sqrt{a} < 0$ or $\sqrt{Ka} < 0$, replace $\sqrt{Ka}$ by $-\sqrt{Ka}$ so that $F \to 1$ as $\pm\sqrt{Ka} \to \infty$.
- By analytic continuation, $F$ can be extended to complex $\sqrt{Ka}$, keeping the branch where $F \to 1$ for $|\pm\sqrt{Ka}| \to \infty$.
- The proper branch is defined for $-\frac{3\pi}{4} < \arg\sqrt{Ka} < \frac{\pi}{4}$, or equivalently for $\sqrt{a} > 0$.
- On the improper branch ($\frac{\pi}{4} < \arg\sqrt{Ka} < \frac{5\pi}{4}$ or $\sqrt{a} < 0$), one replaces $F(\sqrt{Ka})$ by $F(-\sqrt{Ka})$.




Reading, p. 1090

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

Thus, a more general expression for $F$ can be expressed as

$$F(\pm\sqrt{Ka}) = \pm 2i\sqrt{Ka}\, e^{iKa} \int_{\pm\sqrt{Ka}}^{\infty} e^{-i\tau^2}\, d\tau, \quad \begin{bmatrix} -\dfrac{3\pi}{4} < \arg\sqrt{Ka} < \dfrac{\pi}{4}, \\ \dfrac{\pi}{4} < \arg\sqrt{Ka} < \dfrac{5\pi}{4} \end{bmatrix}.$$

For small $\sqrt{Ka}$,

$$F(\pm\sqrt{Ka})\big|_{(\pm\sqrt{Ka})\,\text{small}} \;\approx\; \pm\sqrt{\pi i Ka}\, e^{iKa}.$$

For large $\sqrt{Ka}$, that is

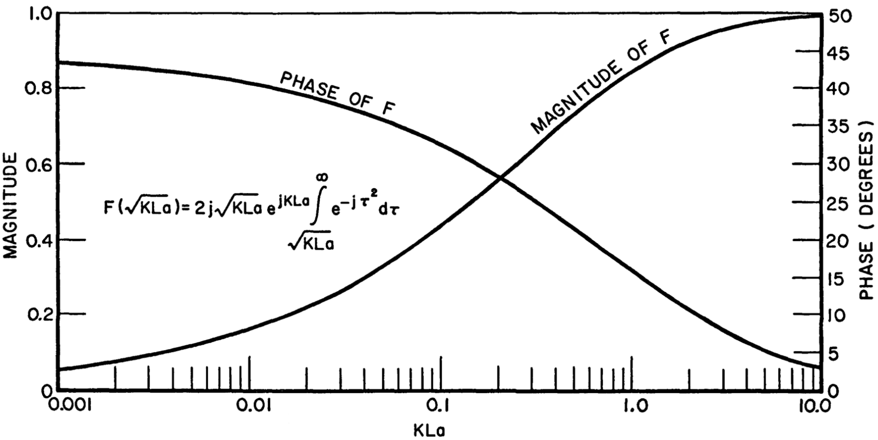
$$|\pm\sqrt{Ka}| \;\to\; \infty.$$

The asymptotic behavior of $F(\pm\sqrt{X})$ for $X = Ka$ is

$$F(\pm\sqrt{X}) \;\sim\; 1 - \frac{1}{2iX} + \cdots, \qquad |X| \to \infty.$$

Physically, large $|\sqrt{Ka}|$ means the pole is far from the saddle point.

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)




Reading, p. 880

# Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM)

From the above analysis, the leading term of the PCM-based SDP evaluation is

$$I_{\text{SDP}}^{\text{PCM}}(K) \sim g(\xi_s) \left| \sqrt{\frac{-2\pi}{Kf''(\xi_s)}} \right| e^{i\phi_s} e^{Kf(\xi_s)} F(\pm\sqrt{Ka}).$$

Also from $\alpha_p^2 = -ia$ or $a = e^{i\pi/2}\alpha_p^2$, so

$$\sqrt{Ka} = \alpha_p e^{i\pi/4}\sqrt{K},$$

$$\arg \alpha_p = \arg \sqrt{Ka} - \tfrac{\pi}{4}.$$

Thus,

$$I_{\text{SDP}}^{\text{PCM}}(K) \sim g(\xi_s) \left| \sqrt{\frac{-2\pi}{Kf''(\xi_s)}} \right| e^{i\phi_s} e^{Kf(\xi_s)} F(\pm\sqrt{Ka}), \quad \Im\alpha_p \lesseqgtr 0.$$

# Table of contents


1. Steepest Descent for Radiation Integrals
2. Saddle Point Topology
3. Asymptotic Evaluation: First-Order Saddle Point (No Singularities)
4. Asymptotic Evaluation: Saddle Point Near a Simple Pole (PCM))
5. Summary

# Summary

### Case A: No singularities in $g(\xi)$

$$I(K) \sim g(\xi_s) \sqrt{\frac{-2\pi}{K f''(\xi_s)}}\, e^{i\phi_s}\, e^{Kf(\xi_s)}.$$

### Case B: Simple pole of $g(\xi)$ at $\xi = \xi_p$ (PCM)

$$I^{\text{PCM}}(K) = 2\pi i\, R_p\, Q + g(\xi_s) \sqrt{\frac{-2\pi}{K f''(\xi_s)}}\, e^{i\phi_s}\, e^{Kf(\xi_s)}\, F(\pm\sqrt{Ka}).$$

### Fresnel transition function $F(Ka)$

$$F(\pm\sqrt{Ka}) = \pm\, 2i\sqrt{Ka}\, e^{iKa} \int_{\pm\sqrt{Ka}}^{\infty} e^{-i\tau^2}\, d\tau, \quad \begin{bmatrix} -3\pi/4 < \arg\sqrt{Ka} < \pi/4, \\ \pi/4 < \arg\sqrt{Ka} < 5\pi/4 \end{bmatrix}.$$

### Parameters/definitions

- $I(K) = \int_C g(\xi)\, e^{Kf(\xi)}\, d\xi$, $K \gg 1$.
- $\xi_s$: first-order saddle, $f'(\xi_s) = 0$, $f''(\xi_s) \neq 0$.
- Mapping on SDP: $f(\xi) = f(\xi_s) - \alpha^2$, $G(\alpha) = g(\xi)\, \dfrac{d\xi}{d\alpha}$.
- $\phi_s = \arg\left[ \dfrac{d\xi}{d\alpha}\Big|_{\alpha=0} \right]$ (SDP direction at saddle).
- Pole separation: $\alpha_p^2 = f(\xi_s) - f(\xi_p) \equiv -ia$, $\sqrt{Ka} = \alpha_p e^{i\pi/4}\sqrt{K}$.
- Residue term: $R_p = \lim_{\xi \to \xi_p} (\xi - \xi_p)\, g(\xi)\, e^{Kf(\xi)}$.
- Enclosure index $Q \in \{+1, -1, 0\}$ ( +1: CCW, -1: CW, 0: none ).
- Asymptotics of $F$

$$F(\pm\sqrt{X}) \sim 1 - \frac{1}{2iX} + \cdots, \quad X = Ka,\ |X| \to \infty,$$

$$F(\pm\sqrt{Ka}) \approx \pm\sqrt{\pi i Ka}\, e^{iKa} \quad (\text{small } \sqrt{Ka}).$$