Advanced Electromagnetics

Chapter 4 –Scattering

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Outline

4.1 Cross Sections

In the analysis of scattering problems, the concept of cross section is frequently used to quantitatively characterize how an object scatters electromagnetic waves in the far field.

The cross section provides a measure of the effective area that intercepts and scatters the incident energy. Various types of cross sections are defined depending on the nature of the interaction. These quantities are fundamental in describing the strength and angular distribution of scattered fields, and are especially useful when comparing the scattering behavior of different objects or materials.

4.1 Cross Sections

Consider an incident plane wave propagating in the $\hat{\imath}$ direction

$$\vec{E}_i = \vec{E}_0 e^{-ik\hat{\imath}\cdot\vec{r}} \tag{4.1.1}$$

We define the scattered field as the difference between the total field and the incident field

$$\vec{E}_s = \vec{E}_t - \vec{E}_i \tag{4.1.2}$$

A scattering problem involves solving finding the scattered field subject to appropriate boundary conditions applied to the total electric or magnetic field.

4.1 Cross Sections

In the far field, the scattered field takes the form

$$\vec{E}_s = \begin{cases} E_0 \dfrac{e^{-ikr}}{r} \vec{f}(\hat{s}, \hat{\imath}) \ \ (3\mathrm{D}) \\ E_0 \dfrac{e^{-ik\rho}}{\sqrt{\rho}} \vec{f}(\hat{s}, \hat{\imath}) \ \ (2\mathrm{D}) \end{cases} \qquad (4.1.3)$$

where $E_0 = |\vec{E}_0|$ and $\vec{f}(\hat{s}, \hat{\imath})$ is called the scattering amplitude function representing the scattered wave in the $\hat{s}$ direction.

From (1.6.7), we have the incident and scattered power flux density:

$$\vec{S}_i = \frac{1}{2}\left(\vec{E}_i \times \vec{H}_i^*\right) = \frac{|\vec{E}_i|^2}{2\eta}\hat{\imath}, \quad \vec{S}_s = \frac{1}{2}\left(\vec{E}_s \times \vec{H}_s^*\right) = \frac{|\vec{E}_s|^2}{2\eta}\hat{s} \qquad (4.1.4)$$

4.1 Cross Sections

We define the differential scattering cross section as

$$\sigma_d(\hat{s},\hat{\imath}) = \begin{cases} \lim\limits_{r\to\infty} r^2 \dfrac{|\vec{S}_s|}{|\vec{S}_i|} & (3\mathrm{D}) \\ \lim\limits_{\rho\to\infty} \rho \dfrac{|\vec{S}_s|}{|\vec{S}_i|} & (2\mathrm{D}) \end{cases} = \left|\vec{f}(\hat{s},\hat{\imath})\right|^2 \tag{4.1.5}$$

The bistatic radar cross section (RCS) is defined as

$$\sigma_{bi}(\hat{s},\hat{\imath}) = \begin{cases} 4\pi\sigma_d(\hat{s},\hat{\imath}) & (3\mathrm{D}) \\ 2\pi\sigma_d(\hat{s},\hat{\imath}) & (2\mathrm{D}) \end{cases} \tag{4.1.6}$$

In 2D, σ_{bi} is also called echo width.

4.1 Cross Sections

The bistatic RCS represents the hypothetical area that, when illuminated by the incident power density and scattering that power isotropically, would produce the same reflected power at the radar as the actual target. The RCS is defined to be independent of the distance between the radar and the target. However, it strongly depends on factors such as incidence angle, observation angle, polarization, frequency, and the target's material and shape.

In particular, the monostatic or backscattering RCS refers to the case where the radar transmitter and receiver are co-located, measuring the power reflected directly back toward the source:

$$\sigma_{mono}(\hat{\imath}) = \sigma_{bi}(-\hat{\imath}, \hat{\imath}) \qquad (4.1.8)$$

4.1 Cross Sections

The scattering cross section, quantifying the amount of incident power that is scattered by an object in all directions, is defined as

$$\sigma_{sca} = \begin{cases} \int_{4\pi} \sigma_d \, d\Omega = \int_{4\pi} r^2 \frac{|\vec{S}_s|}{|\vec{S}_i|} d\Omega & \text{(3D)} \\ \int_{2\pi} \sigma_d \, d\phi = \int_{2\pi} \rho \frac{|\vec{S}_s|}{|\vec{S}_i|} d\phi & \text{(2D)} \end{cases} \qquad (4.1.9)$$

where $d\Omega$ is the differential solid angle.

4.2 Cylindrical Waves

Introduction

In this chapter, we focus on solving the wave and Helmholtz equations in cylindrical and spherical coordinates, along with the corresponding scattering phenomena.

As noted at the end of Section 1.5, in a source-free region, the scalar Helmholtz equation (1.5.17) applies to all three field components in Cartesian coordinates. In cylindrical coordinates, only the z-unit vector remains constant, so (1.5.17) holds for the z-component alone. In spherical coordinates, since none of the unit vectors are constant, additional care is required in the analysis, which is pointed out in Section 4.4.1.

4.2.1 Cylindrical Wave Solution

As mentioned above, we can simplify the vector Helmholtz equation to a scalar one for the z component in cylindrical coordinates:

$$\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\psi\right) + \frac{1}{\rho^2}\partial_\phi^2\psi + \partial_z^2\psi + k^2\psi = 0 \qquad (4.2.1)$$

Seperation of variables are used to solve (2.2.1) by assuming

$$\psi = B(\rho)\Phi(\phi)Z(z) \qquad (4.2.2)$$

Subsituting back to (4.2.1), we get

$$\frac{1}{B\rho}\frac{d}{d\rho}\left(\rho\frac{dB}{d\rho}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0 \qquad (4.2.3)$$

4.2.1 Cylindrical Wave Solution

Variable z is only found in the thrid term of (4.2.3) alone. Thus, similar to the rectangular case, we have

$$\frac{d^2Z}{dz^2} + k_z^2 Z = 0 \tag{4.2.4}$$

with the elementary solution of

$$Z = e^{\pm i k_z z} \tag{4.2.5}$$

Now let

$$k_\rho^2 = k^2 - k_z^2 \tag{4.2.6}$$

and multiply (4.2.3) with ρ^2, we get

$$\frac{\rho}{B}\frac{d}{d\rho}\left(\rho\frac{dB}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + k_\rho^2\rho^2 = 0 \tag{4.2.7}$$

4.2.1 Cylindrical Wave Solution

Similarly, we can separate out the Φ term as

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0 \tag{4.2.8}$$

with the elemenatry solution of

$$\Phi = e^{\pm i\nu\phi} \tag{4.2.9}$$

where $\nu = m$ is an integer constant if we assume Φ is periodic over 2π. Thus, (4.2.7) can now be simplified as

$$\frac{d^2B}{d\rho^2} + \frac{1}{\rho}\frac{dB}{d\rho} + \left(k_\rho^2 - \frac{\nu^2}{\rho^2}\right)B = 0 \tag{4.2.10}$$

(4.2.10) is known as the Bessel equation of order ν.

4.2.1 Cylindrical Wave Solution

The first solution of (4.2.10) with finite value at $\rho = 0$ is called the Bessel function of the first kind

$$J_\nu(k_\rho \rho) = \sum_{m=0}^{\infty} \frac{(-1)^m (k_\rho \rho/2)^{\nu+2m}}{m!(m+\nu)!} \qquad (4.2.11)$$

For non-integer ν, we have a second independent solution $J_{-\nu}(k_\rho \rho)$ which is infinite at $\rho = 0$. If ν is an integer $\nu = n$, then

$$J_{-n}(k_\rho \rho) = (-1)^n J_n(k_\rho \rho) \qquad (4.2.12)$$

A second independent solution called Neumannn function is constructed by

$$N_n(k_\rho \rho) = \lim_{\nu \to n} \frac{J_\nu(k_\rho \rho)\cos(\nu\pi) - J_{-\nu}(k_\rho \rho)}{\sin(\nu\pi)} \qquad (4.2.13)$$

4.2.1 Cylindrical Wave Solution

The Hankel functions of the first and second kind are conbinations of (4.2.11) and (4.2.13):

$$H_n^{(1)}(k_\rho \rho) = J_n(k_\rho \rho) + iN_n(k_\rho \rho) \tag{4.2.14}$$

$$H_n^{(2)}(k_\rho \rho) = J_n(k_\rho \rho) - iN_n(k_\rho \rho) \tag{4.2.15}$$

For $k_\rho \rho \gg 1$, the Hankel functions approximate aymptotically as

$$H_n^{(1)}(k_\rho \rho) \cong \sqrt{\frac{2}{\pi k_\rho \rho}} e^{i(k_\rho \rho - n\pi/2 - \pi/4)} \tag{4.2.16}$$

$$H_n^{(2)}(k_\rho \rho) \cong \sqrt{\frac{2}{\pi k_\rho \rho}} e^{-i(k_\rho \rho - n\pi/2 - \pi/4)} \tag{4.2.17}$$

which behave as an incoming and outgoing travelling waves.

4.2.2 Cylindrical Wave Transformation

Consider a *z*-polarized plane wave propergating in *x*-direction.

$$\vec{E} = \hat{z}E_0 e^{-ikx} = \hat{z}E_0 e^{-ik\rho\cos\phi} \tag{4.2.18}$$

Let us expand the exponential term into an infinite sum of cylincrical waves

$$e^{-ik\rho\cos\phi} = \sum_{n=-\infty}^{\infty} a_n J_n(k\rho) e^{in\phi} \tag{4.2.19}$$

To find the coefficients a_n, we need the following identities

$$\int_0^{2\pi} e^{-i(k\rho\cos\phi + m\phi)} d\phi = 2\pi i^{-m} J_{-m}(-k\rho) = 2\pi i^{-m} J_m(k\rho) \tag{4.2.20}$$

$$\int_0^{2\pi} e^{i(m-n)\phi} d\phi = 2\pi\delta_{mn} \tag{4.2.21}$$

4.2.2 Cylindrical Wave Transformation

Multiply (4.2.19) with $e^{im\phi}$ and integrate over ϕ from 0 to 2π, we get

$$a_m = i^{-m} \tag{4.2.22}$$

Thus, we have the following expansion

$$E_z = E_0 e^{-ikx} = E_0 e^{-ik\rho\cos\phi} = E_0 \sum_{n=-\infty}^{\infty} i^{-n} J_n(k\rho) e^{in\phi} \tag{4.2.23}$$

This is called the cylindrical wave transformation which expands a plane wave to a sum of cylindrical waves.

4.3 Scattering from PEC Cylinders

Introduction

Consider a perfectly conducting cylinder of radius a, aligned along the z-axis and illuminated by a plane wave. Two distinct incident polarizations are defined with respect to the cylinder's axis: E-polarization (transverse magnetic to z, or TM) and H-polarization (transverse electric to z, or TE). In this section, we analyze the TM case, and TE case is left as exercise.

4.3 Scattering from PEC Cylinders

Consider a plane wave propagating in x direction. Using the expansion in (4.2.23), we have the incident field

$$\vec{E}_i = E_0 e^{-ikx}\hat{z} = \hat{z}E_0 \sum_{n=-\infty}^{\infty} i^{-n} J_n(k\rho)e^{in\phi} \qquad (4.3.1)$$

Suppose the scattered field takes the form of

$$\vec{E}_s = \hat{z}E_0 \sum_{n=-\infty}^{\infty} i^{-n} a_n H_n^{(2)}(k\rho)e^{in\phi} \qquad (4.3.2)$$

We use the Hankel function of the second kind to represent outgoing propagating waves. Applying the boundary condition $E_z^t = 0$ at $\rho = a$, we get

$$E_z^t = E_0 \sum_{n=-\infty}^{\infty} i^{-n} \left[J_n(ka) + a_n H_n^{(2)}(ka)\right] e^{in\phi} = 0 \qquad (4.3.3)$$

4.3 Scattering from PEC Cylinders

Thus, the unknown coefficient is

$$a_n = -J_n(ka)/H_n^{(2)}(ka) \tag{4.3.4}$$

Then, scattered field is:

$$\vec{E}_s = -\hat{z}E_0 \sum_{n=-\infty}^{\infty} i^{-n} \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho)e^{in\phi} \tag{4.3.5}$$

and the total field is:

$$\vec{E}_t = \hat{z}E_0 \sum_{n=-\infty}^{\infty} i^{-n} \left[\frac{H_n^{(2)}(ka)J_n(k\rho)-J_n(ka)H_n^{(2)}(k\rho)}{H_n^{(2)}(ka)}\right] e^{in\phi} \tag{4.3.6}$$

4.3 Scattering from PEC Cylinders

From (4.3.17), the far scattered field is approximated as

$$\vec{E}_s \cong -\hat{z}E_0\sqrt{\frac{2i}{\pi k}}\frac{e^{-ik\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{in\phi} \tag{4.3.7}$$

And from (4.1.6), we can compute the echo width:

$$\sigma_{bi} = \frac{4}{k}\left|\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{in\phi}\right|^2 \tag{4.3.8}$$

The result in (4.3.8) is often normalized according to λ as

$$\sigma_{bi}/\lambda = \frac{2}{\pi}\left|\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{in\phi}\right|^2 \tag{4.3.9}$$

4.4 Spherical Waves

4.4.1 Spherical Wave Solution

The application of the vector Helmholtz equation in spherical coordinates is not straightforward, as none of the unit vectors are constant throughout space. As a result, the vector Helmholtz equation cannot be directly reduced to separate scalar Helmholtz equations for each component.

To illustrate this, consider (3.1.4) $\vec{\nabla} \times \vec{\nabla} \times \vec{A} - k^2\vec{A} = \mu\vec{J} - i\omega\mu\epsilon\vec{\nabla}\mathcal{V}$ in spherical coordinates. By assuming that $\vec{A}$ and $\vec{J}$ have only radial component, that is

$$\vec{A} = A_r\hat{r}, \quad \vec{J} = J_r\hat{r} \tag{4.4.1}$$

4.4.1 Spherical Wave Solution

Then, the r, θ, and ϕ components of (3.1.4) are given by

$$\left[\frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\ \ \partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 + k^2\right]A_r = i\omega\mu\epsilon\partial_r\mathcal{V} - \mu J_r \tag{4.4.2}$$

$$-\frac{1}{r}\partial_r\partial_\theta A_r = \frac{i\omega\mu\epsilon}{r}\partial_\theta\mathcal{V} \tag{4.4.3}$$

$$-\frac{1}{r\sin\theta}\partial_r\partial_\theta A_r = \frac{i\omega\mu\epsilon}{r\sin\theta}\partial_\phi\mathcal{V} \tag{4.4.4}$$

We notice that the Lorenz gauge condition (3.1.8) cannot be used to simplify the (4.4.3-4); instead, we choose

$$\partial_r A_r + i\omega\mu\epsilon\mathcal{V} = 0 \tag{4.4.5}$$

4.4.1 Spherical Wave Solution

By doing so, (4.4.3-4) is satisfied and (4.4.2) is rewritten as

$$\left[\partial_r^2 + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\ \ \partial_\theta) + \frac{1}{r^2 \sin^2\theta}\partial_\phi^2 + k^2\right] A_r = -\mu J_r \quad (4.4.6)$$

Let us introduce the Debye Hertz potential:

$$\pi_e = A_r / i\omega\mu\epsilon r \quad (4.4.7)$$

Then (4.4.6) is transformed into the scalar Helmholtz equation in spherical coordinates with $\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{r^2 \sin^2\theta}\partial_\phi^2 + k^2$:

$$(\nabla^2 + k^2)\pi_e = -\frac{J_r}{i\omega\epsilon r} \quad (4.4.8)$$

4.4.1 Spherical Wave Solution

By duality, we also have

$$(\nabla^2 + k^2)\pi_m = -\frac{M_r}{i\omega\mu r} \tag{4.4.10}$$

Once the solutions of (4.4.9-10) are found, the fields are given by

$$\vec{E} = \vec{\nabla} \times \vec{\nabla} \times \vec{\Pi}_e - i\omega\mu\vec{\nabla} \times \vec{\Pi}_m - \frac{\vec{J}}{i\omega\epsilon} = \left(\vec{\nabla}\partial_r + k^2\right)\vec{\Pi}_e - i\omega\mu\vec{\nabla} \times \vec{\Pi}_m \tag{4.4.11}$$

$$\vec{H} = \vec{\nabla} \times \vec{\nabla} \times \vec{\Pi}_m + i\omega\epsilon\vec{\nabla} \times \vec{\Pi}_r - \frac{\vec{M}}{i\omega\mu} = \left(\vec{\nabla}\partial_r + k^2\right)\vec{\Pi}_m + i\omega\mu\vec{\nabla} \times \vec{\Pi}_e \tag{4.4.12}$$

where $\vec{\Pi}_e = r\pi_e\hat{r} = \Pi_e\hat{r}$ and $\vec{\Pi}_m = r\pi_m\hat{r} = \Pi_m\hat{r}$.

4.4.1 Spherical Wave Solution

In terms of spherical components:

$$E_r = (\partial_r^2 + k^2)\Pi_e \tag{4.4.13a}$$

$$E_\theta = \frac{1}{r}\partial_r\partial_\theta\Pi_e - \frac{i\omega\mu}{\sin\theta}\partial_\phi\Pi_m \tag{4.4.13b}$$

$$E_\phi = \frac{1}{r\sin\theta}\partial_r\partial_\phi\Pi_e + i\omega\mu\partial_\theta\Pi_m \tag{4.4.13c}$$

$$H_r = (\partial_r^2 + k^2)\Pi_m \tag{4.4.14a}$$

$$H_\theta = \frac{1}{r}\partial_r\partial_\theta\Pi_m + \frac{i\omega\epsilon}{\sin\theta}\partial_\phi\Pi_e \tag{4.4.14b}$$

$$H_\phi = \frac{1}{r\sin\theta}\partial_r\partial_\phi\Pi_m - i\omega\epsilon\partial_\theta\Pi_e \tag{4.4.14c}$$

4.4.1 Spherical Wave Solution

We considered here only the radial electric and magnetic current sources, where π_e and π_m are directly related to J_r and M_r. For sources with θ and ϕ-components, the relations become more complex. Nonetheless, the general electromagnetic field in spherical coordinates can still be fully described by the two scalar functions.

Now consider the scalar Helmholtz equation in spherical coordinates

$$\frac{1}{r^2}\partial_r(r^2\partial_r\psi)+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\ \partial_\theta\psi)+\frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi+k^2\psi=0 \tag{4.4.15}$$

4.4.1 Spherical Wave Solution

Using separation of variables, let

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \qquad (4.4.16)$$

Plug into (4.4.15), multiply $r^2 \sin\theta$ and divide it by ψ, we get

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + (kr\sin\theta)^2 = 0$$

(4.4.17)

Similar tot the cylindrical case, for the Φ term, we have

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0, \quad \Phi = e^{\pm i\nu\phi} \qquad (4.4.18)$$

where ν is a constant.

4.4.1 Spherical Wave Solution

Plug into (4.4.17) and divide it by $\sin^2\theta$, we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{\nu^2}{\sin^2\theta}+(kr)^2=0 \quad (4.4.19)$$

Introducing the constant

$$\frac{d^2\Phi}{d\phi^2}+\nu^2\Phi=0, \quad \Phi=e^{\pm i\nu\phi} \quad (4.4.18)$$

where ν is a constant. Now, let

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{\nu^2}{\sin^2\theta}=-\mu^2 \quad (4.4.20)$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+(kr)^2=\mu^2 \quad (4.4.21)$$

4.4.1 Spherical Wave Solution

Since Φ is periodic over 2π, we let $\nu = m$ an integer value, and for (4.4.20), we get

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\mu^2-\frac{m^2}{\sin^2\theta}\right)\Theta=0 \tag{4.4.22}$$

Let us consider two cases: the solution is azimuthal symmetric or non-azimuthal symmetric. If it is azimuthal symmetric, we have $m = 0$ and (4.2.21) is reduced to

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\mu^2\Theta=0 \tag{4.4.23}$$

To ensure that all solutions of the equation remain finite at $\sin\theta = \pm 1$, the parameter μ^2 must take the specific form $\mu^2 = n(n+1)$, where n is a non-negative integer.

4.4.1 Spherical Wave Solution

Thus, we arrive at the following equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\Theta=0 \tag{4.4.24}$$

which is known as the Legendre equation, and their solutions are known as the Legendre polynomials:

$$P_n(\cos\theta)=\frac{1}{2^n n!}\left(\frac{d}{d\cos\theta}\right)^n(\cos^2\theta-1)^n \tag{4.4.25}$$

The solutions form a complete set in the interval $-1\leq\cos\theta\leq 1$. The orthogonality relation is

$$\int_{-1}^{1}P_n(\cos\theta)P_{n'}(\cos\theta)\,d\cos\theta=\frac{2}{2n+1}\delta_{nn'} \tag{4.4.26}$$

4.4.1 Spherical Wave Solution

Hence, if we have a function that lies in $0 \le \theta \le \pi$ $(-1 \le \cos\theta \le 1)$, we can expand it as

$$f(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta), \quad 0 \le \theta \le \pi \tag{4.4.27}$$

where

$$a_n = \frac{2n+1}{2} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta \tag{4.4.28}$$

On the other hand, if the solution is non-azimuthal symmetric, we have the associated Legendre equation

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \tag{4.4.29}$$

4.4.1 Spherical Wave Solution

The solutions to (4.4.29) is the associated Legendre polynomials of the first $P_n^m(\cos\theta)$ and second kind $Q_n^m(\cos\theta)$. Notice that all the solutions are necessarily singular at $\cos\theta = \pm 1$ except for $P_n^m(\cos\theta)$ with integer m and n.

For positive integer m and n, they are related to the ordinary Legendre polynomials as

$$P_n^m(\cos\theta) = \sin^m\theta \, P_n^{(m)}(\cos\theta), \quad m \le n \tag{4.4.30}$$

where $P_n^{(m)}$ is the m^{th} derivative with respect to $\cos\theta$ of the n^{th}-order Legendre polynomial.

4.4.1 Spherical Wave Solution

Now let us consider (4.4.21). Substituting $\mu^2 = n(n+1)$, and let

$$R = R_n(kr) = \sqrt{\frac{\pi}{2kr}} B_{n+1/2}(kr) = \sqrt{\frac{\pi}{2kr}} B \qquad (4.4.31)$$

(4.4.21) becomes

$$\frac{d^2B}{dr^2} + \frac{1}{r}\frac{dB}{dr} + \left(k^2 - \frac{n+1/2}{r^2}\right) B = 0 \qquad (4.4.32)$$

which is the Bessel equation and $B_{n+1/2}$ is the Bessel function of order $n+1/2$. (4.3.31) is known as the spherical Bessel function. We often use lowercase letter to denote the spherical modification of the Bessel function, for example, $h_n^{(2)}(kr) = \sqrt{\frac{\pi}{2kr}} H_{n+1/2}^{(2)}(kr)$.

4.4.2 Spherical Wave Transformation

Similar to (4.2.18), consider an *x*-polarized plane wave propergating in *z*-direction:

$$\vec{E} = \hat{x} E_0 e^{-ikz} = \hat{x} E_0 e^{-ikr\cos\theta} \tag{4.4.33}$$

Let us expand the exponential term into an infinite sum of spherical waves. Since (4.4.33) is azimuthal symmetric, we have $m = 0$, and since we a finite value at the origin, we have

$$e^{-ikr\cos\theta} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\theta) \tag{4.4.34}$$

To find a_n, let us find use the orthogonality relation (4.4.26)

$$a_n j_n(kr) = \frac{2n+1}{2} \int_{-1}^{1} e^{-ikr\cos\theta} P_n(\cos\theta)\, d\cos\theta \tag{4.4.35}$$

4.4.2 Spherical Wave Transformation

Using the following identity

$$\int_{-1}^{1} e^{-ikr\cos\theta} P_n(\cos\theta)\, d\cos\theta = 2i^{-n} j_n(kr) \qquad (4.4.36)$$

we have

$$a_n = (2n+1)i^{-n} \qquad (4.4.37)$$

Thus, we have the following expansion

$$E_x = E_0 e^{-ikz} = E_0 e^{-ikr\cos\theta} = E_0 \sum_{n=0}^{\infty} (2n+1) i^{-n} j_n(kr) P_n(\cos\theta) \qquad (4.2.38)$$

This is called the spherical wave transformation which expands a plane wave to a sum of spherical waves.

4.5 Scattering from Dielectric Spheres

4.5 Scattering from Dielectric Spheres

The exact solution for the scattering of a plane electromagnetic wave by an isotropic, homogeneous dielectric sphere of arbitrary size is commonly known as Mie theory. Consider a dieletric sphere with radius a and permitivity / permeability equal to ϵ_d/μ_d placed in a medium with permitivity / permeability equal to $\epsilon\,/\mu$.

Let the incident electric field be an *x*-polarized plane wave propergating in *z*-direction. The radial component of incident electric field can be express as

$$E_r^i = E_0 e^{-ikz}\hat{x}\cdot\vec{r} = E_0 \sin\theta\cos\phi\, e^{-ikr\cos\theta}$$

$$= \frac{1}{ikr}E_0\cos\phi\,\partial_\theta e^{-ikr\cos\theta} \qquad (4.5.1)$$

4.5 Scattering from Dielectric Spheres

Using (4.2.38), the radial component of incident electric field can be express as

$$E_r^i = \frac{E_0 \cos\phi}{ikr} \partial_\theta \sum_{n=0}^{\infty} (2n+1) i^{-n} j_n(kr) P_n(\cos\theta)$$

$$= \frac{iE_0 \cos\phi}{(kr)^2} \sum_{n=1}^{\infty} (2n+1) i^{-n} \hat{J}_n(kr) P_n^1(\cos\theta) \qquad (4.5.2)$$

where

$$-\partial_\theta P_n(\cos\theta) = P_n^1(\cos\theta) \qquad (4.5.3)$$

and

$$\hat{J}_n(kr) = kr j_n(kr) \qquad (4.5.4)$$

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To find the Debye potential of the incident field, (4.4.13a) must be satisfied:

$$E_r^i = (\partial_r^2 + k^2)\Pi_e^i = (\partial_r^2 + k^2)\left(r\pi_e^i\right) \tag{4.5.5}$$

Let us expand the Debye potential in terms of spherical harmonics

$$\pi_e^i = \sum_{n=0}^{\infty}\sum_{m=0}^{n} j_n(k_2 r)P_n^1(\cos\theta)(A_{mn}\cos m\phi + B_{mn}\sin m\phi) \tag{4.5.6}$$

Substituting (4.5.6) into (4.5.5), we get

$$E_r^i = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{n(n+1)}{kr^2}\hat{j}_n(kr)P_n^m(\cos\theta)(A_{mn}\cos m\phi + B_{mn}\sin m\phi) \tag{4.5.7}$$

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Here, the standard idenitiy is used for the derivation

$$\frac{d^2}{dr^2}[rj_n(kr)] + (kr)^2 j_n(kr) = n(n+1)j_n(kr) \tag{4.5.8}$$

which came directly from (4.4.21). Comparing (4.5.7) and (4.5.2), we get

$$\begin{cases} A_{mn} = 0, & m \neq 1 \\ A_{1n} = E_0(-i)^{n-1}\frac{2n+1}{kn(n+1)} \end{cases} \tag{4.5.9}$$

and

$$B_{mn} = 0 \tag{4.5.10}$$

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Thus, we have

$$\pi_e^i = \frac{E_0 \cos\phi}{k^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} \hat{j}_n(kr) P_n^1(\cos\theta) \qquad (4.5.11)$$

Following a similar process, we can obtain

$$\pi_m^i = \frac{E_0 \sin\phi}{\eta k^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} \hat{j}_n(kr) P_n^1(\cos\theta) \qquad (4.5.12)$$

To find the scattered field inside and outside the sphere, let us first specify the boundary conditions to be applied at $r = a$:

$$E_\theta = E_\theta^d, \quad E_\phi = E_\phi^d \qquad (4.5.13)$$

$$H_\theta = H_\theta^d, \quad H_\phi = H_\phi^d \qquad (4.5.14)$$

where the superscript *d* denotes the field inside the sphere.

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As seen from (4.4.13-14) and (4.5.13-14), the boundary conditions involve both $\pi_e = \pi_e^i + \pi_e^s$ and $\pi_{mr} = \pi_m^i + \pi_m^s$, making them coupled. To simplify the analysis, it is convenient to decouple them by deriving boundary conditions for π_e and π_m individually. To achieve this, we consider a linear combination of E_θ and E_ϕ in such a way that all terms involving π_m cancel out: $\partial_\theta(\sin\theta\, E_\theta) + \partial_\phi E_\phi = \left[\partial_\theta(\sin\theta\, \partial_\theta) + \frac{1}{\sin\theta}\partial_\phi^2\right]\frac{1}{r}\partial_r(r\pi_e)$. Using similar technique, at $r = a$, we have:

$$\frac{1}{r}\partial_r(r\pi_e) = \frac{1}{r}\partial_r\left(r\pi_e^d\right), \qquad \mu\pi_e = \mu_d\pi_e^d \tag{4.5.15}$$

$$\frac{1}{r}\partial_r(r\pi_m) = \frac{1}{r}\partial_r\left(r\pi_m^d\right), \quad \epsilon\pi_{mr} = \epsilon_d\pi_m^d \tag{4.5.16}$$

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The boundary conditions ensure that each Debye potential function outside the sphere couples exclusively to its corresponding Debye potential inside the sphere. As a result, if the incident field involves terms with $\cos\phi$ dependence, both the scattered and internal fields will exhibit the same $\cos\phi$ dependence. Likewise, all terms involving $\sin\phi$ will retain their $\sin\phi$ dependence.

Thus, for the scattered field potential, we let

$$\pi_e^s = \frac{-E_0\cos\phi}{k^2 r}\sum_{n=1}^{\infty}(-i)^{n-1}\frac{2n+1}{n(n+1)}a_n\hat{h}_n^{(2)}(kr)P_n^1(\cos\theta) \quad (4.5.17)$$

$$\pi_m^s = \frac{-E_0\sin\phi}{\eta k^2 r}\sum_{n=1}^{\infty}(-i)^{n-1}\frac{2n+1}{n(n+1)}b_n\hat{h}_n^{(2)}(kr)P_n^1(\cos\theta) \quad (4.5.18)$$

where $\hat{h}_n^2(kr)$ is used to meet the radiation condition.

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The total potential outside the sphere is thus written as

$$\pi_e = \frac{E_0 \cos\phi}{k^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} \left[\hat{j}_n(kr) - a_n \hat{h}_n^2(kr)\right] P_n^1(\cos\theta) \tag{4.5.19}$$

$$\pi_m = \frac{\eta E_0 \sin\phi}{k^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} \left[\hat{j}_n(kr) - b_n \hat{h}_n^2(kr)\right] P_n^1(\cos\theta) \tag{4.5.20}$$

For the Debye potential inside the sphere, we let

$$\pi_e^d = \frac{E_0 \cos\phi}{k_d^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} c_n \hat{j}_n(k_d r) P_n^1(\cos\theta) \tag{4.5.21}$$

$$\pi_m^d = \frac{E_0 \sin\phi}{\eta_d k_d^2 r} \sum_{n=1}^{\infty} (-i)^{n-1} \frac{2n+1}{n(n+1)} d_n \hat{j}_n(k_d r) P_n^1(\cos\theta) \tag{4.5.22}$$

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Applying the boundary conditions (4.5.15-16) to (4.5.19-22), we can find the coefficients

$$a_n = \frac{\sqrt{\epsilon\mu_d}\,\hat{j}_n(\alpha)\hat{j}_n'(\beta) - \sqrt{\epsilon_d\mu}\,\hat{j}_n'(\alpha)\hat{j}_n(\beta)}{\sqrt{\epsilon\mu_d}\,\hat{h}_n^{(2)}(\alpha)\hat{j}_n'(\beta) - \sqrt{\epsilon_d\mu}\,\hat{h}_n^{(2)\prime}(\alpha)\hat{j}_n(\beta)} \tag{4.5.23}$$

$$b_n = \frac{\sqrt{\epsilon\mu_d}\,\hat{j}_n'(\alpha)\hat{j}_n(\beta) - \sqrt{\epsilon_d\mu}\,\hat{j}_n(\alpha)\hat{j}_n'(\beta)}{\sqrt{\epsilon\mu_d}\,\hat{h}_n^{(2)\prime}(\alpha)\hat{j}_n(\beta) - \sqrt{\epsilon_d\mu}\,\hat{h}_n^{(2)}(\alpha)\hat{j}_n'(\beta)} \tag{4.5.24}$$

$$c_n = \frac{i\,\sqrt{\epsilon_d\mu}}{\sqrt{\epsilon\mu_d}\,\hat{h}_n^{(2)}(\alpha)\hat{j}_n'(\beta) - \sqrt{\epsilon_d\mu}\,\hat{h}_n^{(2)\prime}(\alpha)\hat{j}_n(\beta)} \tag{4.5.25}$$

$$d_n = \frac{-i\sqrt{\epsilon\mu_d}}{\sqrt{\epsilon\mu_d}\,\hat{h}_n^{(2)\prime}(\alpha)\hat{j}_n(\beta) - \sqrt{\epsilon_d\mu}\,\hat{h}_n^{(2)}(\alpha)\hat{j}_n'(\beta)} \tag{4.5.26}$$

where $\alpha = ka$ and $\beta = k_d a$.

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Let us consider the case in far field. From (4.2.17), when $kr \to \infty$,

$$\hat{h}_n^{(2)}(kr) = \sqrt{\frac{\pi kr}{2}} J_{n+1/2}(kr) \cong i^{n+1} e^{-ikr} \qquad (4.5.27)$$

and we get

$$\Pi_e^s = r\pi_e^s \cong e^{-ikr} \frac{E_0 \cos\phi}{k^2} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} a_n P_n^1(\cos\theta) \qquad (4.5.28)$$

$$\Pi_m^s = r\pi_m^s \cong e^{-ikr} \frac{E_0 \sin\phi}{\eta k^2} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} b_n P_n^1(\cos\theta) \qquad (4.5.29)$$

Noting that in the far field

$$\partial_r \Pi_e^s \cong -ik\Pi_e^s, \quad \partial_r \Pi_m^s \cong -ik\Pi_m^s \qquad (4.5.30)$$

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we can thus express

$$E_\theta \cong f_\theta(\theta,\phi)e^{-ikr}/r \tag{4.5.31}$$

$$E_\phi \cong f_\phi(\theta,\phi)e^{-ikr}/r \tag{4.5.32}$$

$$f_\theta(\theta,\phi) = -i\cos\phi\ S_2(\theta)/k \tag{4.5.33}$$

$$f_\phi(\theta,\phi) = i\sin\phi\ S_1(\theta)/k \tag{4.5.34}$$

$$S_1(\theta) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}[a_n\pi_n(\cos\theta) + b_n\tau_n(\cos\theta)] \tag{4.5.35}$$

$$S_2(\theta) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}[a_n\tau_n(\cos\theta) + b_n\pi_n(\cos\theta)] \tag{4.5.36}$$

$$\pi_n(\cos\theta) = \frac{P_n^1(\cos\theta)}{\sin\theta}, \quad \tau_n(\cos\theta) = \frac{d}{d\theta}P_n^1(\cos\theta) \tag{4.5.37}$$

4.5 Scattering from Dielectric Spheres

By (4.1.5), the differential cross section is

$$\sigma_d(\theta,\phi) = \left|\vec{f}(\theta,\phi)\right|^2 = \left|\cos\phi\,\frac{S_2(\theta)}{k}\right|^2 + \left|\sin\phi\,\frac{S_1(\theta)}{k}\right|^2 \quad (4.5.38)$$

And by (4.1.9), the scattering cross section is

$$\sigma_{sca} = \int_{4\pi} \sigma_d(\theta,\phi)d\Omega = \frac{\pi}{k^2}\int_0^\pi \left(|S_2(\theta)|^2 + |S_1(\theta)|^2\right)\sin\theta\,d\theta \quad (4.5.39)$$

$$|S_1(\theta)|^2 = \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} \frac{2n'+1}{n'(n'+1)}\frac{2n+1}{n(n+1)}\left[a_n\tau_n a_{n'}^*\tau_{n'} + b_n\pi_n b_{n'}^*\pi_{n'} + a_n\tau_n b_{n'}^*\pi_{n'} + b_n\pi_n a_{n'}^*\tau_{n'}\right] \quad (4.5.40)$$

$$|S_2(\theta)|^2 = \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} \frac{2n'+1}{n'(n'+1)}\frac{2n+1}{n(n+1)}\left[a_n\pi_n a_{n'}^*\pi_{n'} + b_n\tau_n b_{n'}^*\tau_{n'} + a_n\pi_n b_{n'}^*\tau_{n'} + b_n\tau_n a_{n'}^*\pi_{n'}\right] \quad (4.5.41)$$

4.5 Scattering from Dielectric Spheres

Using the following orthogonal property

$$\int_0^{\pi}(\pi_n\pi_{n'} + \tau_n\tau_{n'}) \sin\theta \, d\theta = \begin{cases} 0, & \text{if } n \neq n' \\ \frac{2}{2n+1}\frac{(n+1)!}{(n-1)!} n(n+1), & \text{if } n = n' \end{cases} \quad (4.5.42)$$

we get

$$\sigma_{sca} = \frac{2\pi a}{\alpha^2}\sum_{n=1}^{\infty}(2n+1)(|a_n|^2 + |b_n|^2) \quad (4.5.43)$$

Problems

1. From (4.2.1) and (4.2.6), plot $\left|\vec{E}_t/\vec{E}_i\right|$ with $k\rho$ for $ka = 1$.
2. From (4.2.5-6) and Maxwell equations, find the solution of scattered and total magnetic field.
3. From Problem 2, find the induced surface current.
4. Derive the case of TE scattering from a PEC cylinder.
5. Complete the derivation to obtain (4.5.23-26).
6. Compute the first term of (4.5.23-24) with $\beta \ll 1$.